On the exact structure of multidimensional sets with small doubling property

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1. Direct and inverse problems of additive and combinatorial number theory

Additive number theory is the study of sums of sets and we can distinguish two main lines of research.

In a **direct problem** of additive number theory we start with a particular known set $A$ and attempt to determine the structure and properties of the $h$-folds sumset $hA$. These are the classical direct problems in additive number theory: Waring’s problem, Goldbach conjecture...

As a counterbalance to this direct approach, an **inverse problem** in additive number theory is a problem in which we study properties of a set $A$, if some characteristic of the $h$-fold sumset $hA$ is given.
Sumsets can be defined in any Abelian group $G$, for example in

- $\mathbb{Z}$
  the group of integers,

- $\mathbb{Z}/m\mathbb{Z}$
  the group of congruence classes modulo $m$,

- $\mathbb{Z}^n$
  the group of integer lattice points,

- $\mathbb{R}^d$
  the $d$-dimensional Euclidean space.
Freiman proposed an unifying “algorithm” for solving inverse additive problems:

- **Step 1.** Consider some (usually numerical) characteristic of the set under study.

- **Step 2.** Find an extremal value of this characteristic within the framework of the problem that we are studying.

- **Step 3.** Study the structure of the set when its characteristic is equal to its extremal value.

- **Step 4.** Study the structure of the set when its characteristic is near to its extremal value.

- **Step 5.** ....Continue, taking larger and larger neighborhoods for the characteristic.
Let us choose as characteristic the *cardinality of the sumset*:

\[ 2K = K + K, \]

or equivalently the "*measure of doubling*":

\[ \sigma = \frac{|K + K|}{|K|}. \]

We will examine in detail the **exact structure** of a finite set

\[ K \subseteq G, \]

in the case of a torsion free Abelian group

\[ G = \mathbb{Z}^n \quad \text{or} \quad G = \mathbb{R}^d, \]

assuming that the doubling constant is small.

**REMARK:** If \( \sigma \) is an *arbitrary* doubling constant, then *Freiman’s fundamental result (1966)* asserts that such a set is a large subset of a multidimensional arithmetic progression; see also Freiman (1987), Bilu (1993), Ruzsa (1994), Nathanson (1996), or Tao and Vu (2006).
2. Small doubling property on the plane \( \mathbb{Z}^2 \)

Let us describe some results concerning the structure of planar sets with small sumset.

We begin with the following basic inequality:

**Theorem 1** (Freiman 1966). If \( \mathcal{K} \subseteq \mathbb{Z}^2 \) lies on exactly \( s \geq 2 \) parallel lines, then

\[
|\mathcal{K} + \mathcal{K}| \geq (4 - \frac{2}{s})|\mathcal{K}| - 2s + 1 \geq 3k - 3. \tag{1}
\]

Moreover, using Freiman’s \( 3k - 4 \) theorem we easily conclude that a planar set of lattice points \( \mathcal{K} \subseteq \mathbb{Z}^2 \) with

\[
|\mathcal{K} + \mathcal{K}| < 3|\mathcal{K}| - 3
\]

lies on a straight line and is contained in an arithmetic progression of no more than

\[
v = |\mathcal{K} + \mathcal{K}| - |\mathcal{K}| + 1
\]

terms. Step 2 is completely solved.
Therefore, a natural problem is to concentrate on the study of Steps 3 and 4.

We ask for the structure of a finite planar set of lattice points with small doubling $|\mathcal{K} + \mathcal{K}|$. As one can expect, this question is easier to answer when the cardinality $|\mathcal{K} + \mathcal{K}|$ is close to its minimal possible value $3|\mathcal{K}| - 3$, and becomes much more complicated if we choose bigger values for $|\mathcal{K} + \mathcal{K}|$. To be more specific, we may ask the following

**Problem.**

*Find the exact structure of planar sets of lattice points under the doubling hypothesis:*

$$|\mathcal{K} + \mathcal{K}| < (4 - \frac{2}{s + 1})|\mathcal{K}| - (2s + 1).$$
Let us examine the first case $s = 2$.

Though, the Freiman’s $(2^n - \varepsilon)$ theorem gives a first indication on the structure of $\mathcal{K}$, still this is not so precise as the following

**Theorem 2** (Freiman 1966, S. 1998). Let $\mathcal{K} \subseteq \mathbb{Z}^2$ be a finite of dimension $\dim \mathcal{K} = 2$.

