Resource-Efficient Linear Optics Quantum Computation

via the Cluster State Approach

Dan Browne and Terry Rudolph

Imperial College London
Introduction

- *Photons* make excellent carriers of quantum information and *single qubit operations* can be achieved with linear optical elements – (polarising) beam splitters and phase shifters.

- *Two-qubit gates* cannot be achieved deterministically by linear optics alone.

- Need strong non-linear materials or implement *non-deterministic* gates via photo-detection.
Introduction

- *Photons* make excellent carriers of quantum information and *single qubit operations* can be achieved with linear optical elements – (polarising) beam splitters and phase shifters.

- *Two-qubit gates* cannot be achieved deterministically by linear optics alone.

- Need strong non-linear materials or implement *non-deterministic* gates via photo-detection.

- Schemes\(^\alpha\) for *scalable* (near)-deterministic gates are complicated and need a large amount of resources (entangled modes, feed-forward detection) to implement even the simplest gate.

- Here we describe a new scheme which employs the measurement-based *cluster state quantum computation* approach and achieves significant gains in resource efficiency.

Cluster States

A cluster state is an entangled multi-qubit state which may be represented by a graph.

- **Vertices** represent qubits prepared in state $|+\rangle = |0\rangle + |1\rangle$.
- **Edges** represent the application of the entangling quantum CPHASE gate $|0\rangle^a \langle 0|^b + |1\rangle^a \langle 1|\sigma_z^b$ between the connected qubits.
- We will refer to the graph edges as **bonds**.
- Known extra Pauli’s on any cluster qubit can be accounted for.

---

$^a$Briegel and Raussendorf, PRL, 86, 910 (2001)

$^b$Normalisation factors will be omitted.
Some Properties of Cluster States

- We review here a few properties of cluster states, which we shall use later on.

- A computational basis ($\sigma_z$) measurement on a cluster qubit removes the qubit from the cluster breaking all bonds.
Some Properties of Cluster States

- We review here a few properties of cluster states, which we shall use later on.

- A computational basis ($\sigma_z$) measurement on a cluster qubit removes the qubit from the cluster breaking all bonds.

- **Redundant encoding** is to encode a logical qubit in several qubits. The 2-qubit redundant encoding is $|0\rangle_{\text{log.}} \equiv |0\rangle|0\rangle$, $|1\rangle_{\text{log.}} \equiv |1\rangle|1\rangle$.

- A $\sigma_x$ measurement $\{(|0\rangle + |1\rangle), (|0\rangle - |1\rangle)\}$ on a qubit in a linear cluster combines the neighbouring qubits into a single logical qubit (redundantly encoded).
Cluster State Quantum Computation

On a cluster state with sufficient size and bond layout, an arbitrary quantum network can be simulated by *adaptive single-qubit measurements* alone\(^a\). The following cluster state layout\(^b\):

\[(2 \times 1)\]
rectangular grid pattern:

\[
\begin{align*}
|+\rangle & \quad \text{single qubit rotation:} \\
\text{CPHASE:} &
\end{align*}
\]

\(^{a}\) Raussendorf and Briegel, PRL 86, 5188; Raussendorf, Browne and Briegel, PRA 68, 022312

\(^{b}\) Nielsen, accepted PRL (2004). See also Childs, Leung and Nielsen, quant-ph/0404132.
Cluster State Quantum Computation

On a cluster state with sufficient size and bond layout, an arbitrary quantum network can be simulated by *adaptive single-qubit measurements* alone. The following cluster state layout

Measurements of single qubit observables e.g. $U_z(\gamma_1)\sigma_x U_z(-\gamma_1)$

with the above measurements, simulates the quantum network:

$a$ Raussendorf and Briegel, PRL 86, 5188; Raussendorf, Browne and Briegel, PRA 68, 022312

Outline of our scheme

- Qubits will be single photon polarisation $|0\rangle \equiv |H\rangle$, $|1\rangle \equiv |V\rangle$.
- Polarisation measurement in arbitrary bases is trivial. The main part of our scheme is the cluster state generation.
- Instead of using CPHASE gates between qubits, we (probabilistically) fuse clusters.$^a$

