Rigid subsets of weights for semisimple Lie algebras

Tim Ridenour

Advisor: Dr. Vyjayanthi Chari

Preliminary Notation, Definitions, and Results

- *For a finite dimensional semisimple Lie algebra $\mathfrak{g}$ of rank n with fixed Cartan subalgebra $\mathfrak{h}$, we denote the set of roots (resp. the root lattice, the weight lattice) of $\mathfrak{g}$ by R (resp. $Q,\ P$).*
- *Let I be an indexing set for the simple roots of $\mathfrak{g}$.*
- *The set of $\mathfrak{g}$-invariant elements of a $\mathfrak{g}$-module V is denoted by $V^{\mathfrak{g}}$.*
- *Let $wt(V)$ denote the set of weights for the $\mathfrak{g}$-module V.*
- *If $V = V(\lambda)$ is the unique finite-dimensional irreducible highest weight $\mathfrak{g}$-module with highest weight λ, set $wt(\lambda) = wt(V)$.*
- *For any finite subset $S \subset \mathfrak{h}^*$, set $\rho_S := \sum_{\mu \in S} \mu$.*
- *Let $0 \neq \lambda = \sum_{i \in K} m_i \omega_i \in P^+$ with $K \subset I$ and $m_i > 0$ for all $i \in K$. Let $I_1,\ \ldots,\ I_r$ be the (indexing sets for) connected components of the Dynkin diagram such that $K \cap I_j \neq \emptyset\ \forall\ j$. Define $I_\lambda := \coprod_{j=1}^{r} I_j$.*
- *An element $\lambda \in P^+$ is **quasi-regular** if $K = I_\lambda$. If $K = I_\lambda = I$, then λ is **regular**.*
- $S_J := \{\mu = \lambda - \sum_{j \in J} r_j \alpha_j | r_j \in \mathbb{Z}^+\} \cap wt(\lambda)$ *for $J \subset I$.*
- $S(\xi) := \{\mu \in wt(\lambda) \mid (\xi, \mu) \geq (\xi, \nu)\ \forall\ \nu \in wt(\lambda)\}$ *for $0 \neq \xi \in P$.*
- *A subset $\Psi \subset wt(V)$ is said to be **rigid** if given $\eta \in \mathbb{Z}^+\Psi$ and $\eta = \sum_{\mu \in wt(\lambda)} m_\mu \mu$ with $m_\mu \in \mathbb{Z}^+$, then $\sum_{\mu \in wt(\lambda)} m_\mu$ is the least possible if and only if $m_\mu = 0\ \forall\ \mu \notin \Psi$.*
- *Similarly, $\Psi \subset wt(V)$ is **2-rigid** if $\gamma + wt(\lambda) \cap (\Psi + \Psi) = \emptyset = (\Psi + \Psi) \cap wt(\lambda)\ \forall\ \gamma \in wt(\lambda) \setminus \Psi$.*
- *If a subset $\Psi \subset wt(\lambda)$ contains a non-empty $\mathcal{W}$-invariant subset T, then Ψ is not rigid. In particular, $wt(\lambda)$ is not rigid.*
- *If $\Psi \subset wt(\lambda)$ is rigid, then $w(\Psi)$ is rigid for all $w \in \mathcal{W}$.*
- *If $\Psi \subset wt(\lambda)$ is a non-empty rigid set, then $w(\lambda) \in \Psi$ for some $w \in \mathcal{W}$. In particular, if $|\Psi| = 1$, then $w(\Psi) = \{\lambda\}$ for some $w \in \mathcal{W}$.*
- *For all $0 \neq \lambda \in P^+$ and $0 \neq \xi \in P$, the set $S(\xi) \subset wt(\lambda)$ is (2-)rigid.*
- *If $J \subset I$, $\rho_{S_J} \in P^+$. Furthermore, $S_J = S(\rho_{S_J})\ \forall\ J \subset I$, and, thus, S_J is rigid if and only if $J \cap I_\lambda \neq I_\lambda$.*

Rigid subsets of the adjoint representation

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra of finite type. Let $V = V(\theta) = \mathfrak{g}_{ad}$ be the adjoint representation of $\mathfrak{g}$. In this case, $wt(V) = R \cup \{0\}$. The (2-)rigid subsets of $wt(V)$ are as follows:

- Let $\mathfrak{g}$ be of type A_n. We can write the simple roots as $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $i \in [n]$, so $R = \{\epsilon_i - \epsilon_j \mid i \neq j\}$. Then, $\Psi \subset R$ is rigid if and only if $\Psi = \{\epsilon_i - \epsilon_j \mid i \in \mathbf{i},\ j \in \mathbf{j}\}$ for some subsets $\mathbf{i}, \mathbf{j} \subset [n+1]$ such that $\mathbf{i} \cap \mathbf{j} = \emptyset$.

