An Exact Connection between two Solvable SDEs and a Non Linear Utility Stochastic PDEs

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- No clear idea how to specify the utility function
- Classical or recursive utilities are defined in isolation to the investment opportunities given to an agent.
- Explicit solutions to optimal investment problems can only be derived under very restrictive model and utility assumptions, as Markovian assumption which yields to HJB PDEs.
- The investor may want to use intertemporal diversification, i.e., implement short, medium and long term strategies
- Can the same utility function be used for all time horizons?
Consistent Dynamic Utility

Let $\mathcal{X}$ be a convex family of positive portfolios, called Test portfolios.

**Definition:** An $\mathcal{X}$-Consistent progressive utility $U(t, x)$ process is a positive adapted random field s.t.

- **Concavity assumption:** For $t \geq 0$, $x > 0 \mapsto U(t, x)$ is an increasing concave function, (in short utility function).

- **Consistency with the class of test portfolios:** For any admissible wealth process $X \in \mathcal{X}$, $\mathbb{E}(U(t, X_t)) < +\infty$ and

  $$\mathbb{E}(U(t, X_t) / \mathcal{F}_s) \leq U(s, X_s), \forall s \leq t.$$ 

- **Existence of optimal:** For any initial wealth $x > 0$, there exists an optimal wealth process (benchmark) $X^* \in \mathcal{X}$ ($X_0^* = x$),

  $$U(s, X_s^*) = \mathbb{E}(U(t, X_t^*) / \mathcal{F}_s) \forall s \leq t.$$ 

  **In short** for any admissible wealth $X \in \mathcal{X}$, $U(t, X_t)$ is a supermartingale, and a martingale for the optimal-benchmark wealth $X^*$. 

The General Market Model

- The security market consists of one riskless asset $S^0$, $dS^0_t = S^0_t r_t dt$, and $d$ continuous risky assets $S^i$, $i = 1..d$ defined on a filtered Brownian space $(\Omega, \mathcal{F}_{t \geq 0}, \mathbb{P})$
  \[ \frac{dS^i_t}{S^i_t} = b^i_t dt + \sigma^i_t dW_t, \quad 1 \leq i \leq d \]

- Risk premium vector, $\eta_t$ with \[ b(t) - r(t)1 = \sigma_t \eta_t \]

Def A positive wealth process is defined as a pair $(x, \pi)$, $x > 0$ is the initial value of the portfolio and $\pi = (\pi^i)_{1 \leq i \leq d}$ is the (predictable) proportion of each asset held in the portfolio, assumed to be $S$-integrable process.

- Thanks to AOA in the market, wealth process with $\pi$-strategy is driven by
  \[ \frac{dX^\pi_t}{X^\pi_t} = r_t dt + \sigma_t \pi_t (dW_t + \eta_t dt), \]

For simplicity we denote by $\mathcal{R}^\sigma$ the range of the matrix $\sigma := (\sigma^i)_{i=1...d}$, $\kappa := \sigma \pi$, $\pi \in \mathbb{R}^d$. The class of Test portfolio in what follows is
  \[ \mathcal{K}^\sigma := \{(X^K) : \frac{dX^K_t}{X^K_t} = r_t dt + \kappa_t (dW_t + \eta^\sigma_t dt), \quad \kappa_t \in \mathcal{R}^\sigma_t \} \]
Consistent Utility of Itô’s Type

Let $U$ be a dynamic utility (concave, increasing),

$$dU(t, x) = \beta(t, x)dt + \gamma(t, x)dW_t$$

such that $U(t, X_t^\kappa)$ is a supermartingale for $X_t^\kappa \in \mathcal{X}$ and a martingale for the optimal one.

Open questions

- What about the drift $\beta$ of the utility?
- What about the volatility $\gamma$ of the utility?
- Under which assumptions on $(\beta, \gamma)$ can one be sure that solutions are concave and increasing,

Main difficulties come from the forward definition.
Let \( U \) be a progressive utility of class \( C^{(2)} \) in the sense of Kunita with local characteristics \((\beta, \gamma)\) and risk tolerance coefficient \( \alpha^U_t(t, x) = -\frac{U_x(t, x)}{U_{xx}(t, x)} \). We introduce the utility risk premium \( \eta^U_t(t, x) = \frac{\gamma_x(t, x)}{U_x(t, x)} \). Then, for any admissible portfolio \( X^\kappa \),

\[
dU(t, X^\kappa_t) = \left( U_x(t, X^\kappa_t)X^\kappa_{t} + \gamma(t, X^\kappa_t) \right) . dW_t
\]

\[
+ \left( \beta(t, X^\kappa_t) + U_x(t, X^\kappa_t) r_t X^\kappa_t + \frac{1}{2} U_{xx}(t, X^\kappa_t) Q(t, X^\kappa_t, \kappa_t) \right) dt,
\]

where \( x^2 Q(t, x, \kappa) := \|x\kappa_t\|^2 - 2\alpha^U(t, x)(x\kappa_t). (\eta^\sigma_t + \eta^{U,\sigma}(t, x)) \).