(i) $|\mathcal{K}| \geq 11$ and $|\mathcal{K} + \mathcal{K}| < \frac{10}{3}|\mathcal{K}| - 5$ then $\mathcal{K}$ lies on two parallel lines.

(ii) If $\mathcal{K}$ lies on two parallel lines and

$$|\mathcal{K} + \mathcal{K}| < 4|\mathcal{K}| - 6$$

then $\mathcal{K}$ is included in two parallel arithmetic progressions with the same common having together no more than $v = |2\mathcal{K}| - 2k + 3$ terms.

This means that the total number of holes satisfies

$$h \leq |2\mathcal{K}| - (3k - 3).$$
The following theorem incorporates Freiman’s previous result as a particular case:

**Theorem 3 (S. 1998).** Let $\mathcal{K}$ be a finite set of $\mathbb{Z}^2$ and $s \geq 1$ be a natural number. If $|\mathcal{K}|$ is sufficiently large, i.e. $k \geq O(s^3)$, and

$$|\mathcal{K} + \mathcal{K}| < \left(4 - \frac{2}{s + 1}\right)|\mathcal{K}| - (2s + 1),$$

then there exist $s$ parallel lines which cover the set $\mathcal{K}$.

This is a best possible result, because it cannot be improved by increasing the upper bound for $|\mathcal{K} + \mathcal{K}|$, or by reducing the number of lines that cover $\mathcal{K}$. 


EXAMPLE: ...
The theorem is effective and recently Serra and Grynkiewicz obtained an explicit value for the constant $k_0(s) = 2s^2 + s + 1$. They also succeeded to extend the result for sums of different sets $A + B$:

**Theorem 4** (Grynkiewicz and Serra 2007). Let $A, B \subseteq \mathbb{R}^2$ be finite subsets and $s \geq 1$ be a natural number.

(i) If $|A| - |B| \leq s + 1, |A| + |B| \geq 4s^2 + 2s + 1$ and

$$|A + B| < (2 - \frac{1}{s + 1})(|A| + |B|) - (2s + 1)$$

then there exist $2s$ (not necessarily distinct) parallel lines which cover the sets $A$ and $B$.

(ii) If $|A| > |B| + s, |B| \geq 2s^2 + \frac{s}{2}$ and

$$|A + B| < |A| + (3 - \frac{2}{s + 1})|B| - (s + 1)$$

then there exist $2s$ (not necessarily distinct) parallel lines which cover the sets $A$ and $B$. 
The next natural question is to consider a finite set $\mathcal{K}$ of lattice points on a plane having the \textit{small doubling property}

$$|2\mathcal{K}| < (4 - \frac{2}{s + 1})|\mathcal{K}| - (2s + 1)$$

and ask for a reasonable estimate for the number of lattice points of a "minimal" parallelogram that covers the set $\mathcal{K}$.

More precisely, if $\mathcal{L}$ is a lattice generated by $\mathcal{K}$, we are interested in precise upper bounds for the number of points of $\mathcal{L}$ that lie in the convex hull of $\mathcal{K}$. Our main result asserts that $\mathcal{K}$ is located inside a parallelogram that lies on a few lines which are well filled:
**Theorem 5** (S. 2007). Let \( s \geq 19 \) be an integer and let \( K \) be a finite subset of \( \mathbb{Z}^2 \) that lies on exactly \( s \) parallel lines. If

\[
|2K| < (4 - \frac{2}{s+1})|K| - (2s + 1),
\]

then there is a lattice \( L \subseteq \mathbb{Z}^2 \) and a parallelogram \( P \) such that

\[
K \subseteq (P \cap L) + v
\]

and

\[
|P \cap L| \leq 24\left(|K + K| - 2|K| + 1\right),
\]

for some \( v \in \mathbb{Z}^2 \).