- Let $\mathfrak{g}$ be of type C_n. We can write the simple roots as $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $i \in [n-1]$ and $\alpha_n = 2\epsilon_n$. This gives $R^+ = \{\epsilon_i - \epsilon_j \mid 1 \leq i < j \leq n\} \cup \{\epsilon_i + \epsilon_j \mid 1 \leq i \leq j \leq n\}$. Then, $\Psi \subset R$ is rigid if and only if $\Psi = \{\epsilon_i - \epsilon_j \mid i \in \mathbf{i},\ j \in \mathbf{j}\} \cup \{\epsilon_{i_1} + \epsilon_{i_2} \mid i_1, i_2 \in \mathbf{i}\} \cup \{-(\epsilon_{j_1} + \epsilon_{j_2} \mid j_1, j_2 \in \mathbf{j}\}$ for some subsets $\mathbf{i}, \mathbf{j} \subset [n]$ such that $\mathbf{i} \cap \mathbf{j} = \emptyset$.

- Let $\mathfrak{g}$ be of type B_n. We can write the simple roots as $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $i \in [n-1]$ and $\alpha_n = \epsilon_n$. This gives $R^+ = \{\epsilon_i - \epsilon_j \mid 1 \leq i < j \leq n\} \cup \{\epsilon_i + \epsilon_j \mid 1 \leq i < j \leq n\} \cup \{\epsilon_i \mid i \in [n]\}$. Then, $\Psi \subset R$ is rigid if and only if one of the following holds:
  1. $\Psi = \{\epsilon_i - \epsilon_j \mid i \in \mathbf{i},\ j \in \mathbf{j}\} \cup \{\epsilon_{i_1} + \epsilon_{i_2} \mid i_1, i_2 \in \mathbf{i},\ i_1 \neq i_2\} \cup \{-(\epsilon_{j_1} + \epsilon_{j_2} \mid j_1, j_2 \in \mathbf{j},\ j_1 \neq j_2\}$ for some subsets $\mathbf{i}, \mathbf{j} \subset [n]$ such that $\mathbf{i} \cap \mathbf{j} = \emptyset$
  2. $\pm\Psi = \{\epsilon_i\} \cup \{\epsilon_i \pm \epsilon_j \mid j \in [n],\ i \neq j\}$.

- Let $\mathfrak{g}$ be of type D_n. We can write the simple roots as $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $i \in [n-1]$ and $\alpha_n = \epsilon_{n-1} + \epsilon_n$. This gives $R^+ = \{\epsilon_i - \epsilon_j \mid 1 \leq i < j \leq n\} \cup \{\epsilon_i + \epsilon_j \mid 1 \leq i < j \leq n\}$. Then, $\Psi \subset R$ is rigid if and only if one of the following holds:
  1. $\Psi = \{\epsilon_i - \epsilon_j \mid i \in \mathbf{i},\ j \in \mathbf{j}\} \cup \{\epsilon_{i_1} + \epsilon_{i_2} \mid i_1, i_2 \in \mathbf{i},\ i_1 \neq i_2\} \cup \{-(\epsilon_{j_1} + \epsilon_{j_2} \mid j_1, j_2 \in \mathbf{j},\ j_1 \neq j_2\}$ for some subsets $\mathbf{i}, \mathbf{j} \subset [n]$ such that $\mathbf{i} \cap \mathbf{j} = \emptyset$
  2. $\pm\Psi = \{\epsilon_i \pm \epsilon_j \mid j \in [n],\ i \neq j\}$.

Main Result

Theorem 1. *Suppose that $\lambda \in P^+$ is quasi-regular. Then, the following are equivalent for a set $\Psi \subset wt(\lambda)$:*

1. *$\Psi = w(S_J)$ for some $w \in \mathcal{W}$ and $J \subset I$ with $J \cap I_\lambda \neq I_\lambda$;*
2. *$\Psi = S(\rho_\Psi)$ with $\rho_\Psi \neq 0$;*
3. *$\Psi = S(\nu)$ for some $0 \neq \nu \in P$ with $(\nu, \mu) \neq 0$ for some $\mu \in wt(\lambda)$;*
4. *Ψ is rigid.*
5. *Ψ is a 2-rigid set with $\Psi \neq wt(\lambda)$.*

- The implications $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 5$ in the theorem are all true for any nonzero $\lambda \in P^+$.

