Quantum Measurement Approach to a Non-Markovian Master Equation*

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Open System Dynamics

Exact Solution (Kraus Rep.):

\[ \rho_S(t) = \sum_{\alpha} E_{\alpha}(t) \rho_S(0) E_{\alpha}^\dagger(t), \]
\[ \sum_{\alpha} E_{\alpha}^\dagger(t) E_{\alpha}(t) = I. \]

\[ E_{\alpha}(t) = \sqrt{\nu} \langle \mu | U(t) | \nu \rangle, \quad |\nu\rangle \text{ and } |\mu\rangle \text{ are bath states.} \]

Markovian Regime (Lindblad Eq.):

\[ \frac{\partial \rho_S}{\partial t} = \mathcal{L} \rho_S = -\frac{1}{2} \sum_{\alpha} a_{\alpha} ([F_{\alpha}, \rho_S F_{\alpha}^\dagger] + [F_{\alpha} \rho_S, F_{\alpha}^\dagger]), \quad \rho_S(t) = e^{\mathcal{L} t} \rho_S(0). \]
### Dynamics in Two Limits

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<thead>
<tr>
<th></th>
<th>Exact Solution</th>
<th>Markovian Approximation</th>
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<tbody>
<tr>
<td><strong>Advantages</strong></td>
<td>No approximation</td>
<td>1. Closed form of dynamical map.</td>
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<td>2. Effective numerical solution.</td>
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<td><strong>Disadvantages</strong></td>
<td>Analytically solvable only for simple models.</td>
<td>Inadequate description for a bath with a significant memory effect.</td>
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**Goal of the presented work:**

“To develop a dynamical master equation beyond the Markovian regime that is analytically solvable and the resulting map is completely positive.”

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Measurement Theory Picture of Dynamics

Kraus Sum Representation: \[ \rho_{\text{out}} = \sum_k M_k \rho_{\text{in}} M_k^\dagger, \quad \sum_k M_k^\dagger M_k = I. \]

Non-Selective Generalized Measurement (GM): \[ \rho_{\text{out}} = \sum_k p_k \rho_k, \quad \text{k'th outcome:} \quad \rho_k = \frac{M_k \rho_{\text{in}} M_k^\dagger}{\text{Tr}(M_k^\dagger M_k \rho_{\text{in}})} \]

Probability: \[ p_k = \text{Tr}(M_k^\dagger M_k \rho_{\text{in}}) \]

Exact Solution: \[ \rho_S(t) = \sum_\alpha E_\alpha(t) \rho_S(0) E_\alpha^\dagger(t). \]

GM operators: \{E_\alpha\}

Lindblad (Quantum Jump):
\[ \tau \ll \|\mathcal{L}\|^{-1}, \quad \rho_S(t + \tau) \approx (I - \frac{T}{2} \sum_\alpha F_\alpha^\dagger F_\alpha) \rho_S(t) (I - \frac{T}{2} \sum_\alpha F_\alpha^\dagger F_\alpha) + \tau \sum_\alpha F_\alpha \rho_S(t) F_\alpha^\dagger. \]

GM operators: \{\sqrt{\tau} F_\alpha, I - \frac{T}{2} \sum_\beta F_\beta^\dagger F_\beta\}
Single-Shot Measurement Process

Exact Solution:

Single-Shot Measurement:

Markovian Approximation (Quantum Trajectories):

Measurement:  
Preparation:  

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Non-Markovian Master Equation

Probabilistic Procedure:

Probability of an extra measurement at time $t_1$: $w(t_1)$,

$$\rho_S(t = N\epsilon) = \sum_{m=1}^{N-1} w(m\epsilon)\Lambda(m\epsilon)\rho_S(t_1) = (N-m)\epsilon$$

Non-Markovian Master Equation (Newton Iteration Method):

$$\frac{\partial \rho_S}{\partial t} = \int_0^t dt'k(t')\Lambda(t')\dot{\Lambda}(t')\Lambda^{-1}(t')\rho_S(t-t')$$
Post-Markovian Master Equation:

\[ \Lambda(t) = e^{\mathcal{L}t} : \quad \begin{array}{cccc}
0 & \text{P} & \text{M} & \text{M} & \text{M} \\
& t_i & t_i + \tau & t - 2\tau & t - \tau & t
\end{array} \]

1) S.Daffer et al., quant-ph/309081.
2) S.M.Barnett et al., PRA, 2001, 64, 33808.