**Conjecture.** We believe that for a best possible result, the constant factor 24 of Theorem 5 should be replaced by \( \frac{1}{2} + \frac{1}{s-1} \), i.e.

\[
|P \cap L| \leq \frac{s}{2(s-1)}\left(|K + K| - 2|K| + 2s - 1\right).
\]

So far inequality this estimate has been proved only for \( s = 2 \) (Freiman 1966) and \( s = 3 \) (S. 1999).
3. Planar sets with no three collinear points on a line

Let $A \subseteq \mathbb{Z}^2$ be a finite set, not containing any three collinear points. Freiman asked in 1966 for a lower bound for $|A + A|$. As a first step in the investigation of this problem we showed that $\frac{|A \pm A|}{|A|}$ is unbounded, as $\lim |A| = \infty$:

**Theorem 6 (S.2002).** Let $A \subseteq \mathbb{Z}^2$ be a finite set of $n$ lattice points. If $A$ does not contain any three collinear points, then there is a positive absolute constant $\delta > 0$ such that

$$|A \pm A| \gg n(\log n)^\delta.$$ (3)

The constant $\delta$ can be easily computed: for instance, any positive $\delta$ smaller than 0.125 will do.
There is an intimate connection between two seemingly unrelated problems:

(i) non-averaging sets of integers of order \( t \) and
(ii) planar sets with no three points on a line.

**Definition.** A finite set of integers \( B \subseteq \mathbb{Z} \) is called a **non-averaging set of order** \( t \), if for every \( 1 \leq m, n \leq t \) the equation

\[
mX_1 + nX_2 = (m + n)X_3,
\]

have no nontrivial solutions with \( X_i \in B \).

Let

\[
s_t(n)
\]

be the maximal cardinality of a **non-averaging set of order** \( t \) included in the interval \([1, n]\).
It is clear that a non-averaging set of order 1 is simply an integer set containing no arithmetic progressions. Bourgain’s bound for Roth’s theorem gives:

\[ s_t(n) \leq s_1(n) = r_3(n) \ll \frac{n}{(\log \log n)^{1/2}}. \]

**Remark.** We also obtained a *more exact* inequality, valid for sets \( A \subseteq \mathbb{Z}^2 \) containing no \( k \)--terms arithmetic progressions: for every integer \( t \geq 1 \) we have

\[ |A \pm A| \geq \frac{1}{2} |A| \left( \frac{n}{s_t(n)} \right)^{\frac{1}{4t}}. \]  

(4)
We formulate the following:

**Problem S.** *Suppose that* \( t \geq 1 \) *is a fixed, positive, but rather large integer. Is it true that* \( s_t(n) \ll \frac{n}{(\log n)^4t} \), *or at least* \( s_t(n) \ll \frac{n}{(\log n)^c} \), *for a positive absolute constant* \( c \geq \frac{1}{2} \) ?

Note that Freiman’s question asks for a non-trivial lower estimate of \( |A + A| \) for a set \( A \subseteq \mathbb{Z}^2 \) containing no three collinear points and in Problem S we want to estimate the density of a sequence of natural numbers \( B \), assuming that \( t \) linear equations does not hold for \( B \). Inequality (4) shows that any upper bound for \( s_t(n) \), *better* than the trivial one \( r_3(n) \) will lead to a corresponding sharpening of (3) and (4).
As regards lower bounds, we have:

**Theorem 7 (S. 2002).**

(i) For every $t \geq 1$, there is a positive constant $c_t$ such that for every $n$ one has

$$s_t(n) \geq n \exp(-c_t \sqrt{\log n}).$$

(ii) There is no $\epsilon_0 > 0$ such that the inequality

$$|A + A| \gg |A|^{1+\epsilon_0}$$

holds for every finite set $A \subseteq \mathbb{Z}^2$ containing no three collinear points.

The proof uses Freiman’s fundamental concept of isomorphism, Behrend’s method and a result of Ruzsa about sets of integers containing no non-trivial three term arithmetic progressions.

A recent improvement of the lower bound (3), was obtained by T. Sanders (2006):

$$|A + A| \gg\epsilon |A|(\log |A|)^{\frac{1}{3} - \epsilon}.$$

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4. The simplest inverse problem for sums of sets in several dimensions

It is a well known fact that $|A+B| \geq |A|+|B|-1$ for every two finite sets $A$ and $B$ of $\mathbb{Z}^d$, equality being attained when $A$ and $B$ are arithmetic progressions with the same difference.

It is possible to obtain a much better estimate. The first result connecting geometry and additive properties is

**Theorem 8** (Freiman 1966). *For every finite set $A \subseteq \mathbb{Z}^d$ of affine dimension $\dim A = d$, one has*

$$|A+A| \geq (d+1)|A| - \frac{1}{2}d(d+1).$$

(5)

This lower bound is tight, i.e. Step 2 is solved.
EXAMPLE:
Let us investigate now Step 3. What is the exact structure of multi-dimensional sets having the smallest cardinality of the sumset?