---

$^a$The same idea underlies the “valence bond model” of Verstraete and Cirac, quant-ph/0311130

$^b$Can be generated e.g. via non-linear processes or linear optics and feed-forward.
Outline of our scheme

• Qubits will be single photon polarisation $|0\rangle \equiv |H\rangle$, $|1\rangle \equiv |V\rangle$.
• Polarisation measurement in arbitrary bases is trivial. The main part of our scheme is the cluster state generation.
• Instead of using CPHASE gates between qubits, we (probabilistically) fuse clusters $^a$.
• Let us introduce fusion operator $|0\rangle\langle 00| + |1\rangle\langle 11|$.
• This replaces two qubits with a single one while retaining all cluster state bonds on each qubit.

$^a$The same idea underlies the “valence bond model” of Verstraete and Cirac, quant-ph/0311130

$^b$Can be generated e.g. via non-linear processes or linear optics and feed-forward.
Outline of our scheme

- Qubits will be single photon polarisation $|0\rangle \equiv |H\rangle$, $|1\rangle \equiv |V\rangle$.
- Polarisation measurement in arbitrary bases is trivial. The main part of our scheme is the cluster state generation.
- Instead of using CPHASE gates between qubits, we (probabilistically) fuse clusters.
- Let us introduce fusion operator $|0\rangle\langle00| + |1\rangle\langle11|$.
- This replaces two qubits with a single one while retaining all cluster state bonds on each qubit.
- The initial resource will be photon Bell-pairs $|H\rangle(|H\rangle + |V\rangle) + |V\rangle(|H\rangle - |V\rangle)$. These are 2-qubit cluster states.

\[\text{fusion}\]

---

\[a\] The same idea underlies the “valence bond model” of Verstraete and Cirac, quant-ph/0311130

\[b\] Can be generated e.g. via non-linear processes or linear optics and feed-forward.
The fusion operation

- The fusion can be seen explicitly if we write out each bond CPHASE:

\[
\left| 0 \right\rangle \prod_{i=1}^{m} 1^{(i)} \left| \psi \right\rangle + \left| 1 \right\rangle \prod_{i=1}^{m} \sigma_{z}^{(i)} \left| \psi \right\rangle \otimes \left( \left| 0 \right\rangle \prod_{i=1}^{n} 1^{(i)} \left| \psi' \right\rangle + \left| 1 \right\rangle \prod_{i=1}^{n} \sigma_{z}^{(i)} \left| \psi' \right\rangle \right)
\]

which, after the fusion operation \( |0\rangle\langle 00| + |1\rangle\langle 11| \), becomes

\[
\left| 0 \right\rangle \prod_{i=1}^{m} 1^{(i)} \left| \psi \right\rangle \prod_{i=1}^{n} 1^{(i)} \left| \psi' \right\rangle + \left| 1 \right\rangle \prod_{i=1}^{m} \sigma_{z}^{(i)} \left| \psi \right\rangle \prod_{i=1}^{n} \sigma_{z}^{(i)} \left| \psi' \right\rangle
\]
Polarising Beam Splitter - (PBS)

The key component for realising the fusion operation is the PBS.

Vertically polarised light is reflected

Horizontally polarised light is transmitted
Building Linear Clusters - Type-I Fusion

- The fusion operation can be realised non-deterministically using the illustrated setup:\(^a\)
- With a photon incident in each port, there are 4 possible outcomes, each with probability 25%.

Building Linear Clusters - Type-I Fusion

- The fusion operation can be realised non-deterministically using the illustrated setup:\(^a\)

- With a photon incident in each port, there are 4 possible outcomes, each with probability 25%.

- **Two** outcomes give us the desired fusion operators
  \[ |0\rangle\langle 00| + |1\rangle\langle 11| \quad \text{or} \quad |0\rangle\langle 00| - |1\rangle\langle 11| \], (one and only one photon, H or V). The second of these adds an extra \(\sigma_z\), but this is naturally accounted for. Thus, overall success probability is 50%.

