Generalised Fractional-Black-Scholes Equation: pricing and hedging

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Outline

- Lévy processes
- Fractional calculus
- Fractional-Black-Scholes
**Definition 1 Lévy process.** Let \( X(t) \) be a random variable dependent on time \( t \). Then the stochastic process \( X(t) \), for \( 0 < t < \infty \) and \( X(0) = 0 \), is a Lévy process iff it has independent and stationary increments.

**Theorem 1 Lévy-Khintchine representation.** Let \( X(t) \) be a Lévy process. Then the natural logarithm of the characteristic function can be written as

\[
\ln \mathbb{E}[e^{i\theta X(t)}] = ait\theta - \frac{1}{2}\sigma^2t\theta^2 + t \int \left( e^{i\theta x} - 1 - i\theta x \mathbf{1}_{|x|<1} \right) W(dx),
\]

where \( a \in \mathbb{R} \), \( \sigma \geq 0 \), \( \mathbf{1} \) is the indicator function and the Lévy measure \( W \) must satisfy

\[
\int_{\mathbb{R}/0} \min\{1, x^2\} W(dx) < \infty. \quad (1)
\]
Which Lévy process? Why?

- Brownian motion; Bachelier.
- $\alpha$-Stable or Lévy Stable; Mandelbrot.
- Jump Diffusion; Merton.
- GIG and Generalised Hyperbolic Distribution; Barndorff-Nielsen.
- Variance Gamma; Madan et al.
When specifying a particular Lévy process we are basically asking how do we want to specify the ‘behaviour’ of the jumps, in other words how is the Lévy density \( w(x) \) \( (\text{i.e} \ W(dx) = w(x)dx) \) chosen. For example

- CGMY, Carr et al.
- KoBol, Tempered Stable; Koponen.
- FMLS; Carr and Wu.
- others.
• Size and sign of jumps

• Frequency of jumps

• Existence of moments

• Simplicity
The CGMY process

A simple answer is then to consider a Lévy density of the form

\[ w_{CGMY}(x) = \begin{cases} 
C|x|^{-1-Y}e^{-G|x|} & \text{for } x < 0, \\
Cx^{-1-Y}e^{-Mx} & \text{for } x > 0,
\end{cases} \]

and the log of the characteristic function is given by

\[
\Psi_{CGMY}(\theta) = tC\Gamma(Y)\{(M - i\theta)^Y - M^Y + (G + i\theta)^Y - G^Y\}.
\]

Here \( C > 0, \ G \geq 0, \ M \geq 0 \) and \( Y < 2 \).
The Damped-Lévy process

\[
\begin{align*}
\mathcal{D}_L(x) = \begin{cases} 
C q |x|^{-1-\alpha} e^{-\lambda |x|} & \text{for } x < 0, \\
C p x^{-1-\alpha} e^{-\lambda x} & \text{for } x > 0,
\end{cases}
\end{align*}
\]

and the natural logarithm of the characteristic equation is given by

\[
\Psi_{DL}(\theta) = \kappa^\alpha \left\{ p(\lambda - i\theta)^\alpha + q(\lambda + i\theta)^\alpha - \lambda^\alpha - i\theta \alpha \lambda^{\alpha-1} (q - p) \right\},
\]

for \( 1 < \alpha \leq 2 \) and \( p + q = 1 \).
The Lévy-Stable process

Is a pure jump process with Lévy density

\[ w_{LS}(x) = \begin{cases} 
Cq|x|^{-1-\alpha} & \text{for } x < 0, \\
Cpx^{-1-\alpha} & \text{for } x > 0,
\end{cases} \]

Hence the log of the characteristic function is \( \Psi(\theta) = \)

\[ \begin{cases} 
-\kappa^\alpha|\theta|^\alpha \left\{ 1 - i\beta \ \text{sign}(\theta) \tan(\alpha\pi/2) \right\} & \text{for } \alpha \neq 1, \\
-\kappa|\theta| \left\{ 1 + \frac{2i\beta}{\pi} \ \text{sign}(\theta) \ln|\theta| \right\} & \text{for } \alpha = 1,
\end{cases} \]

here \( C > 0 \) is a scale constant, \( p \geq 0 \) and \( q \geq 0 \), with \( p + q = 1 \) and \( \beta = p - q \) is the skewness parameter.
Fractional Integrals

For an $n$-fold integral there is the well known formula

$$\int_a^x \int_a^x \cdots \int_a^x f(x)dx = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} f(t)dt.$$ 

Note that since $(n-1)! = \Gamma(n)$ the expression above may have a meaning for non-integer values of $n$.