Let \( \gamma^\sigma_x \) be the orthogonal projection of \( \gamma_x \) on \( R^\sigma \). Let \( Q^*(t, x) = \inf_{\kappa \in R^\sigma} Q(t, x, \kappa) \); the minimum of this quadratic form is achieved at the optimal policy \( \kappa^*_t \) given by

\[
\begin{cases}
  x\kappa^*_t(x) = -\frac{1}{U_{xx}(t, x)} (U_x(t, x) \eta^\sigma_t + \gamma^\sigma_x(t, x)) = \alpha^U(t, x)(\eta^\sigma_t + \eta^{U,\sigma}(t, x)) \\
x^2 Q^*(t, x) = -\frac{1}{U_{xx}(t, x)^2} \|U_x(t, x) \eta^\sigma_t + \gamma^\sigma_x(t, x)\|^2 = -\|x\kappa^*_t(x)\|^2.
\end{cases}
\]
Verification Theorem: 1

Let $U$ be a progressive utility of class $C^{(2)}$ in the sense of Kunita with local characteristics $(\beta, \gamma)$.

**Hyp** Assume the drift constraint to be Hamilton-Jacobi-Bellman nonlinear type

$$\beta(t, x) = -U_x(t, x)r_t x + \frac{1}{2} U_{xx}(t, x)\|x \kappa_t^*(t, x)\|^2$$  \hspace{1cm} (1)

where $\kappa^*$ is the optimal policy given by

$$x \kappa_t^*(x) = - \frac{1}{U_{xx}(t, x)} (U_x(t, x) \eta_t^\sigma + \gamma_x^\sigma(t, x))$$

Then the progressive utility is solution of the following forward HJB-SPDE

$$dU(t, x) = \left(- U_x(t, x)r_t x + \frac{1}{2} \frac{(U_x(t, x))^2}{U_{xx}(t, x)} \|\eta_t^\sigma + \frac{\gamma_x^\sigma(t, x)}{U_x(t, x)}\|^2\right)dt + \gamma(t, x).dW_t,$$

and for any admissible wealth $X_t^\kappa$, the process $U(t, X_t^\kappa)$ is a supermartingale.
Verification Theorem: II

Under previous hypothesis,

- **Assume** that $\kappa^*(t, x)$ is sufficiently smooth so that the equation

$$dX_t^* = X_t^*(r_t dt + \kappa^*(t, X_t^*). (dW_t + \eta_t^\sigma dt)$$

has a (unique? strong?) positive solution for any initial wealth $x > 0$.

⇒ Then, the progressive increasing utility $U$ is a $\mathcal{X}$-consistent utility, with optimal wealth $X_t^*$. 
Inverse flows

Let $\phi$ be a strictly monotone Itô-Ventzel regular flow with inverse process $\xi(t, y) = \phi(t, .)^{-1}(y)$. Assume $d\phi(t, x) = \mu(t, x)dt + \gamma(t, x)dW_t$,

i) The inverse flow $\xi(t, y)$ has as dynamics in old variable

$$d\xi(t, y) = -\xi'_y(t, y)(\mu(t, \xi)dt + \gamma(t, \xi)dW_t) + \frac{1}{2} \partial_y \frac{\|\gamma(t, \xi)\|^2}{\phi'_x(t, \xi)} dt$$

ii) In terms of new variable, with $\nu^\xi(t, y) = -\xi'_y \gamma(t, \xi)$

$$d\xi(t, y) = \nu^\xi(t, y)dW_t + \left(\frac{1}{2} \partial_y \left(\frac{\|\nu^\xi(t, y)\|^2}{\xi'_y}\right) - \mu(t, \xi) \xi'_y(t, y)\right) dt$$

iii) If $\phi = \Phi'_x(t, x)$ with $d\Phi(t, x) = M(t, x)dt + C(t, x)dW_t$, then $\xi = \Xi'_y(t, y)$

$$d\Xi(t, y) = -C(t, \xi)dW_t - M(t, \xi)dt + \frac{1}{2} \frac{\|C'_x(t, \xi)\|^2}{\Phi''_x(t, \xi)} dt$$
Duality: Convex conjugate SPDE

