Theory and Computation in the Search for Special Surfaces

Let $\mathbb{K}$ be an algebraically closed field, let $\mathbb{P}^n$ be the $n$-dimensional projective space over $\mathbb{K}$, and let $V$ and $W$ be two disjoint irreducible projective varieties in $\mathbb{P}^n$. We denote by $J(V,W)$ the union of the lines in $\mathbb{P}^n$ joining $V$ to $W$. Then $J(V,W)$ is a projective variety. This variety is called the \textit{join} of $V$ and $W$.

**Exercise 1.** (a) Write a \textit{Macaulay 2} script for computing the ideal of the join of two given varieties. Let $L_1$ and $L_2$ be two skew lines in $\mathbb{P}^4$. Compute the ideal of $J(L_1, L_2)$.

(b) Make a \textit{Macaulay 2} function for computing the ideal of the secant variety to a given variety (use the script you wrote in Exercise 1). Use your script to compute the ideal of the secant variety to the Veronese variety in $\mathbb{P}^5$.

\textit{Hint.} Let $A = [a_0 : \cdots : a_n]$ and $B = [b_0 : \cdots : b_n]$ be points of $V$ and $W$ respectively. Then any point $R = [z_0 : \cdots : z_n]$ of the line passing through $A$ and $B$ is given by

\[
\begin{pmatrix}
  z_0 \\
  \vdots \\
  z_n
\end{pmatrix} = s \begin{pmatrix}
  a_0 \\
  \vdots \\
  a_n
\end{pmatrix} + t \begin{pmatrix}
  b_0 \\
  \vdots \\
  b_n
\end{pmatrix}
\]

for some $[s : t] \in \mathbb{P}^1$. Let $\{f_1, \ldots, f_l\}$ and $\{g_1, \ldots, g_m\}$ be the generating sets for $I(V)$ and $I(W)$ respectively. Then the ideal defining $J(V,W)$ is obtained by solving the system of equations $f_1 = \cdots = f_l = 0$, $g_1 = \cdots = g_m = 0$ and (1) for $z_0, \ldots, z_{n-1}$ and $z_n$. By replacing $x_i$'s by $a_i$'s and $b_i$'s, we obtained new ideals $I(V)_a$ in $\mathbb{K}[a_0, \ldots, a_n]$ and $I(W)_b$ in $\mathbb{K}[b_0, \ldots, b_n]$ respectively. Let $I$ be the ideal in the following new ring:

$\mathbb{K}[a_0, \ldots, a_n, b_0, \ldots, b_n, s, t, z_0, \ldots, z_n]$ generated by the equations in (1), and let $J = I + I(V)_a + I(W)_b$. Saturate $J$ with respect to the ideal $(s, t)$:

$J' = J : (s, t)^\infty$,

because we do not want $s$ and $t$ to vanish at the same time (recall that $[s : t] \in \mathbb{P}^1$). Take the intersection $\overline{J} = J' \cap \mathbb{K}[z_0, \ldots, z_n]$. Then $V(\overline{J}) = J(V,W)$.

**Exercise 2.** Construct a smooth rational surface $X$ in $\mathbb{P}^4$, which is isomorphic to $\mathbb{P}^2$ blown up in eight points embedded by the linear system of the following type:

$4L - 2E_0 - \sum_{i=1}^7 E_i$,

where $L$ is the pullback from $\mathbb{P}^2$ a line, while the $E_i$ are the exceptional curves of the blowup. Compute the degree and sectional genus of $X$. 

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Exercise 3. Construct the union $P$ of four planes $P_0$, $P_1$, $P_2$ and $P_3$ in $\mathbb{P}^4$ such that

(a) $P_0$ intersects each of $P_1$, $P_2$ and $P_3$ in a line;
(b) For $1 \leq i < j \leq 3$, $P_i$ and $P_j$ meet at a single point.

Check that the ideal of $U$ is generated by 7 cubics. Take two general cubics to get the ideal of the surface $X$ linked to $U$ in a $(3,3)$ complete intersection. What is this surface? Any line in $P_i$, $i \in \{1, 2, 3\}$, is a 3-secant line to $X$.

