The Valuation of Basket Credit Derivatives: 
A Copula Function Approach

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Outline

- The Construction of Credit Curves
- On Default Correlation: The Joy of Copula Functions
- The Valuation of Credit Default Swaps
- The Valuation of Basket Credit Derivatives
  - First default / first loss
  - CBOs/CLOs
Credit Markets Are Being Transformed

- Shrinking Loan Profit Margin
- Low interest rate environment
- Huge amount of investment money
- Changing regulatory environment
- Theoretical and analytical advancements
- Technology
## Credit Derivative Products

<table>
<thead>
<tr>
<th>Structures</th>
<th>Underlying Assets</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Total return swap</td>
<td>• Corporate loans</td>
</tr>
<tr>
<td>• Default contingent forward</td>
<td>• Corporate bonds</td>
</tr>
<tr>
<td></td>
<td>• Sovereign bonds/loans</td>
</tr>
<tr>
<td>• Credit swap</td>
<td>• Specified loans or bonds</td>
</tr>
<tr>
<td>• Credit linked note</td>
<td>• Portfolio of loans or bonds</td>
</tr>
<tr>
<td>• Spread forward</td>
<td></td>
</tr>
<tr>
<td>• Spread option</td>
<td></td>
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</tbody>
</table>
Credit Swap Pricing: Illustration

Reference Credit: Company X
Swap Tenor: 3 Years
Event Payment: Par - Post Default Market Value
Bond Insurance v.s. Credit Default Swaps

<table>
<thead>
<tr>
<th>Bond insurance</th>
<th>Credit Default Swaps</th>
</tr>
</thead>
<tbody>
<tr>
<td>● Player</td>
<td>Banking</td>
</tr>
<tr>
<td>Insurance company</td>
<td>long and short</td>
</tr>
<tr>
<td>● which side of credit risk</td>
<td>long</td>
</tr>
<tr>
<td>long</td>
<td>How to price</td>
</tr>
<tr>
<td>● How to price</td>
<td>Relative pricing based</td>
</tr>
<tr>
<td>Actuarial approach based</td>
<td>asset swap spread</td>
</tr>
<tr>
<td>on historical data</td>
<td></td>
</tr>
</tbody>
</table>
Default Probabilities

- **From Historical Data**
  - Moody’s and S&P publish historical data

- **From Merton’s Option Framework**
  - data
  - method to address term structure of default rates

- **From Market Observed Credit Spread or Asset Spread**
Asset Swaps of Bonds

- **Bank ABC**
  - Bank receives fund
  - Need to pay to swap counterparty if the bond issuer defaults
  - Receive S

- **Swap Counterparty**
  - Pay all cash flows promised by the bond
  - Pay amount M to buy the bond
  - Pay coupons and Principle

- **Bond Holder**
  - Pay coupons and Principle

- **Funding Source**
  - Bank pay a funding cost of \( L \)
  - Bank receives fund
  - receive \( L + S \)
Credit Swap Pricing:

A *credit curve* gives instantaneous default probabilities of a credit at any time in the future conditional on the survival at that time

- Construct a discount curve, such as LIBOR
- Construct a credit curve for the reference credit
- Construct a credit curve for the counterparty
- Calculate the NSP of the protection
- Amortize the NSP into a number of years
The Characterization of Default

- Define a random variable called the *time-until-default* to denote the survival time \( \Pr[T < t] = F(t) \)

- Use survival function or hazard rate function to describe this survival time

\[
S(t) = 1 - F(t) \\
h(t) = \frac{f(t)}{1 - F(t)} = -\frac{S'(t)}{S(t)} \\
- \int_{0}^{t} h(s) ds \\
S(t) = e^{-\int_{0}^{t} h(s) ds} \\
t q_x = \Pr[T - t \leq t | T > x] \\
t p_x = 1 - t q_x
\]
Constructing a Credit Curve

- Valuation of Risky Bond -- Duffie and Singleton Approach
- Default Treatment: Recover a fixed % $R$ of the value just before default
- One period

\[ V = \left[ p + (1 - p) R \right] e^{-r \Delta t} \]

\[ - \int_{0}^{\Delta t} \left[ r(s) + (1 - R(s)) h(s) \right] ds \]

\[ \cong e \]
Multiperiod

General Case

\[ V(t_0) = \sum_{i}^{n} C_i \cdot e^{-\int_{t_0}^{t_i} [r(s) + (1 - R(s)) h(s)] \, ds} \]
## Asset Swap Spreads

