Sheaves on subanalytic sites and $\mathcal{D}$-modules

Luca Prelli

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1. Sheaves

2. Sheaves on subanalytic sites

3. $\mathcal{D}$-modules
Categories

**Definition:** A category $\mathcal{C}$ is the data of a set $\text{Ob}(\mathcal{C})$ of objects of $\mathcal{C}$, and for any $X, Y \in \text{Ob}(\mathcal{C})$ a set of morphisms $\text{Hom}_\mathcal{C}(X, Y)$, with a composition $\circ$ which is associative and satisfying $f \circ \text{id} = f$ and $\text{id} \circ g = g$. 
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Examples of categories
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Examples of categories

1. Set: the objects are sets and morphisms are maps between sets.
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Examples of categories

1. **Set**: the objects are sets and morphisms are maps between sets.
2. **Mod($k$) ($k$ a field)**: the objects are $k$-vector spaces and morphisms are linear maps.
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Examples of categories

1. $\text{Set}$: the objects are sets and morphisms are maps between sets.
2. $\text{Mod}(k)$ ($k$ a field): the objects are $k$-vector spaces and morphisms are linear maps.
3. $\text{Op}(X)$ ($X$ a topological space): the objects are the open subsets of $X$ and the morphisms are the inclusions.
**Definition:** Given two categories $\mathcal{C}, \mathcal{C}'$, a **functor** $F : \mathcal{C} \to \mathcal{C}'$ is the data of a morphism

$$F_o : \text{Ob}(\mathcal{C}) \to \text{Ob}(\mathcal{C}')$$
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and for each $X, Y \in \text{Ob}(\mathcal{C})$, a morphism

$$F_m : \text{Hom}_{\mathcal{C}}(X, Y) \to \text{Hom}_{\mathcal{C}'}(F_o X, F_o Y)$$

commuting with the composition law.
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commuting with the composition law.

**Definition:** Two categories are **equivalent** if there is a functor $F$ such that $F_o$ is a bijection between the isomorphism classes of objects and $F_m$ is a bijection between the set of morphisms.
What is a sheaf?

Let $X$ be a topological space and let $k$ be a field.
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**Definition:** A sheaf of $k$-vector spaces is the data of:

Open sets of $X$ \[ \rightarrow \] Mod($k$)

$U$ \[ \mapsto \] $\Gamma(U; F)$ ($= F(U)$)
What is a sheaf?

Let $X$ be a topological space and let $k$ be a field. **Definition:** A sheaf of $k$-vector spaces is the data of:

- Open sets of $X$ map to $\text{Mod}(k)$
  - $U \mapsto \Gamma(U; F) \ (= F(U))$
- $(V \subset U) \mapsto (F(U) \to F(V))$ (restriction)
  - $s \mapsto s|_V$
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Satisfying the following gluing conditions. Let $U$ be open and let $\{U_j\}_{j \in J}$ be a covering of $U$. We have the exact sequence

$$0 \rightarrow F(U) \rightarrow \prod_{j \in J} F(U_j) \rightarrow \prod_{j,k \in J} F(U_j \cap U_k)$$
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- if \( s \in \Gamma(U; F) \) and \( s|_{U_j} = 0 \) for each \( j \) then \( s = 0 \)
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- if \( s \in \Gamma(U; F) \) and \( s|_{U_j} = 0 \) for each \( j \) then \( s = 0 \)
- if \( s_j \in \Gamma(U_j; F) \) such that \( s_j = s_k \) on \( U_j \cap U_k \) then they glue to \( s \in \Gamma(U; F) \) (i.e. \( s|_{U_j} = s_j \))
Let us consider

\[ \mathbb{R}_X : \text{Open sets of } X \rightarrow \text{Mod}(\mathbb{R}) \]