Theorem 1 (Beilinson, 1978). For any sheaf $\mathcal{F}$ on $\mathbb{P}^n$, there is a complex $\mathcal{K}$ with

$$\mathcal{K}^i = \bigoplus H^{i-j}(\mathbb{P}^n, \mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^n}(j)) \otimes \Omega^{-j}(-j)$$

such that

$$H^i(\mathcal{K}) = \begin{cases} \mathcal{F} & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Exercise 4. Let $\mathcal{E}$ be a locally free sheaf on $\mathbb{P}^3$ whose cohomology table for $h^i(\mathbb{P}^3, \mathcal{E}(j))$ in the range $-3 \leq j \leq 0$ of twists is of the following type:

\[
\begin{array}{cccc}
  & h^i & 3 & 2 \\
 2 & 2 & 2 & 1 \\
-3 & -2 & -1 & 0 \\
\end{array}
\]

Use Beilinson’s theorem to determine the type of the monad for $\mathcal{E}$. Then construct the differentials of the monad. How do you check that your sheaf is locally free?

Exercise 5. Let $X$ be an elliptic conic bundle in $\mathbb{P}^4$. Compute the image of the adjunction map $\Phi_{|H+K|} : X \rightarrow \mathbb{P}^2$. (Use the script for the ideal of $X$).

You can find more exercises in the last two pages!
hilbertPolynomial I

-- Example: Veronese surface in P^4 --

ringP4=KK[y_0..y_4];
veronese=basis(2,ringP2)*random(ringP2^6,ringP2^5);
I=ker map(ringP2,ringP4,veronese);
hilbertPolynomial I
-- Is V(I) smooth?
singI=minors(codim I,jacobian I)+I;
codim singI
betti I

-- Example: del Pezzo surface --

randomPoints=(N)->(
  i:=1;
  idl:=ideal random(ringP2^{0},ringP2^{2:-1});
  while i<N do (idl=intersect(idl,ideal random(ringP2^{0},ringP2^{2:-1})); i=i+1; )
  idl)
points=randomPoints(5);
hilbertPolynomial points
del=gens points*map(source gens points,basis(3,points));
delPezzo=trim ker map(ringP2,ringP4,del);
betti res delPezzo

-- Example: Veronese surface in P^4 and elliptic quintic scroll --

veronese=basis(2,ringP2)*random(ringP2^6,ringP2^5);
I=ker map(ringP2,ringP4,veronese);
betti I
V=ideal(gens I*random(source gens I,ringP4^{2:-3}));
J=V:I;
\texttt{betti J}
\texttt{X=Proj(ringP4/J);}
\texttt{HH^{-1}(00_X)}

\texttt{------------------------------------}
\texttt{--Example: nullcorellation bundle --}
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\texttt{sortedBasis=(i,E)->(}
\texttt{ m:=basis(i,E);}
\texttt{ p:=sortColumns (m, MonomialOrder=>Descending); m_p);}

\texttt{beilinson1=(e,dege,i,S)->(}
\texttt{ E:=ring e;}
\texttt{ mi:=if i<0 or i>=numgens E then map(E^1,E^0,0)
else if i === 0 then id_(_E^1)
else sortedBasis(i+1,E);}
\texttt{ r:=i-dege;}
\texttt{ mr:= if r<0 or r>=numgens E then map(E^1,E^0,0)
else sortedBasis (r+1,E);}
\texttt{ s:=numgens source mr;}
\texttt{ if i===0 and r===0 then substitute(map(E^1,E^1,{{e}}),S)
else if i>0 and r===i then substitute(e*id_(_E^s),S)
else if i>0 and r===0 then
   (vars S)*substitute(contract(diff(e,mi),transpose mr),S)
else substitute(contract(diff(e,mi),transpose mr),S));}

\texttt{U=(i,S)->(}
\texttt{ if i<0 or i>=numgens S then S^0
else if i===0 then S^1
else coker koszul(i+2,vars S)**S^{i});}

\texttt{beilinson=(o,S)->(}
\texttt{ coldegs:=degrees source o;}
\texttt{ rowdegs:=degrees target o;}
\texttt{ mats:=table(numgens target o, numgens source o,}
\texttt{ (r,c)->(}
\texttt{ rdeg=first rowdegs#r;}
\texttt{ cdeg=first coldegs#c;}
\texttt{ overS=beilinson1 (o_(r,c),cdeg-rdeg,cdeg,S);
map(U(rdeg,S),U(cdeg,S),overS));}
\texttt{ if #mats===0 then matrix(S,{{}})
else matrix(mats));}

\texttt{ringP3=KK[x_0..x_3];}
E=KK[e_0..e_3,SkewCommutative=>true];