**Definition 2 The Riemann-Liouville Fractional Integral.** The fractional integral of order $\alpha > 0$ of a function $f(x)$ is given by

$$D^{-\alpha}_{a+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-\xi)^{\alpha-1} f(\xi)d\xi,$$

and

$$D^{-\alpha}_{b-} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (\xi-x)^{\alpha-1} f(\xi)d\xi.$$
Definition 3 The Riemann-Liouville Fractional Derivative.

\[ D_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_{a}^{x} (x-\xi)^{n-\alpha-1} f(\xi) d\xi, \]

and

\[ D_{b-}^{\alpha} f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_{x}^{b} (\xi-x)^{n-\alpha-1} f(\xi) d\xi. \]

The Fourier Transform View

Note that if we let \( a = -\infty \) and \( b = \infty \) we have

\[ \mathcal{F}\{D_{+}^{\alpha} f(x)\} = (-i\xi)^{\alpha} \hat{f}(\xi) \]

and

\[ \mathcal{F}\{D_{-}^{\alpha} f(x)\} = (i\xi)^{\alpha} \hat{f}(\xi). \]
The Lévy-Stable Fractional-Black-Scholes. Under the physical measure the price process follows a geometric LS process

\[ d(\ln S) = \mu dt + \sigma dL_{LS}, \]

where \( L \sim S_\alpha(dt^{1/\alpha}, \beta, 0) \) with \( 1 < \alpha < 2, \ -1 \leq \beta \leq 1 \) and \( \sigma > 0 \).

And under the risk-neutral measure (McCulloch) it follows

\[ d(\ln S) = (r - \beta \sigma^\alpha \sec(\alpha\pi/2)) dt + dL_{LS} + dL_{DL} \]

where \( dL_{LS} \) and \( dL_{DL} \) are independent.

\[
\begin{align*}
rV &= \frac{\partial V(x, t)}{\partial t} + (r - \beta \sigma^\alpha \sec(\alpha\pi/2)) \frac{\partial V(x, t)}{\partial x} - \kappa_2^\alpha \sec(\alpha\pi/2) D_+^\alpha V(x, t) \\
&\quad + \kappa_1^\alpha \sec(\alpha\pi/2) \left( V(x, t) - e^x D_-^\alpha e^{-x} V(x, t) \right),
\end{align*}
\]

where

\[
\kappa_2^\alpha = \frac{1 - \beta}{2} \sigma^\alpha \quad \text{and} \quad \kappa_1^\alpha = \frac{1 + \beta}{2} \sigma^\alpha.
\]
Two cases: classical Black-Scholes and the fractional FMLS

Case $\alpha = 2$, Black-Scholes

$$rV(x, t) = \frac{\partial V(x, t)}{\partial t} + (r - \sigma^2) \frac{\partial V(x, t)}{\partial x} + \sigma^2 \frac{\partial^2 V(x, t)}{\partial x^2}.$$

Case $\alpha > 1$ and $\beta = -1$, FMLS

$$rV(x, t) = \frac{\partial V(x, t)}{\partial t} + (r + \sigma^\alpha \sec(\alpha \pi/2)) \frac{\partial V(x, t)}{\partial x}$$
$$-\sigma^\alpha \sec(\alpha \pi/2) D^\alpha_+ V(x, t).$$
Proposition 1 CGMY Fractional-Black-Scholes equation.

Let the risk-neutral log-stock price dynamics follow a CGMY process

\[ d(\ln S) = (r - w)dt + dL_{\text{CGMY}}. \]  \hspace{1cm} (2)

The value of a European-style option with final payoff \( \Pi(x, T) \) satisfies the following fractional differential equation

\[
\begin{align*}
    rV(x, t) &= \frac{\partial V(x, t)}{\partial t} + (r - w)\frac{\partial V(x, t)}{\partial x} \\
    &\quad\quad + \sigma (M^Y + G^Y)V(x, t) \\
    &\quad\quad + \sigma e^{Mx} D^{-}_- (e^{-Mx}V(x, t)) \\
    &\quad\quad + \sigma e^{-Gx} D^+_+ (e^{Gx}V(x, t)),
\end{align*}
\]

where \( \sigma = C\Gamma(-Y) \).
Proof

1

\[ V(x, t) = e^{-r(T-t)} E_t[\Pi(x_T, T)]. \]

2

\[ V(x, t) = \frac{e^{-r(T-t)}}{2\pi} E_t \left[ \int_{-\infty + i\nu}^{\infty + i\nu} e^{-i\xi x} \Pi(\xi) d\xi \right]. \]

3

\[ \hat{V}(\xi, t) = e^{-r(T-t)} e^{-i\xi \mu(T-t)} e^{(T-t)\psi(-\xi)} \hat{\Pi}(\xi). \]

4

\[ \frac{\partial \hat{V}(\xi, t)}{\partial t} = (r + i\xi \mu - \psi(-\xi)) \hat{V}(\xi, t) \]

with boundary condition \( \hat{V}(\xi, T) = \hat{\Pi}(\xi, T). \)
**Dynamic Hedging:** Delta hedging, Delta-Gamma hedging, Variance minimisation.