\[ U \mapsto \Gamma(U; \mathbb{R}_X) = \{ \text{constant functions on } U \} \]
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- If \( s \) is zero on a covering of \( U \) then \( s = 0 \).
- For example, let \( X = \mathbb{R}, U_1 = (1, 2), U_2 = (2, 3) \). We have \( U_1 \cap U_2 = \emptyset \). The constant functions \( s_1 = 0 \) on \( U_1 \) and \( s_2 = 1 \) on \( U_2 \) do not glue on a constant function on \( U_1 \cup U_2 \).
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⇒ The correspondence

\[ U \mapsto \Gamma(U; \mathbb{R}_X) = \{\text{constant functions on } X\} \text{ does not define a sheaf on } X. \]
Let us consider

\[ R_X : \text{Open sets of } X \rightarrow \text{Mod}(\mathbb{R}) \]

\[ U \mapsto \Gamma(U; R_X) = \{\text{constant functions on } U\} \]

\[ (V \subset U) \mapsto (R_X(U) \rightarrow R_X(V)) \text{ (restriction)} \]

\[ s \mapsto s|_V \]

⇒ The correspondence

\[ U \mapsto \Gamma(U; R_X) = \{\text{constant functions on } X\} \text{ does not define a sheaf on } X. \text{ We have to consider locally constant functions.} \]
Examples

Let us consider

$$\mathcal{C}_X : \text{Open sets of } X \rightarrow \text{Mod}(\mathbb{R})$$

$$U \mapsto \{ \text{continuous real valued functions on } U \}$$
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Let us consider

\[ \mathcal{C}_X : \text{Open sets of } X \to \text{Mod}(\mathbb{R}) \]

\[ U \mapsto \{ \text{continuous real valued functions on } U \} \]

- If \( s \) is a continuous function and \( s \) is zero on a covering of \( U \) then \( s = 0 \).
- If \( \{ s_i \} \) are continuous functions on a covering \( \{ U_i \} \) of \( U \), such that \( s_i = s_j \) on \( U_i \cap U_j \), then there exists \( s \) continuous on \( U \) with \( s = s_i \) on each \( U_i \).
Let us consider

\[ C_X : \text{Open sets of } X \rightarrow \text{Mod}(\mathbb{R}) \]

\[ U \mapsto \{ \text{continuous real valued functions on } U \} \]

⇒ The correspondence

\[ U \mapsto \Gamma(U; C_X) = \{ \text{continuous real valued functions on } U \} \]

defines a sheaf on \( X \)
Examples

Let us consider

\[ C_b^X : \text{Open sets of } X \rightarrow \text{Mod}(\mathbb{R}) \]

\[ U \mapsto \{ \text{continuous bounded functions on } U \} \]
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Let us consider

\[ C^b_X : \text{Open sets of } X \rightarrow \text{Mod}(\mathbb{R}) \]
\[ U \mapsto \{ \text{continuous bounded functions on } U \} \]

For example, let \( X = \mathbb{R} \), \( U_n = (-n, n) \), \( n \in \mathbb{N} \), and \( s_n : U_n \rightarrow \mathbb{R}, x \mapsto x^2 \). Then \( s_n \) is bounded on \( U_n \) for each \( n \in \mathbb{N} \), but \( x \mapsto x^2 \) is not bounded on \( \mathbb{R} \).
Examples

Let us consider

\[ \mathcal{C}_X^b : \text{Open sets of } X \to \text{Mod}(\mathbb{R}) \]
\[ U \mapsto \{ \text{continuous bounded functions on } U \} \]

⇒ The correspondence \[ U \mapsto \Gamma(U; \mathcal{C}_X^b) = \{ \text{continuous bounded real valued functions on } U \} \]
does not define a sheaf on \( X \).
More Examples

**Sheaves**: holomorphic functions, $C^\infty$ functions, distributions.
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**Not sheaves:** $L^2$ functions, tempered distributions.
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If we consider “less open subsets” and “less coverings” they may become sheaves.
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Sheaves: holomorphic functions, $C^\infty$ functions, distributions.

Not sheaves: $L^2$ functions, tempered distributions. In fact they do not satisfy gluing conditions.