-- indecomposable

alpha=map(E^{-1},E^{-3},\{e_0*e_2+e_1*e_3\});
alpha=beilinson(alpha,ringP3);
betti alpha
omega1=U(1,ringP3)
F=prune coker(presentation omega1|map(target presentation omega1,,alpha));
fF=res F;
betti fF
rank F
Ft=syz transpose fF.dd_1;
-- Check F is locally free.
codim minors(2,Ft)
codim minors(3,Ft)

-- decomposable

alpha=map(E^{-1},E^{-3},\{e_0*e_2\});
alpha=beilinson(alpha,ringP3);
betti alpha
omega1=U(1,ringP3)
F=prune coker(presentation omega1|map(target presentation omega1,,alpha));
fF=res F;
betti fF
rank F
Ft=syz transpose fF.dd_1;
codim minors(2,Ft)
codim minors(3,Ft)

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-- Example: elliptic conic bundle --
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ringP4=KK[x_0..x_4];
E=KK[e_0..e_4,SkewCommutative=>true];
beta=random(E^{-1:0},E^{-1:-1,1:-2})
betti syz beta
alpha=(syz beta)*random(source syz beta,E^{-3});
beta=beilinson(beta,ringP4);
alpha=beilinson(alpha,ringP4);
F=prune homology(beta,alpha);
fF=res F;
betti fF
phi=presentation F|random(target presentation F,ringP4^{4:-1});
fphit=res (prune coker transpose phi,LengthLimit=>2);
betti fphit
I=ideal fphit.dd_2;
hilbertPolynomial I

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-- Example: Bordiga surface --
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restart
kk=ZZ/32003
S=kk[a..e]
I=ideal(
  490*a^3+12221*a^2*b+1663*a*b^2-14146*b^3-6390*a^2*c-8829*a*b*c
  +6322*b^2*c-9839*a*c^2-7186*b*c^2-4124*c^3-6141*a^2*d+15804*a*b*d
  -10963*b^2*d-14244*a*c*d+12528*c^2*d+1270*a*d^2+9748*b*d^2-24690*c*d^2
  -11431*d^3-30663*a^2*e+965*a*b*e+15018*b^2*e+3510*a*c*e-13207*b*c*e
  -12323*c^2*e-5496*a*d*e-1110*b*d*e+4737*d^2*e+11286*a*e^2+3759*b*e^2
  -12124*c^2*e^2-8044*d*e^2-8400*e^3,
  -4212*a^3+6354*a^2*b+8359*a*b^2-15620*b^3-7605*a^2*c-14070*a*b*c
  +1533*b^2*c-6538*a*c^2+1929*b*c^2+6483*c^3+2503*a^2*d
  -13769*a*b*d-10716*b^2*d+15442*a*c*d+10241*b*c*d+4046*c^2*d
  -7595*a*d^2+2455*b*d^2+5535*c*d^2-12507*d^3-8300*a^2*e
  -9255*a*b*e-166*b^2*e-3948*a*c*e-887*b*c*e+12166*c^2*e
  -12019*a*d*e+4986*b*d*e-9965*c*d*e-2795*d^2*e-11300*a*e^2
  -9200*b*e^2+2201*c*e^2+771*d*e^2-10878*e^3,
  -13422*a^3-13098*a^2*b+165*a*b^2-15751*b^3-9062*a^2*c
  +3931*a*b*c+10465*b^2*c+3947*a*c^2+15856*b*c^2-13765*c^3
  +12661*a^2*d-759*a*b*d+2367*b^2*d-5578*a*c*d-10985*b*c*d
  +987*c^2*d-9772*a*d^2-1366*b*d^2+2179*c*d^2-13964*d^3
  +11350*a^2*e-1598*a*b*e+2754*b^2*e+9928*a*c*e+13439*b*c*e
  +15037*c^2*e+9065*a*d*e+12521*b*d*e+9348*c*d*e-10519*d^2*e
  -13803*a*e^2+6223*b*e^2+6660*c*e^2+9736*d*e^2+5263*e^3,
  14635*a^3+2059*a^2*b-5482*a*b^2-14747*b^3+1035*a^2*c-4988*a*b*c
  +11895*b^2*c+13195*a*c^2+7097*b*c^2-8995*c^3+1144*a^2*d
  -11329*a*b*d+14481*b^2*d+4991*a*c*d+14404*b*c*d+10242*c^2*d
  -6921*a*d^2-6661*b*d^2-15585*c*d^2+9999*d^3+4902*a^2*e
\[-12029*a*b*e-681*b^2*e-2020*a*c*e+14081*b*c*e+15876*c^2*e+7994*a*d*e+7138*b*d*e-3671*c*d*e+11424*d^2*e+380*a*e^2-11858*b*e^2+9754*c*e^2+13383*d*e^2+6735*e^3);\]