Let $U$ be a consistent progressive utility of class $C^{(3)}$, in the sense of Kunita, satisfying the $\beta$ constraint (1), then the convex conjugate $\tilde{U}(t, y) \overset{\text{def}}{=} \inf_{x \in Q^*_+} (U(t, x) - x y)$ satisfies

$$d\tilde{U}(t, y) = \left[ \frac{1}{2\tilde{U}_{yy}(t, y)} (\|\tilde{\gamma}_y(t, y)\|^2 - \|\tilde{\gamma}^\sigma_y(t, y) + y\tilde{U}_{yy}(t, y)\eta^\sigma_t\|^2) + y\tilde{U}_y(t, y)r_t \right] dt$$

$$+ \tilde{\gamma}(t, y).dW_t \quad \text{with} \quad \tilde{\gamma}(t, y) = \gamma(t, -\tilde{U}_y(t, y)).$$

- The drift $\tilde{\beta}(t, y)$ is the value of an optimization program achieved on the optimal policy $\nu^*(t, y) = -\tilde{\gamma}^\perp_y(t, y)/y\tilde{U}_{yy}(t, y)$.
- $\tilde{\beta}$ can be written as the solution of the following optimization program

$$\hat{\beta}(t, y) = y\tilde{U}_y(t, y)r_t - \frac{1}{2} y^2 \tilde{U}_{yy}(t, y) \inf_{\nu_t \in \mathcal{R}^\sigma, \perp} \{ \|\nu_t - \eta^\sigma_t\|^2 + 2(\nu_t - \eta^\sigma_t). \left( \frac{\tilde{\gamma}_y(t, y)}{y\tilde{U}_{yy}(t, y)} \right) \}$$

with $-\tilde{\gamma}_y(t, y)/y\tilde{U}_{yy}(t, y) = \eta^U(t, -\tilde{U}(t, y)) = \gamma_x(t, -\tilde{U}(t, y))/y$. 
Convex conjugate forward Utility

Under previous assumption,

- The conjugate Utility $\tilde{U}(t, y)$ is a convex decreasing stochastic flows,
- consistent with the family $\mathcal{Y}$ of semimartingales $Y^\nu$, defined from

$$
\frac{dY_t}{Y_t} = -r_t dt + (\nu_t - \eta_t^\sigma) dW_t, \quad \nu_t \in \mathcal{R}_t^\sigma, \perp
$$

- There exists a dual optimal choice $Y^*_t = Y^\nu^*$ satisfying the dual identity

$$
Y^*(t, y) = U_x(t, X^*_t((U'_x)^{-1}(0, y))), \quad \mathcal{Y}(t, x) := U_x(t, X^*_t(x))
$$

Assume $X^*_t(x)$ is strictly monotone in $x$, by taking the inverse $\mathcal{X}(t, x)$,

$$
\Rightarrow U_x(t, x) = Y^*_t(u_x(\mathcal{X}_t(x)))
$$

$$
\Rightarrow U(t, x) = \int_0^x Y^*_t(u_x(\mathcal{X}_t(z))) dz
$$

Req: $x \mapsto X^*_t(x)$ is increasing $\Rightarrow y \mapsto Y^*_t(y)$ is increasing.
Let $X^*(x)$ be any wealth process and $Y^*(y)$ be any state price density assumed to be continuous and increasing in $x$ (resp. in $y$) from 0 to $+\infty$. Moreover, $X^*$ and $Y^*$ are Itô-Ventzel regular

$$dX^*_t(x) = X^*_t(x) r_t dt + X^*_t(x) \kappa^*(t, X^*) (dW_t + \eta^*_{t} dt), \quad \kappa^*(t, x) \in \mathcal{R}^\sigma_t$$

$$dY^*_t(y) = -Y^*_t(y) r_t dt + (\nu^*(t, Y^*_t) - \eta^*_{t}) dW_t, \quad \nu^*(t, y) \in \mathcal{R}^\sigma_t, \perp$$

Note that the Monotony Assumption is

- true in a lot examples,
- may be a consequence of no arbitrage opportunity.
- from flows point of view, it is implied by coefficient regularity.
Theorem: Utility Characterization, Basic Example