<table>
<thead>
<tr>
<th>Maturity Year</th>
<th>LIBOR Yield</th>
<th>Asset Swap Spread</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5.89%</td>
<td>200 bp</td>
</tr>
<tr>
<td>2</td>
<td>6.13%</td>
<td>200</td>
</tr>
<tr>
<td>3</td>
<td>6.30%</td>
<td>200</td>
</tr>
<tr>
<td>4</td>
<td>6.40%</td>
<td>200</td>
</tr>
<tr>
<td>5</td>
<td>6.48%</td>
<td>200</td>
</tr>
<tr>
<td>7</td>
<td>6.62%</td>
<td>200</td>
</tr>
<tr>
<td>10</td>
<td>6.78%</td>
<td>200</td>
</tr>
</tbody>
</table>
### An Example

<table>
<thead>
<tr>
<th>Maturity</th>
<th>Coupon</th>
<th>Spread</th>
<th>Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 year</td>
<td>7.89%</td>
<td>200</td>
<td>100.00</td>
</tr>
<tr>
<td>2 year</td>
<td>8.13%</td>
<td>200</td>
<td>100.00</td>
</tr>
<tr>
<td>3 year</td>
<td>8.30%</td>
<td>200</td>
<td>100.00</td>
</tr>
<tr>
<td>4 year</td>
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<td>8.78%</td>
<td>200</td>
<td>100.00</td>
</tr>
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</table>

**Credit Curve B: Instantaneous Default Probability**

(Spread = 300 bp, Recovery Rate = 50%)
Default Correlation

- What is the default correlation?
- Traditional Correlation defined in the current finance literature

\[ \text{Corr}(A,B) = \frac{\Pr [A \cap B] - P[A] \cdot P[B]}{\sqrt{P(A)[1 - P(A)]P(B)[1 - P(B)]}} \]
One year is an arbitrary choice, useful information about the term structure of default rates could be lost.

Default correlation is a time dependent variable.

Need correlation over a number of years instead of only one year.

Estimation of default correlation has its problem.
Default Correlation: The Joy of Copulas

- We first know the marginal distribution of survival time for each credit.
- We need to construct a joint distribution with given marginals and a correlation structure.
- Copula function in multivariate statistics can be used.
- The correlation parameters used in copula function can be interpreted as the asset correlation between two credits used in CreditMetrics.
What is a Copula Function?

- Function that join or couple multivariate distribution functions to their one-dimensional marginal distribution functions

- For m uniform r. v., U1, U2, ...., Um

\[ C(u_1,u_2,\ldots,u_m) = \Pr[U_1 \leq u_1, U_2 \leq u_2, \ldots, U_m \leq u_m] \]

- Suppose we have m marginal distributions with distribution function \( F_i(x_i) \)

- Then the following defines a multivariate distribution function

\[ F(x_1, x_2, \ldots, x_m) = C(F_1(x_1), F_2(x_2), \ldots, F_m(x_m)) \]
A Few Copula Functions

• Normal Copula Function

\[ C(u,v) = \Phi_2(\Phi^{-1}(u), \Phi^{-1}(v), \rho) \]

• Frank Copula Function

\[ C(u,v) = \frac{1}{\alpha} \ln \left[ 1 + \frac{(e^{\alpha u} - 1)(e^{\alpha v} - 1)}{e^\alpha - 1} \right] \]

• Mixture Copula Function

\[ C(u,v) = (1 - \rho)uv + \rho \min(u,v) \]
Credit Swap Pricing:

- Calculate the PV of Payment
  - \(100 - Q(t_i)\) if bond issuer defaults, but the seller does not
  - \([100 - Q(t_i)]R_c\) if both the bond issuer and the default protection seller defaults

\[
\sum_{i=1}^{n} \left( \left[100 - Q(t_i)\right] \Pr[t_{i-1} < \tau_B \leq t_i, \tau_c > t_i] + R_C \left[100 - Q(t_i)\right] \Pr[t_{i-1} < \tau_B \leq t_i, \tau_c \leq t_i] \right) \cdot D(t_i)
\]

- Calculate the PV of Premium

\[
X \sum_{i=0}^{n-1} \Pr[\tau_B > t_i, \tau_C > t_i] \cdot D(t_i)
\]

- The Periodic or Level Premium \(X\) can be solved by equating the above two equations
Numerical Examples of Default Swap Pricing

Default Correlation vs Credit Swap Value

Credit Swap Value vs Default Correlation

-1.50 -1.00 -0.50 0.0 0.50 1.00 1.50

1.00 2.00 3.00 4.00 5.00 6.00
How do we simulate the default time?