---

The fusion operation can be realised non-deterministically using the illustrated setup:\(^a\)

With a photon incident in each port, there are 4 possible outcomes, each with probability 25%.

Two outcomes give us the desired fusion operators

\[ |0\rangle\langle 00| + |1\rangle\langle 11| \quad \text{or} \quad |0\rangle\langle 00| - |1\rangle\langle 11| \]

(one and only one photon, H or V). The second of these adds an extra \(\sigma_z\) but this is naturally accounted for. Thus, overall success probability is 50%.

If 0 or 2 photons are detected, this is a failure, equivalent to measuring both qubits in the \((\sigma_z)\) computational basis. The qubits are thus both cut from their respective clusters.

Linear Clusters

- Adding Bell Pairs to a linear cluster:
- On average, the length of the cluster does not increase.

\[ \begin{align*}
\text{Bell Pair} & \quad 50\% \quad \quad 50\%\\
\text{3 photon cluster} & \quad 2 \text{ qubits gained} \quad \quad 1 \text{ qubit lost} \\
\end{align*} \]

\[^a\text{Actually due to the “reflective boundary” at length 2, one reaches the desired length in quadratic steps.}\]
Linear Clusters

- Adding Bell Pairs to a linear cluster:

- On average, the length of the cluster does not increase. 

- The solution is to first combine Bell pairs into 3-photon clusters (requires on average 4 Bell pairs).

- On average, cluster length increases by 1/2 qubit. Therefore, to add one qubit to the cluster you need $2 \times 4 - 1 = 7$ Bell Pairs.

\[ ^a \text{Actually due to the “reflective boundary” at length 2, one reaches the desired length in quadratic steps.} \]
Linear Clusters

- Adding Bell Pairs to a linear cluster:

- On average, the length of the cluster does not increase.

- The solution is to first combine Bell pairs into 3-photon clusters (requires on average 4 Bell pairs).

- On average, cluster length increases by 1/2 qubit. Therefore, to add one qubit to the cluster you need $2 \times 4 - 1 = 7$ Bell Pairs.

- The best protocol we have found uses 5-photon clusters as building blocks. This gives a rate: 6.5 Bell pairs per added qubit.

\(^a\) Actually due to the “reflective boundary” at length 2, one reaches the desired length in quadratic steps.
Joining Clusters into the 2-D Pattern

- We now need to join these linear clusters into the desired 3-dimensional layout.

- With a deterministic fusion, this pattern of fusions produces the cluster state layout required.

- However, our Type-I fusion is only successful half the time. Failure is equivalent to $\sigma_z$ measurement, which would break up hard won existing bonds!
Joining Clusters - Type-II Fusion

- Recall that a $\sigma_x$ measurement on a cluster state does not cut the qubits bonds, but merges neighbouring qubits into a single redundantly encoded qubit.
- If we modify our fusion operation by introducing extra $45^\circ$ rotations to each qubit, the failure outcomes will be $\sigma_x$ measurements.
- However, the “success” projection is then no longer diagonal in the computational basis and does not perform the required fusion.
Joining Clusters - Type-II Fusion

- Recall that a $\sigma_x$ measurement on a cluster state does not cut the qubits bonds, but merges neighbouring qubits into a single redundantly encoded qubit.

- If we modify our fusion operation by introducing extra $45^\circ$ rotations to each qubit, the failure outcomes will be $\sigma_x$ measurements.

- However, the “success” projection is then no longer diagonal in the computational basis and does not perform the required fusion.

- We get round this by measuring both outputs.

- This leads to projections onto states $|++\rangle + |--\rangle = |00\rangle + |11\rangle$
  or $|++\rangle - |--\rangle = |01\rangle + |10\rangle$.

- If one of the qubits this is applied to is redundantly encoded this gives us the desired fusion! We call this a Type-II fusion.