\[\text{betti } I\]

\[fI = \text{res } I;\]

\[\text{betti } fI\]

\[\text{ringP2} = \text{kk}[s_0..s_2];\]

\[\text{ringP4xP2} = \text{kk}[a..e, s_0..s_2, \text{MonomialOrder}=>\text{Eliminate 5}];\]

\[\omegaX = \text{coker transpose } fI.\dd_2;\]

\[\phi = \text{sub}(\text{vars } \text{ringP2, ringP4xP2}) * \text{sub}(\text{presentation } \omegaX, \text{ringP4xP2});\]

\[\text{adjunction} = \text{ideal}(\omegaX);\]

\[\text{adjunctionsat} = \text{saturate}(\text{adjunction}, \text{ideal}(a..e));\]

\[J = \text{ideal sub}(\text{selectInSubring}(1, \text{gens } \text{gb } \omegaX), \text{ringP2});\]

\[\text{codim } J\]

\[\psi = \text{contract}(\text{sub}(\text{vars } S, \text{ringP4xP2}), \text{transpose } \phi);\]

\[\text{points} = \text{sub}(\text{minors}(4, \psi), \text{ringP2});\]

\[\text{hilbertPolynomial } \text{points}\]

\[\text{betti } \text{points}\]

**Lemma 1** (Terracini’s lemma). Let \(X\) be a smooth variety in \(\mathbb{P}^n\) and let \(p_1, \ldots, p_s\) be \(s\) generic points on \(X\). Then for a generic \(p \in \langle p_1, \ldots, p_s \rangle\),

\[\dim \sigma_s(X) = \dim \langle T_{p_1} X, \ldots, T_{p_s} X \rangle,\]

where \(T_{p_i} X\) is the projectivized tangent space to \(X\) in \(\mathbb{P}^n\) at \(p_i\).

Let \(X\) be a smooth variety in \(\mathbb{P}^n\) and let \(p_1\) and \(p_2\) be two generic points on \(X\). Then for a generic \(p \in \langle p_1, p_2 \rangle\),

\[\dim \text{Sec}(X) = \dim \langle T_{p_1} X, T_{p_2} X \rangle,\]

where \(T_{p_i} X\) is the projectivized tangent space to \(X\) in \(\mathbb{P}^n\) at \(p_i\).

**Exercise 6.** Let \(X = G(3,6)\) be the Grassmaniann of 3-planes in a 6-dimensional vector space. Use Terracini’s lemma to prove that \(\text{Sec}(X)\) has the expected dimension.

*Hint.* Let \(V\) be a 6-dimensional vector space with basis \(e_0, \ldots, e_5\) and let \(p_1\) and \(p_2\) be the points of \(X\) corresponding to \(e_0 \wedge e_1 \wedge e_2\) and \(e_3 \wedge e_4 \wedge e_5\). Try to prove that

\[\dim \langle T_{p_1} X, T_{p_2} X \rangle = 19.\]

You can use the following fact: Let us denote by \(W\) the following subspace of \(\bigwedge^3 V\):

\[V \wedge e_1 \wedge e_2 + e_0 \wedge V \wedge e_2 + e_0 \wedge e_1 \wedge V;\]

Note that \(T_{\omegaX} X = \mathbb{P}(W)\). Let \(W^\perp = \{w \in \bigwedge^3 V | w \wedge e_0 \wedge e_1 \wedge e_2\}\). The orthogonal complement \(W^\perp\) can be identified with \([e_0, e_1, e_2]_3\), where \((e_0, e_1, e_2)\) is the ideal of the exterior algebra \(\bigoplus_{i=0}^3 \bigwedge^i V\) generated by \(e_0, e_1\) and \(e_2\) and \([e_0, e_1, e_2]_3\) is the degree 3 part of \((e_0, e_1, e_2)^2\).
Exercise 7. Let $s \in H^0(\mathbb{P}^3, \mathcal{E}(1))$ be a general section of $\mathcal{E}(1)$. Compute the ideal of the zero locus $Z$ of $s$. Answer the following questions:

(a) What is the dimension of $Z$? How about the degree and arithmetic genus of $Z$?
(b) Is $Z$ smooth?
(c) How many components does $Z$ have?

Exercise 8. Let $M$ be the family of vector bundles whose cohomology tables are as given in Exercise 4. Compute the dimension of the Zariski tangent space to $M$ at the point corresponding to the example you constructed in Exercise 4.