If we consider “less open subsets” and “less coverings” they may become sheaves. We need the notion of site.
Let $F \in \text{Mod}(k_X)$ we define the fiber of $F$ at $x$ as

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$U \ni x$

It means that the elements of $F_x$ are equivalence classes, i.e. $f \in F_x$ is represented by $f \in F(U)$ where $U$ is a neighborhood of $x$. 
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It means that the elements of $F_x$ are equivalence classes, i.e. $f \in F_x$ is represented by $f \in F(U)$ where $U$ is a neighborhood of $x$.

Moreover, given $U_1, U_2 \ni x$ and $f_i \in U_i$, we have $f_1 \equiv f_2$ in $F_x$ if $f_1 = f_2$ on a neighborhood of $x$ $W \subset U_1 \cap U_2$. 
Fibers

Two sheaves $F, G$ are isomorphic if

$$F_x \cong G_x$$

for any $x \in X$. More generally a sequence of sheaves

$$0 \to F' \to F \to F'' \to 0$$

is exact if the sequence

$$0 \to F'_x \to F_x \to F''_x \to 0$$

is exact in $\text{Mod}(k_X)$. 
The definition of sheaf depends only on
  - open subsets
  - coverings
Topological sites

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- open subsets
- coverings

One can generalize this notion by choosing a subfamily of open subsets $\mathcal{T}$ of $X$ and for each $U$ a subfamily $\text{Cov}(U)$ of coverings if $U$ satisfying suitable hypothesis (defining a site $X_\mathcal{T}$).
One can generalize this notion by choosing a subfamily of open subsets $\mathcal{T}$ of $X$ and for each $U$ a subfamily $\text{Cov}(U)$ of coverings if $U$ satisfying suitable hypothesis (defining a site $X_{\mathcal{T}}$).

Then $F : \mathcal{T} \to \text{Mod}(k)$ is a sheaf on $X_{\mathcal{T}}$ if for each $U \in \mathcal{T}$ and each $\{U_j\}_{j \in J} \in \text{Cov}(U)$ we have the exact sequence

$$0 \to F(U) \to \prod_{j \in J} F(U_j) \to \prod_{j,k \in J} F(U_j \cap U_k)$$
Example

For example, let us consider the site $X_{\mathcal{T}}$ where

- $\mathcal{T} =$ open subsets of $X$
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For example, let us consider the site $X_T$ where

- $T =$ open subsets of $X$
- $Cov(U) =$ finite coverings of $U$

and consider the correspondence $U \mapsto \Gamma(U; C^b_X)$ (continuous bounded functions).
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For example, let us consider the site $X_{\mathcal{T}}$ where

- $\mathcal{T}$ = open subsets of $X$
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and consider the correspondence $U \mapsto \Gamma(U; \mathcal{C}^b_X)$ (continuous bounded functions).

- If $\{s_i\}$ are **bounded** on a finite covering $\{U_i\}$ of $U$, such that $s_i = s_j$ on $U_i \cap U_j$, then there exists $s$ bounded on $U$ with $s = s_i$ on each $U_i$. 

$\Rightarrow$ The correspondence $U \mapsto \Gamma(U; \mathcal{C}^b_X)$ defines a sheaf on $X_{\mathcal{T}}$. 

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and consider the correspondence $U \mapsto \Gamma(U; \mathcal{C}^b_X)$ (continuous bounded functions).

$\Rightarrow$ The correspondence $U \mapsto \Gamma(U; \mathcal{C}^b_X)$ defines a sheaf on $X_{\mathcal{T}}$. 
The general case

Let $X$ be a topological space and consider a family of open subsets $\mathcal{T}$ satisfying:

\[
\begin{align*}
\text{(i)} & \quad U, V \in \mathcal{T} \iff U \cap V, U \cup V \in \mathcal{T}, \\
\text{(ii)} & \quad U \setminus V \text{ has finite numbers of connected components } \forall U, V \in \mathcal{T}, \\
\text{(iii)} & \quad \mathcal{T} \text{ is a basis for the topology of } X.
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**Definition:** The site $X_{\mathcal{T}}$ is defined by:

- open subsets: *elements of* $\mathcal{T}$
The general case

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(i) $U, V \in \mathcal{T} \iff U \cap V, U \cup V \in \mathcal{T}$,

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**Definition:** The site $X_\mathcal{T}$ is defined by:

- open subsets: elements of $\mathcal{T}$
- $\text{Cov}(U)$ (coverings of $U \in \text{Op}(X_\mathcal{T})$): finite coverings of $U$
Examples

1. $\mathcal{T} = \{\text{open semialgebraic subsets of } \mathbb{R}^n\}$
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2. $\mathcal{T} = \{\text{open relatively compact subanalytic subsets of a real analytic manifold}\}$, the subanalytic site $X_{sa}$. 
Examples

1. $\mathcal{T} = \{\text{open semialgebraic subsets of } \mathbb{R}^n\}$
2. $\mathcal{T} = \{\text{open relatively compact subanalytic subsets of a real analytic manifold}\}$, the subanalytic site $X_{sa}$.
3. $\mathcal{T} = \{\text{open definable subsets of } \mathbb{N}^n\}$, given an O-minimal structure $(\mathbb{N}, <, \ldots)$, the site DTOP.
Let $F$ be a presheaf on $X_{\mathcal{T}}$. Assume that

- $F(\emptyset) = 0$
Construction of sheaves on $X_{\mathcal{T}}$

Let $F$ be a presheaf on $X_{\mathcal{T}}$. Assume that

- $F(\emptyset) = 0$
- $\forall U, V \in \mathcal{T}$ the sequence

$$0 \rightarrow F(U \cup V) \rightarrow F(U) \oplus F(V) \rightarrow F(U \cap V)$$

is exact.
Let $F$ be a presheaf on $X_T$. Assume that

- $F(\emptyset) = 0$
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is exact.

Then $F$ is a sheaf on $X_T$. 
Subanalytic sheaves

From now on we will consider the subanalytic site $X_{sa}$.

- open subsets: relatively compact subanalytic open subsets
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Why subanalytic sheaves?

Let us consider as an example the presheaf

\[ U \mapsto \mathcal{D}b_X^t(U) \]

of tempered distribution over a real analytic manifold \( X \). This is not a sheaf with the usual topology.
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of tempered distribution over a real analytic manifold $X$. This is not a sheaf with the usual topology.

For example, if $X = \mathbb{R}$, we can find tempered distributions $s_n$ on $\left\{ \frac{1}{n} < x < 1 \right\}$, $n \in \mathbb{N}$ which do not glue to a tempered distribution $s$ on $\left\{ 0 < x < 1 \right\}$. 
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Anyway for $U$, $V$ open subanalytic relatively compact subsets of $X$ we have the exact sequence

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Anyway for $U, V$ open subanalytic relatively compact subsets of $X$ we have the exact sequence

$$0 \to Db^t_X(U \cup V) \to Db^t_X(U) \oplus Db^t_X(V) \to Db^t_X(U \cap V)$$

This implies that $U \mapsto Db^t_X(U)$ is a sheaf on the subanalytic site $X_{sa}$. 
Let $X$ be a complex manifold and let $U \subset X$ be a relatively compact subanalytic open subset, $f$ holomorphic on $U$ is tempered if $\exists M, C > 0$ such that

$$|f(z)| \leq \frac{C}{\text{dist}(z, \partial U)^M}.$$
In the case of subanalytic sheaves we do not have the notion of fibers in the usual sense, i.e. if we consider

\[ F_x = \lim_{U \ni x} F(U) \]

i.e. there are \( F \not\cong G \) even if \( F_x \cong G_x \ \forall x \in X \).
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i.e. there are \( F \not\cong G \) even if \( F_x \cong G_x \ \forall \ x \in X \).