- **Conjecture** If $\lambda \in P^+$ and $\Psi \subsetneq wt(\lambda)$ is (2-)rigid, then $\Psi = w(S_J)$ for some $w \in \mathcal{W}$ and $J \subsetneq I_\lambda$.

- In *Ideals in parabolic subalgebras of simple Lie algebras*, Chari, Dolbin and Ridenour show that the conjecture is true for the case when $\lambda = \theta$ where θ is the highest weight in the adjoint representation of a finite dimensional simple Lie algebra $\mathfrak{g}$.

- The conjecture has also been verified for all simple Lie algebras of rank 2.

An Application to Associative Algebras

We first introduce some additional notation:

- $\mathbb{V} := \bigoplus_{\mu \in P^+} V(\mu), \quad \mathbb{V}^{\circledast} := \bigoplus_{\mu \in P^+} V(\mu)^*.$

- $\mathbb{A} := A \otimes \mathbb{V}^{\circledast} \otimes \mathbb{V}$, where A is an associative $\mathbb{C}$-algebra with unity 1_A.

- If A is $\mathbb{Z}^+$-graded, then $\mathbb{A}[k] := A[k] \otimes \mathbb{V}^{\circledast} \otimes \mathbb{V}$ gives a $\mathbb{Z}^+$-grading on $\mathbb{A}$.

- For $\mu \in P^+$, 1_μ is the canonical $\mathfrak{g}$-invariant element in $V(\mu)^* \otimes V(\mu)$. Set $\mathbf{1}_\mu := 1_A \otimes 1_\mu$.

- Given $\lambda \in P^+$ and $\Psi \subset wt(\lambda)$, say $\mu \preceq_\Psi \nu$ iff $\nu - \mu \in \mathbb{Z}_+\Psi$ and $d_\Psi(\mu, \nu) = min\{\sum_{\beta \in \Psi} m_\beta \mid \nu - \mu = \sum_{\beta \in \Psi} m_\beta \beta,\ m_\beta \in \mathbb{Z}_+\ \forall \beta\}$.

- For $\mu, \nu \in P^+$, define the following sets: $\preceq_\Psi \mu = \{\eta \in P^+ \mid \eta \preceq_\Psi \mu\}$, $\mu \preceq_\Psi = \{\xi \in P^+ \mid \mu \preceq_\Psi \xi\}$, and $[\mu, \nu]_\Psi = (\preceq_\Psi \mu) \cap (\preceq_\Psi \nu)$.

- If A is $\mathbb{Z}^+$-graded, $\mathbb{A}_\Psi(\mu, \nu) := 1_\mu \mathbb{A}[d_\Psi(\mu, \nu)] 1_\nu$ where $\mu \preceq_\Psi \nu \in P^+$. For $F \subset P^+$, $\mathbb{A}_\Psi(F) := \bigoplus_{\mu, \nu \in F : \mu \preceq_\Psi \nu} \mathbb{A}_\Psi(\mu, \nu)$.

- If A is also a $\mathfrak{g}$-module with $\mathfrak{g}(A[k]) \subset A[k]\ \forall\ k$, $\mathbb{A}_\Psi^{\mathfrak{g}}(\mu, \nu) := (\mathbb{A}_\Psi(\mu, \nu))^{\mathfrak{g}}$ and $\mathbb{A}_\Psi^{\mathfrak{g}}(F) := (\mathbb{A}_\Psi(F))^{\mathfrak{g}}$.

Theorem 2. *Suppose $\Psi \subset wt(V(\lambda))$ is rigid. Let $A = S(V(\lambda))$, the symmetric algebra on $V(\lambda)$. Let $\mathbb{S} := A \otimes \mathbb{V}^{\circledast} \otimes \mathbb{V}$. Given $\mu, \nu \in P^+$, the algebras $\mathbb{S}_\Psi^{\mathfrak{g}}(\preceq_\Psi \nu)$, $\mathbb{S}_\Psi^{\mathfrak{g}}(\mu \preceq_\Psi)$, and $\mathbb{S}_\Psi^{\mathfrak{g}}([\mu, \nu]_\Psi)$ are Koszul with global dimension at most $N_\Psi := \sum_{\xi \in \Psi} dim(V(\lambda)_\xi)$. Moreover, the global dimension is exactly N_Ψ for some choice of $\mu \preceq_\Psi \nu \in P^+$.*

Current Research

- Expanding the results of Theorem 1 to the case when λ is not regular. Working jointly with Apoorva Khare to consider the case when $\mathfrak{g}$ is reductive.

UC Riverside Department of Mathematics

A Chari's Angels Production

Contact Information: tbr4@math.ucr.edu