**The Taylor Expansion View**

\[
dV = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} dS^2 + \ldots.
\]

Portfolio \( P(S, t) = V_1(S, t) - \Delta S - bV_2(S, t) \)

\[
\Delta = \frac{\partial V_1(S, t)}{\partial S} - \frac{\partial V_2(S, t)}{\partial S} b,
\]

\[
b = \frac{\partial^2 V_1(S, t)/\partial S^2}{\partial^2 V_2(S, t)/\partial S^2}.
\]
FMLS Daily Delta hedge with $\alpha=1.5$, $S=K=100$, $T=1\text{ month}$
FMLS Min Variance with $\alpha=1.5$, $S=K=100$, $T=1\text{month}$

Profit and Loss

Frequency
FMLS Delta and Gamma Hedging with $\alpha=1.5$, $S=K=100$, $T=1$ month

Profit and Loss (£)

Frequency

Min = -35.33
Max = 10.6
std = 0.70
mean = -0.0006
FMLS Black-Scholes: the Taylor expansion view

$$dV(x,t) = \frac{\partial V(x,t)}{\partial t} dt + \frac{\partial V(x,t)}{\partial x} dx + \frac{1}{\Gamma(2-\alpha)} D^\alpha V(x,t) (dx)^\alpha + \cdots.$$  

(Samko et al 1993).

Therefore it seems natural, in the FMLS case, to delta and fractional-gamma hedge the portfolio \( P(x,t) = V_1(x,t) - \Delta e^x - bV_2(x,t) \), hence

\[
a = \frac{\partial V_1(x,t)}{\partial x} \frac{1}{e^x} - \frac{\partial V_2(x,t)}{\partial x} \frac{1}{e^x} b
\]

and

\[
b = \frac{e^x D^\alpha + V_1(x,t) - \partial V_1(x,t)/\partial x D^\alpha + e^x}{e^x D^\alpha + V_2(x,t) - \partial V_2(x,t)/\partial x D^\alpha + e^x}.
\]
FMLS Delta and Fractional Hedging with $\alpha=1.5$, $S=K=100$, $T=1$ month

Frequency

Min = -1.2
Max = 8.4
std = 0.17
mean = 0.001
In General might want to do...

\[
    rV(x, t) = \frac{\partial V(x, t)}{\partial t} + \mu \frac{\partial V(x, t)}{\partial x} + \mathcal{G}V(x, t), \quad (3)
\]

where \( \mathcal{G} \) is an operator containing the fractional derivatives.

\[
P(x, t) = V_1(s, t; T_1) - ae^{x} - bV_2(s, t; T_2) \quad (4)
\]

Therefore we require

\[
a = \frac{\partial V_1(x, t)}{\partial x} \frac{1}{e^{x}} - \frac{\partial V_2(x, t)}{\partial x} \frac{1}{e^{x}} b
\]

and

\[
b = \frac{e^{x} \mathcal{G}V_1(x, t) - \partial V_1(x, t)/\partial x \mathcal{G}e^{x}}{e^{x} \mathcal{G}V_2(x, t) - \partial V_2(x, t)/\partial x \mathcal{G}e^{x}}
\]

so the portfolio is both Delta and Fractional-Gamma-neutral, ie

\[
\frac{\partial P(x, t)}{\partial x} = 0 \quad \text{and} \quad \mathcal{G}P(x, t) = 0.
\]
LS Delta Hedging, $\alpha=1.7$, $S=K=100$, $T=1$ month

Min = $-408$
Max = 5.2764
Mean = $-0.055$
LS Delta and Gamma Hedging, $\alpha=1.7$, $S=K=100$, $T=1$ month

- Min = −56
- Max = 2.89
- Mean = 0.0024
Levy–Stable Delta and Fractional Hedging, $\alpha=1.7$, $S=K=100$, $T=1$ month

Min=$-20.64$
Max=$2.29$
Mean=$-0.0001$
CONCLUSIONS and FURTHER WORK

For Lévy processes with Lévy densities that have a polynomial singularity at the origin and exponential decay at the tails we can recast the pricing equation in terms of Fractional derivatives.

The non-local property of the fractional operators can be useful when dynamically hedging options.

Using well established numerical schemes for Fractional operators it might be possible to price American options. Moreover, for these processes we can derive Fractional Fokker-Planck equations that may also be used in the pricing of American options.