- Again, the success probability of this step is 50%.
Joining Clusters - Type-II Fusion

- We now have a recipe to make the inter-cluster fusions.

Apply a $\sigma_x$ measurement to prepare redundantly encoded qubit
Joining Clusters - Type-II Fusion

- We now have a recipe to make the inter-cluster fusions.

  Apply a $\sigma_x$ measurement to prepare redundantly encoded qubit

  Apply Type-II fusion

  With 50% prob: Success!

  50% prob: failure, but a redundantly enc. qubit is ready for next attempt
Joining Clusters - Type-II Fusion

- We now have a recipe to make the inter-cluster fusions.

Apply a $\sigma_x$ measurement to prepare redundantly encoded qubit

Apply Type-II fusion

With 50% prob: Success!

50% prob: failure, but a redundantly encoded qubit is ready for next attempt

- Note that failures only “use up” cluster qubits to the right of the fusion.
- Thus, failures cannot propagate back through the cluster as in the Type-I fusion and other schemes.
Quantifying Resource Requirements

- In our cluster state measurement pattern, there are the same number of:
  - simulated 2-qubit gates:
    - $U_z(\alpha_1) \rightarrow U_x(\alpha_2)$
    - $U_z(\beta_1) \rightarrow U_x(\beta_2)$
  - T-shaped units in the cluster state:

This means the resources required to build the T-shape are a measure of the resources per two-qubit gate.
Quantifying Resource Requirements

• In our cluster state measurement pattern, there are the same number of:
  ○ simulated 2-qubit gates:
    - $U_z(\alpha_1)$ $U_x(\alpha_2)$
    - $U_z(\beta_1)$ $U_x(\beta_2)$
  ○ T-shaped units in the cluster state:

This means the resources required to build the T-shape are a measure of the resources per two-qubit gate.

• If we use the method above, the construction of a T-shape consumes on average 8 bonds from the linear clusters used.

• Thus, per general 2-qubit gate the resource requirements are:

$$8 \times 6.5 = 52 \text{ Bell Pairs.}$$
Other Schemes: Rough Comparison of Resources
Approx. entanglement resources required per (general) 2-qubit gate:

**Knill-Laflamme-Milburn** (KLM), Nature 401, 46 (2001) (for 92.5% gate success prob.)
\[ \sim 100\text{-photon “KLM state”} \]

**Yoran and Reznik**, PRL 91, 037903 (2003). (Measurement based “chain state” q.c., uses KLM gates)
\[ \sim 23 \text{ 12-photon “KLM states”} \]

\[ \sim 54 \text{ 8-photon “KLM states”} \]

**Our scheme**, quant-ph/0405157
\[ 52 \text{ 2-photon Bell states} \]

\(^a\)Note that the KLM resource states require a complicated linear optical network conditional on several / many measurements for their generation.
Summary

- We have presented a scheme for linear optics quantum computation based on the cluster state approach, that is very resource efficient compared to other schemes.
- For the shorter-term, the work provides a recipe for the generation of interesting new entangled states.
- The procedures at the heart of the scheme have already been implemented experimentally.
- The scheme has other advantages. For example, the absence of concatenated beam-splitters, unavoidable in other schemes, makes the mode-matching requirements much less strict.
Future Directions

- Cluster state layout can be optimised for specific algorithms – how much more gains in resource efficiency are possible?
- Can the general scheme be optimised to further reduce its experimental complexity?
- What about fault-tolerance?
Future Directions

- Cluster state layout can be optimised for specific algorithms – how much more gains in resource efficiency are possible?
- Can the general scheme be optimised to further reduce its experimental complexity?
- What about fault-tolerance?

This work can be found in pre-print quant-ph/0405157.

The authors would like to thank for helpful comments and support: Viv Kendon, Michael Nielsen, Jeremy O’Brien, Martin Plenio, Petra Scudo, Andrew White and “ceptimus” of the puzzles forum at randi.org.

This work was supported by the EPSRC, and Hewlett-Packard Ltd.