The following result is an analogue of the well known Vosper’s theorem (1956), \( \mathbb{Z}/p\mathbb{Z} \) being here replaced by the \( d \)-dimensional space \( \mathbb{R}^d \).

**Theorem 9 (S. 1998).** Let \( A \subseteq \mathbb{R}^n \) be a finite set such that \( \dim A \geq d \) and

\[
|A + A| = (d + 1)|A| - \frac{1}{2}d(d + 1).
\]

If \( |A| \neq d + 4 \), then \( A \) is a \( d \)-dimensional set and \( A \) consists of \( d \) parallel arithmetic progressions with the same common difference.

Moreover, if \( |A| = d + 4 \), then

\[
A = \{v_0, v_1, ..., v_d\} \cup \{2v_1, v_1 + v_2, 2v_2\},
\]

where \( v_i \) are the vertices of a \( d \)-dimensional simplex.
EXAMPLE:
Further developments:

**Ruzsa (1994):** If $|A| \geq |B|$ and $\dim(A+B) = d$, then

$$|A + B| \geq |A| + d|B| - \frac{d(d+1)}{2}.$$

**Gardner and Gronchi (2001):** If $|A| \geq |B|$ and $\dim(B) = d$, then

$$|A + B| \geq |A| + (d-1)|B| + \sqrt{d(|A| - d)^{d-1}} - \frac{d(d-1)}{2}.$$

**Green and Tao (2006)**
Suppose that $A \subseteq \mathbb{R}^m$ is a finite set which contains a parallelepiped $P = \{0, 1\}^d \subseteq \mathbb{Z}^d \subseteq \mathbb{R}^m$.

Then

$$|A + A| \geq 2^{d/2}|A|.$$
5. Exact Structure Results for Multidimensional Inverse Additive Problems

A natural question is to generalize Theorem 3 to the multidimensional case $d = \dim(\mathcal{K}) \geq 3$:

Assume that the doubling coefficient of the sum set $2\mathcal{K}$ is not much exceeding the minimal one, i.e.

$$d + 1 \leq \sigma = \frac{|2\mathcal{K}|}{|\mathcal{K}|} < \rho_d.$$

What can be said about the exact structure of $\mathcal{K}$? The expected result is: if

$$\rho_d = d + 1 + \frac{1}{3},$$

then the set $\mathcal{K}$ is contained in $d$ "short" arithmetical progressions.
The problem was first solved for the first open case $d = 3$:

**Theorem 10** (S. 2005). Let $\mathcal{K}$ be a finite subset of $\mathbb{Z}^3$ of affine dimension $\dim \mathcal{K} = 3$.

(i) If $|\mathcal{K}| > 12^3$ and

$$|\mathcal{K} + \mathcal{K}| < \frac{13}{3}|\mathcal{K}| - \frac{25}{3}$$

then $\mathcal{K}$ lies on three parallel lines.

(ii) If $\mathcal{K}$ lies on three parallel lines and

$$|\mathcal{K} + \mathcal{K}| < 5|\mathcal{K}| - 10,$$

then $\mathcal{K}$ is contained in three arithmetic progressions with the same common difference, having together no more than

$$\nu = |\mathcal{K} + \mathcal{K}| - 3|\mathcal{K}| + 6$$

terms.
The structure of $\mathcal{K}$ can be also be described for sets of dimension $d \geq 3$:

**Theorem 11** (S. 2008). Let $\mathcal{K} \subseteq \mathbb{Z}^d$ be a finite set of dimension $d \geq 2$.

(i) If $k > 3 \cdot 4^d$ and

$$|\mathcal{K} + \mathcal{K}| < (d + \frac{4}{3})|\mathcal{K}| - c_d,$$

where $c_d = \frac{1}{6}(3d^2 + 5d + 8)$, then $\mathcal{K}$ lies on $d$ parallel lines.

(ii) If $\mathcal{K}$ lies on $d$ parallel lines and

$$|\mathcal{K} + \mathcal{K}| < (d + 2)|\mathcal{K}| - \frac{1}{2}(d + 1)(d + 2),$$

then $\mathcal{K}$ is contained in $d$ parallel arithmetic progressions with the same common difference, having together no more than

$$v = |\mathcal{K} + \mathcal{K}| - d|\mathcal{K}| + \frac{1}{2}d(d + 1) \text{ terms.}$$
These results are best possible and cannot be sharpened by reducing the quantity $v$ or by increasing the upper bounds for $|\mathcal{K} + \mathcal{K}|$.