**Example:** Let \( X = \mathbb{R} \) and consider the sheaves \( \mathcal{C}^b_\mathbb{R} \) and \( \mathcal{C}^b_\mathbb{R} \). Then \( \mathcal{C}^b_{\mathbb{R}, x} \cong \mathcal{C}^b_{\mathbb{R}, x} \ \forall \ x \in \mathbb{R} \). Indeed, any continuous function \( f \) in \( (x - \varepsilon, x + \varepsilon) \), \( \varepsilon > 0 \) is bounded in \( (x - \varepsilon/2, x + \varepsilon/2) \).
Hence if we consider only the fibers associated to the points of $x$ we loose informations about $F \in \text{Mod}(k_{X_{sa}})$. 
Fibers

Hence if we consider only the fibers associated to the points of $x$ we lose informations about $F \in \text{Mod}(k_{X_{sa}})$.

We need to consider more points.
Let us consider a countable locally finite covering \( \{ U_n \}_{n \in \mathbb{N}} \) of \( X \), with \( U_n \sim \mathbb{R}^n \) relatively compact and subanalytic.
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Let us consider a countable locally finite covering $\{U_n\}_{n \in \mathbb{N}}$ of $X$, with $U_n \sim \mathbb{R}^n$ relatively compact and subanalytic. In $U_n$ consider the ultrafilters of globally subanalytic subsets (i.e. subanalytic in $X$).

A **neighborhood** of an ultrafilter $\alpha$ is a globally subanalytic open subset $U$ contained in $\alpha$. 
Let us consider a countable locally finite covering \( \{ U_n \}_{n \in \mathbb{N}} \) of \( X \), with \( U_n \cong \mathbb{R}^n \) relatively compact and subanalytic. In \( U_n \) consider the ultrafilters of \textbf{globally subanalytic subsets} (i.e. subanalytic in \( X \)).

A \textbf{neighborhood} of an ultrafilter \( \alpha \) is a globally subanalytic open subset \( U \) contained in \( \alpha \).

We call \( \tilde{X} \) the associated topological space. In \( \tilde{X} \) any covering of a \textbf{relatively compact} subanalytic open subset has a finite subcover.
For example, the points of $\tilde{\mathbb{R}}$ are the following. Let $x \in \mathbb{R}$

1. $\{ S \text{ subanalytic, } S \supseteq x \}$ (the point $x$)
2. $\{ S \text{ subanalytic, } S \supseteq (x, x + \varepsilon), \varepsilon > 0 \}$ (the point $x^+$)
3. $\{ S \text{ subanalytic, } S \supseteq (x - \varepsilon, x), \varepsilon > 0 \}$ (the point $x^-$)
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Thanks to these new points we can distinguish \( C_\mathbb{R} \) from \( C^b_\mathbb{R} \) on \( \tilde{\mathbb{R}} \). For example let \( f = x^{-1} \). Then \( f \notin C^b_\mathbb{R}(0, \varepsilon) \forall \varepsilon > 0 \).
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Thanks to these new points we can distinguish \( C_{\mathbb{R}} \) from \( C^b_{\mathbb{R}} \) on \( \tilde{\mathbb{R}} \). For example let \( f = x^{-1} \). Then \( f \notin C^b_{\mathbb{R}}(0, \varepsilon) \) \( \forall \varepsilon > 0 \). Hence \( f \notin C^b_{\mathbb{R},0^+} \), but \( f \in C_{\mathbb{R},0^+} \), this implies \( C^b_{\mathbb{R},0^+} \nsubseteq C_{\mathbb{R},0^+} \).
Theorem:

Let $X$ be a real analytic manifold. The categories $\text{Mod}(k_{X_{sa}})$ and $\text{Mod}(k_{\tilde{X}})$ are equivalent.

Hence, if we want to work on fibers on $X_{sa}$, we have to consider the topological space $\tilde{X}$. 
Theorem:
Let \( f : X \rightarrow Y \) be a morphism of real analytic manifolds. The six Grothendieck operations \( \mathcal{H}om, \otimes, f_*, f^{-1}, f!!, f! \) are well defined for subanalytic sheaves.