Markovian approximation can be recovered by choosing: \( k(t) = \delta(t) \),

The intuitively \(^{(1,2)}\) addressed memory function could also be retrieved in the limit of \( \|\mathcal{L}\| \ll T^{-1} \).
Example: Single Qubit Dephasing

Spin-Boson Hamiltonian:

\[ H_{SB} = \sum_k \sigma_z \otimes (\lambda_k b + \lambda_k^* b^\dagger) \]

\[ \rho(t) = \frac{1}{2} (I + f(t) \alpha_x \sigma_x + f(t) \alpha_y \sigma_y + \alpha_z \sigma_z) \]

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<th>Markovian Regime</th>
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<td>[ f(t) = \exp[-\sum_k</td>
<td>\lambda_k</td>
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Post-Markovian Equation Result:

Memory Function:

\[ k(t) = (1 - \theta) \frac{1}{\gamma} e^{-\gamma t} + \theta \delta(t) \]

\[ f(t) = (1 - \theta) e^{-(\gamma/2 + a)t} \cos(\sqrt{2a\gamma - (\gamma/2 + a)^2 t + \varphi}) + \theta e^{-at} \]
Quantum Dynamical Map

Laplace Transformation:

\[ s\tilde{\rho}_S(s) - \rho_S(0) = [\tilde{k}(s) \ast \frac{\mathcal{L}}{s - \mathcal{L}}] \tilde{\rho}_S(s) \]

Eigenvalue, right and left eigenoperators of the superoperator \( \mathcal{L} : \{\lambda_i, R_i, L_i\} \).

\[
\begin{align*}
\mathcal{L}\rho_i &= \lambda_i \rho_i \\
\rho_S(t) &= \sum_i \mu_i(t) R_i
\end{align*}
\]

\[
\begin{align*}
s\tilde{\mu}_i(s) - \mu_i(0) &= \lambda_i \tilde{k}(s - \lambda_i) \tilde{\mu}_i(s)
\end{align*}
\]

Dynamical Map:

\[
\Phi(t) : \rho \mapsto \sum_i \xi_i(t) \text{Tr}[L_i \rho] R_i , \quad \xi_i(t) = \text{Lap}^{-1}\left[\frac{1}{s - \lambda_i \tilde{k}(s - \lambda_i)}\right]
\]
Complete Positivity:

\[ \Phi((|i\rangle\langle j|))_{1 \leq i, j \leq n} \geq 0 \quad \Rightarrow \quad \sum_k \xi_k(t) L_k^T \otimes R_k \geq 0. \]

Experimental Determination of the Kernel Function:

Quantum state tomography result: \( \rho(t) \).

Kernel function:

\[ k(t) = \text{Lap}^{-1}[(s - 1/\text{Lap}[\xi_i(t)])] e^{-\lambda_i t} / \lambda_i \]

\[ \xi_i(t) = \text{Tr}[L_i \rho(t)] / \text{Tr}[L_i \rho(0)] \]
**Conclusion and Possible Extensions:**

We have introduced:

- Phenomenological picture of a non-Markovian master equation in the measurement theory.

- A post-Markovian master equation which can be analytically solved by applying the Laplace transform.

- A condition on the memory function to preserve the complete positivity of the corresponding dynamical map.

We like to present in the future:

- Improving the introduced non-Markovian equation by going to higher steps of Newton iteration method.

- Exploring the memory function for a set of performed experiment results.