**EXAMPLES:**
We found that a similar inequality can be formulated for $d$–dimensional sets that have a small doubling coefficient $C_d = d + 2 - \frac{2}{s-d+3}$ (where $s \geq d$ is a positive integer). In this case we prove that $\mathcal{K}$ lies on no more than $s$ parallel lines.

These results can be used to make Freiman’s Main Theorem more precise.

In a joint work with Freiman (2008) we study the exact structure of $d$-dimensional sets satisfying the small doubling property

$$|2K| < (d + 2 - \epsilon)|K|.$$
6. Difference Sets

We will present now some results on difference sets in a $d$-dimensional Euclidean space. The need for lower estimates for $|A − A|$ in terms of $|A|$ has been raised by Uhrin (1981), where the trivial $|A − A| \geq 2|A| − 1$ is used to prove theorems sharpening the classical theorem of Minkowski-Blichfeldt in geometry of numbers.

It can be stated that the sharper estimation for $|A − A|$ we have, the sharper results in geometry of numbers can be proved.

Let $A \subseteq \mathbb{R}^d$ be a finite set and (as Step 1 of Freiman’s algorithm requires) we choose as numerical characteristic the cardinality of the difference set $A − A$. 
The following inequality is analogous to (5):

**Theorem 12** (Freiman-Heppes-Uhrin 1989). If \( \dim \mathcal{A} \geq 1 \), then

\[
|\mathcal{A} - \mathcal{A}| \geq (d + 1)|\mathcal{A}| - \frac{1}{2}d(d + 1). 
\] (6)

This immediately yields that if

- \( d = 1 \) and \( \mathcal{A} \subseteq \mathbb{R} \), then \( |\mathcal{A} - \mathcal{A}| \geq 2|\mathcal{A}| - 1 \) and if
- \( d = 2 \) and \( \mathcal{A} \subseteq \mathbb{R}^2 \), then \( |\mathcal{A} - \mathcal{A}| \geq 3|\mathcal{A}| - 3 \).

These two inequalities cannot be strengthened. However, the lower bound (6) is not exact for dimension \( d = 3 \).

Freiman-Heppes-Uhrin (1989) and Ruzsa (1994) conjectured that the “correct” lower bound for \( \dim \mathcal{A} = 3 \) is

\[
|\mathcal{A} - \mathcal{A}| \geq 4.5|\mathcal{A}| - 9. 
\] (7)

This conjecture is correct and (7) is a best possible lower bound for \( |\mathcal{A} - \mathcal{A}| \):
Theorem 13 (S. 1998). Let $A$ be a finite set of $\mathbb{R}^3$ and let $\{e_1, e_2, e_3\}$ be the standard basis of $\mathbb{R}^3$.

(i) If $\dim A = 3$, then $|A - A| \geq 4.5|A| - 9$.

(ii) Equality is attained if and only if $A$ is a union of four parallel arithmetic progressions: $A = \{0, e_1, e_2, e_1 + e_2\} + \{0, e_3, 2e_3, ..., ke_3\}$.

For 2-dimensional sets the situation is similar:

Theorem 14 (S. 1998). Let $D$ be a finite set in $\mathbb{R}^2$ of affine dimension $\dim D = 2$. Then $|D - D| = 3|D| - 3$, if and only if $D$ consists of two parallel arithmetic progressions with the same number of elements and the same common difference.

This solves Steps 2 and 3 of Freiman’s algorithm: it gives the structure of 2 and 3 dimensional sets having the smallest cardinality of the difference set.
Let us give now a short description of the multidimensional case $d \geq 4$.

Let $s_d$ be the maximal positive number for which the inequality

$$|A - A| \geq s_d|A| - t_d$$

holds for every finite set $A$ of affine dimension $\dim A = d$.