L. Prelli *Sheaves on subanalytic sites*, Rendiconti del Seminario Matematico dell’Università di Padova Vol. 120 (2008).
Let $X$ be a complex analytic manifold. We denote by $\mathcal{D}_X$ the sheaf of rings of differential operators. Locally, a section of $\Gamma(U; \mathcal{D}_X)$ may be written as $P = \sum_{|\alpha| \leq m} a_\alpha(z) \partial_\alpha$ with $a_\alpha(z)$ holomorphic on $U$. We denote by $\text{Mod}(\mathcal{D}_X)$ the sheaf of $\mathcal{D}_X$-modules.
Complex of solutions

The sheaf $\mathcal{O}_X$ of **holomorphic functions** has a structure of $\mathcal{D}_X$-module.
Complex of solutions

The sheaf $\mathcal{O}_X$ of holomorphic functions has a structure of $\mathcal{D}_X$-module.

**Definition:** If $U$ is open, $\mathcal{F}$ a $\mathcal{D}_X$-module, $P$ a differential operator, $\text{Sol}_\mathcal{F}(P)$ on $U$ is the complex

$$\Gamma(U; \mathcal{F}) \xrightarrow{P} \Gamma(U; \mathcal{F})$$

**Definition:** $P_1$ and $P_2$ are equivalent if for any

$$\ker P_1 \cong \ker P_2 \quad \text{and} \quad \coker P_1 \cong \coker P_2$$

(i.e. $\text{Sol}_\mathcal{F}(P_1)$ and $\text{Sol}_\mathcal{F}(P_2)$ are quasi-isomorphic).

$$H^0(U; \text{Sol}_\mathcal{F}(P)) = \{ s \in \Gamma(U; \mathcal{F}), \ Ps = 0 \} = \ker P$$

$$H^1(U; \text{Sol}_\mathcal{F}(P)) = \Gamma(U; \mathcal{F})/P\Gamma(U; \mathcal{F}) = \coker P$$
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\end{align*}
$$

**Definition:** $P_1$ and $P_2$ are equivalent if for any $\mathcal{F}$ $\ker P_1 \simeq \ker P_2$ and $\text{coker} P_1 \simeq \text{coker} P_2$ (i.e. $\text{Sol}_\mathcal{F}(P_1)$ and $\text{Sol}_\mathcal{F}(P_2)$ are quasi-isomorphic).
Example

Let $\alpha \in \mathbb{C}$, and consider the operators

$$P_{\alpha} = z \partial_z - \alpha \quad P_{\alpha+1} = z \partial_z - \alpha - 1.$$
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If \( \alpha \neq -1 \), one can verify that we have morphisms

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\text{Sol}_F(P_\alpha) \underoverset{\sim}{z \cdot \frac{\partial}{\alpha+1}}{\longleftrightarrow} \text{Sol}_F(P_{\alpha+1})
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$$
\frac{\partial_z}{\alpha + 1}
$$

the above morphisms induce an isomorphism between the homogeneous solutions $\ker P_\alpha$ and $\ker P_{\alpha + 1}$. 
Moreover one can prove that the above morphisms induce an isomorphism between $\text{coker} P_\alpha$ and $\text{coker} P_{\alpha+1}$ (i.e. the complexes $\text{Sol}_\mathcal{F}(P_\alpha)$ and $\text{Sol}_\mathcal{F}(P_{\alpha+1})$ are quasi-isomorphic).
Example

Moreover one can prove that the above morphisms induce an isomorphism between $\text{coker} P_\alpha$ and $\text{coker} P_{\alpha+1}$ (i.e. the complexes $\text{Sol}_F(P_\alpha)$ and $\text{Sol}_F(P_{\alpha+1})$ are quasi-isomorphic).

Hence $P_\alpha$ and $P_{\alpha+1}$ are equivalent.
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Let \( f \in \mathcal{O}_X(U) \).