Let $\mathcal{X}(t, z)$ be the inverse flow of $X^*(t, z)$, satisfying $X^* Y^\nu (\nu \in \mathcal{R}^{\sigma, \perp})$ is a martingale. Then for any utility function $u$ such that $u_x(\mathcal{X}(t, z))$ is locally integrable near $z = 0$, the stochastic process $U$ defined by

$$U(t, x) = Y_t^\nu (1) \int_0^x u_x(\mathcal{X}(t, z))dz, \quad U(t, 0) = 0 \quad (2)$$

is a $\mathcal{X}$-Consistent utility. The associated optimal wealth process is $X^*$ and the optimal dual choice $Y^*(y) = y Y^\nu (1)$. Moreover

$$\gamma_x(t, x) = U_x(t, x)(\nu_t - \eta_t^\sigma) - U_{xx}(t, x)\kappa^*(t, x).$$

Furthermore, the conjugate process of $U$ denoted by $\tilde{U}$, is given by

$$\tilde{U}(t, y) = \int_y^{+\infty} X^*(t, -\tilde{u}_y(z/Y_t^\nu (1)))dz, \quad (3)$$
Theorem

Let \( (X^*_t(x)) \), and \( Y^*_t(t, y) \) be two regular stochastic flows as above and \( u \) an utility function. Denote by \( \mathcal{X} \) and \( \mathcal{Y} \) the inverse flows and assume that \( x \mapsto Y^*_t(u_x(\mathcal{X}(t, y))) \) is locally integrable near \( z = 0 \). Define the processes \( U \) and \( \tilde{U} \) by

\[
U(t, x) = \int_0^x Y^*_t(u_x(\mathcal{X}(t, z))) dz, \quad \tilde{U}(t, y) = \int_y^{+\infty} X^*_t(-\tilde{u}_y(\mathcal{Y}(t, z))) dz.
\]

Then \( U \) is a consistent utility, whose the convex conjugate is \( \tilde{U} \), and the dynamics

\[
dU(t, x) = \left( -xU_x(t, x) r_t + \frac{1}{2U_{xx}(t, x)} \|\gamma^\sigma_x(t, x) + U_x(t, x)\eta^\sigma_t\|^2 \right) dt + \gamma(t, x).dW_t,
\]

with volatility vector \( \gamma \) given by

\[
\gamma(t, x) = -U(t, x)\eta^\sigma_t - \int_0^x \left( zU_{xx}(t, z)\kappa^*(t, z) - \nu^*_t(U_x(t, z)) \right) dz.
\]

The associated optimal portfolio and the optimal dual process are \( X^* \) and \( Y^* \).
Connection with two Solvable SDEs

Consider a utility stochastic PDE with initial condition \( u(.) \),

\[
\begin{align*}
    dU(t, x) &= \left( -xU_x(t, x)r_t + \frac{1}{2U_{xx}(t, x)} \| \gamma_x^\sigma(t, x) + U_x(t, x)\eta_t^\sigma \|^2 \right) dt + \gamma(t, x).dW_t, 
\end{align*}
\]

where the derivative \( \gamma_x \) of \( \gamma \) is the operator given by

\[
\gamma_x(t, x) = -U_x(t, x)\eta_t^\sigma - xU_{xx}(t, x)\kappa_t^*(t, x) + \nu_t^*(U_x(t, x)), \quad \kappa_t^* \in \mathcal{R}_t^\sigma, \quad \nu_t^* \in \mathcal{R}_t^\sigma, \quad t \geq 0.
\]

Assume that the both equations

\[
\begin{align*}
    \frac{dX_t^*(x)}{X_t^*(x)} &= r_t dt + \kappa_t^*(t, X_t^*(x)).(dW_t + \eta_t^\sigma dt), \\
    \frac{dY_t^*(y)}{Y_t^*(y)} &= -r_t dt + \left( \nu_t^*(Y_t^*(y)) - \eta_t^\sigma \right).dW_t
\end{align*}
\]

admit solutions and that \( X^* \) is monotonous regular flow in the sense of Kunita \( \Rightarrow \)

there exists a solution \( U \) of the SPDE (4) given by

\[
U(t, x) = \int_0^x Y_t^*(u_x(X(t, z)))dz
\]

- If \( X^* \) and \( Y^* \) are increasing regular flows \( \Rightarrow \) \( U \) is an increasing and concave solution of the SPDE (4).
- If \( X^* \) and \( Y^* \) are unique \( \Rightarrow \) \( U \) is the unique solution of (4).
The main assumption is that the optimal portfolio is increasing in $x$, because we have the same characterization in more abstract form (minimal regularities assumption), based on the properties of the optimum.
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Thank you for your attention