What can one say about $s_d$? The exact value of $s_d$ is known only for $d = 1$, $d = 2$ and $d = 3$ and Ruzsa conjectured

**Conjecture.** (Ruzsa, 1994) For every $d \geq 4$ we have

$$s_d = 2d - 2 + \frac{2}{d}.$$
EXAMPLES:
The following upper bound for $s_d$ is true:

**Theorem 15** (S. 2001). For every integer $d$, $d \geq 2$ one has

$$s_d \leq 2d - 2 + \frac{1}{d-1}.$$  

This readily disproves Ruzsa’s conjecture. Moreover, in view of inequality (7) and Theorem 15, it seems that the equality $s_d = 2d - 2 + \frac{1}{d-1}$ is true for every $d \geq 2$. Thus, we suggest the following:

**Conjecture 16** (S. 2001). For every finite set $A$ of affine dimension $\dim A = d \geq 2$, one has

$$|A - A| \geq (2d - 2 + \frac{1}{d-1})|A| - (2d^2 - 4d + 3).$$

Of course, in view of Theorem 15, if the above inequality is true, then is best possible.
EXAMPLES for dimension 2, 3 and 4...
7. Finite Abelian groups

Similar questions can be asked for any group $G$. A short and incomplete list of results for

$$G = \mathbb{F}_p, G = (\mathbb{F}_2)^d, G = \mathbb{Z}/n\mathbb{Z}$$

will show that additive questions in finite abelian groups are generally more difficult than analogous problems in $\mathbb{Z}$.

- Consider for the beginning sums of congruence classes modulo a prime $p$. Take two finite sets $A$ and $B$ in $\mathbb{F}_p$ and choose as characteristic the cardinality of the sum

$$A + B = \{a + b : a \in A, b \in B\}.$$ 

Then the solution of Step 2 is Cauchy-Davenport theorem:

$$|A + B| \geq \min\{p, |A| + |B| - 1\}.$$
The answer to Step 3 is given by Vosper’s theorem (1956), which classify those pairs $A, B$ of sets of residues for which equality holds in Cauchy-Davenport inequality.

The next natural question is to consider Step 4 and to analyze the case when the cardinality of the sum is not much exceeding its extremal value.

Freiman (1966), generalized Vosper’s theorem for sumsets of the form $A + A$ in $\mathbb{F}_p$, by describing the structure of $A$ in the case

$$|2A| < c|A| - 3,$$

with $c < 2.4$; either $|A|$ is large or the set $A$ is located in a short arithmetic progression.

This has been recently extended to any $c$ by Green and Ruzsa (2006), using the rectification principle of Freiman and Bilu-Lev-Ruzsa (1998).
• For sumsets in vector spaces over finite fields, Eliahou and Kervaire proved in (1998) that
\[ |A+B| \geq \min \left\{ p^t \left( \left\lceil \frac{|A|}{p^t} \right\rceil + \left\lceil \frac{|B|}{p^t} \right\rceil - 1 \right) : 0 \leq t \leq d \right\}, \]
for every two sets \( A \) and \( B \) included in \((F_p)^d\). Step 2 is solved.

Deshouillers-Hennecart-Plagne gave in (2004) an answer to Steps 3 and 4 by obtaining a structure theorem under the assumption
\[ A \subseteq F_2^d, |A + A| = c|A|, 1 \leq c < 4. \]
In this instance the set \( A \) is contained in a coset \( a + H \) of order at most \( \frac{|A|}{u(c)} \) where \( u(c) > 0 \) is an explicit function depending only on \( c \).
Recently Step 5 was solved by Ruzsa and Green (2008), not only for $G = \mathbb{F}_p^d$, but also for commutative torsion groups:

If $A$ is a subset of a commutative group $G$ of exponent $r$ and if

$$|A + A| < k|A|,$$

then $A$ is contained in a coset of a subspace of size no more than

$$k^2 r^2 k^2 - 2.$$
• Let $G$ is an arbitrary Abelian group. Kneser (1953) gave a deep generalization of Cauchy-Davenport’s theorem:

Let $A$ and $B$ be two finite subsets of an Abelian group $G$. One has

$$|A + B| \geq |A| + |B| - |H|,$$

where $H$ is the stabilizer of $A + B$.

Important results concerning the equality case in Kneser’s theorem are due to Kemperman (1960) and Lev (1999).

In a step beyond Kneser’s theorem, Deshouillers and Freiman (2003) proved a structural result for the cyclic group

$$G = \mathbb{Z}/n\mathbb{Z}$$

assuming that

$$|A + A| < 2.04|A|$$

and $|A|$ sufficiently small.