The equation \( z \partial_z u = f \) has holomorphic solutions if and only if \( f(0) = 0 \).
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The equation \( \partial_z u = f \) has always solutions.
Example

Let $f \in \mathcal{O}_X(U)$.

The equation $z\partial_z u = f$ has holomorphic solutions if and only if $f(0) = 0$. The equation $\partial_z u = f$ has always solutions. So, even if the kernels of

$$\mathcal{O}_X(U) \xrightarrow{z\partial_z} \mathcal{O}_X(U)$$

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Then $z\partial_z$ and $\partial_z$ are not equivalent.
Let us consider the operators $z(z\partial_z + 1)$ and $z^2\partial_z + 1$. 
Example

Let us consider the operators $z(z\partial_z + 1)$ and $z^2\partial_z + 1$. They have $z^{-1}$ and $\exp(z^{-1})$ as homogeneous solutions. If $\mathcal{M}$ denotes the sheaf of meromorphic functions and $U \not\ni 0$, then
Example

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\[
H^0(U, Sol_{\mathcal{M}}(z(z\partial_z + 1))) \cong \mathbb{C} \cdot z^{-1}
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H^0(U, Sol_{\mathcal{M}}(z^2\partial_z + 1)) \cong 0 \text{ because } \exp(z^{-1}) \not\in \mathcal{M}(U).
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Hence $z(z\partial_z + 1)$ and $z^2\partial_z + 1$ are not equivalent (even if the holomorphic solutions are).
Equivalence for regular operators

**Definition:** \( P = \sum_{\alpha \leq n} a_\alpha(z) \partial^\alpha, \ a_\alpha(0) \neq 0, \) is regular at 0 if for each \( j \leq n, \ n - \operatorname{ord}_0(a_n) \geq j - \operatorname{ord}_0(a_j). \)
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Theorem: Let \( P \) and \( Q \) be regular at 0. The following are equivalent.

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**Theorem:** Let \( P \) and \( Q \) be regular at 0. The following are equivalent.

1. \( P \) and \( Q \) are equivalent.
2. The kernels and cokernels of

\[
\mathcal{O}_X \xrightarrow{P} \mathcal{O}_X \quad \mathcal{O}_X \xrightarrow{Q} \mathcal{O}_X
\]

are isomorphic (i.e. \( \text{Sol}_{\mathcal{O}_X}(P) \) is quasi-isomorphic to \( \text{Sol}_{\mathcal{O}_X}(Q) \)).
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   \[ O_X \xrightarrow{P} O_X, \quad O_X \xrightarrow{Q} O_X \]
   are isomorphic (i.e. \( \text{Sol}_{O_X}(P) \) is quasi-isomorphic to \( \text{Sol}_{O_X}(Q) \)).

In particular the holomorphic solutions are sufficient to establish if two regular equations are equivalent.
The sheaf $\mathcal{O}_X^t$ of tempered holomorphic functions has a structure of $\rho!\mathcal{D}_X$-module. ($\Gamma(U; \rho!\mathcal{D}_X)$ are differential operators $\sum_{|\alpha| \leq m} a_\alpha \partial_z^\alpha$ with $a_\alpha$ holomorphic in $\overline{U}$)
Example

Let us consider the operators $z^2 \partial_z + 1$ and $z^3 \partial_z + 2$. Their solutions are respectively $\exp(z^{-1})$ and $\exp(z^{-2})$.

**Theorem (G. Morando):** There exists an open subanalytic $U$ such that $\exp(z^{-1}) \in \mathcal{O}^t_X(U)$ and $\exp(z^{-2}) \notin \mathcal{O}^t_X(U)$. 
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In particular for such $U$ we have

\[
H^0(U; \text{Sol}_{\mathcal{O}_X^t}(z^2 \partial_z + 1)) \cong \mathbb{C} \cdot \exp(z^{-1})
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Hence thanks to tempered holomorphic solutions we can distinguish irregular differential operators which cannot be distinguished with holomorphic solutions.
Sheaves on subanalytic sites and $\mathcal{D}$-modules

Luca Prelli

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