- Map obligors to countries and industries
- Calculate asset correlation based on the historical data of equity indices, use CreditManager
- Simulate $y_1, y_2, \ldots, y_n$ from a multivariate normal distribution with the asset correlation matrix
- Transform the equity return to survival time by

$$T_i = F_i^{-1} \left( \Phi \left( Y_i \right) \right)$$
Summary of the Simulation

- Use CreditMetrics Approach to Default Correlation
- Simulate correlated multivariate normal distribution with the asset correlation
- Translate the multivariate normal random variable into survival times by using marginal term structure of default rates

Details: CreditMetrics Monitor, May 1999
The Valuation of the First-to-Default

- An Example: The contract pays $1 if the first default of 5-credit portfolio occurs during the first 2 years.
- We use the above approach to construct a credit curve for each credit.
- Using asset correlation and normal copula function we can construct a joint distribution of survival times.
- Then we can simulate the survival times for all 5 credits.
An Numerical Example

Input Parameter
hazard rate = 0.1,
Interest rate = 0.1
Asset Correlation = 0.25
The Impact of Asset Correlation

The Price of the First-to-Default v.s. Asset Correlation

- 5-asset
- 20-asset
CBO/CLO Models: Extraction of Cash Flows from Simulation

For a defaultable bond we can project the cash flow if we know when default occurs.

Actual Cash Flows

Promised Cash Flows

Default
Cash Flow Distribution

- **Interest Proceeds**
  - Pass OC and IC test - payment each tranche consecutively
  - Fail OC and IC test - Retire principal

- **Principal Proceeds**
  - Pass OC and IC test
    - During the reinvestment period - buy additional high yield
    - After the reinvestment period - retire principal from the top to bottom
  - Fail OC and IC test
    - During the reinvestment period - flow through each tranche until tests are passed, remaining one is used to buy additional collateral assets
    - After the reinvestment period - flow through each
Flow Chart of Cash Flow Distribution

Collateral at start of period

Credit Metrics

Collateral at end of period

Cashflow model

Reduced collateral

Principal pre-payments

Triggers

End?

No

Yes

Cashflow model

Principal payments

Interest payments
Simple cashflow CBO

- Collateral pool -- total value of $100M
  - 80 identical assets, face value of $1.25M
  - one year maturity
  - annual coupon of L+180bp
  - in default, recover 40% of face value

- Securitization
  - Senior tranche -- $90M of one year notes paying L+80bp
  - Equity -- $10M held as loss reserve

What is the probability that the Senior notes pay their coupon?
What is the return for the equity investors?
How to characterize risk in general?
Collateral guidelines and ratio tests

- **Overcollateralization**
  - ratio of performing collateral to par value of Senior notes
  - here, $\frac{100}{90} = 1.11$

- **Interest coverage**
  - ratio of collateral interest to interest on Senior notes
  - here, $\frac{100(L+180bp)}{90(L+80bp)} = 1.29$ (assume $L=5.5\%$)

- **Other guidelines on average rating, maturity, diversification**

- **Typically, minimum ratio levels must be maintained throughout the life of the structure**

  *Ratios characterize risk generally, but for more information, we must look at default scenarios.*
The best scenario -- no defaults

- Assume LIBOR is 5.5%

- **Interest**
  - receive $80\times$1.25M*(L+180bp)=$7.30M
  - pay to Senior $90M*(L+80bp)=$5.67M
  - pay remainder ($1.63M) to Equity

- **Principal**
  - receive $80\times$1.25M=$100M
  - pay $90M to Senior notes, $10M to Equity

- **Yield**
  - Senior receives the contracted L+80bp
  - Equity appreciates by 16.3%, or L+1080bp
A moderate scenario -- two defaults

- **Interest**
  - receive $78 \times $1.25M \times (L+180\text{bp}) = $7.12M
  - pay to Senior $90M \times (L+80\text{bp}) = $5.67M
  - pay remainder ($1.45M) to Equity

- **Principal**
  - receive $78 \times $1.25M + 2 \times 40\% \times $1.25M = $98.5M
  - pay $90M to Senior notes, $8.5M to Equity

- **Yield**
  - Senior receives the contracted L+80bp
  - Equity depreciates by 0.5%
At fifteen defaults, Senior investors get hit

- **Receive**
  - Interest -- $65 \times 1.25M \times (L + 180bp) = $5.93M
  - Principal -- $65 \times 2M + 15 \times 40\% \times 2M = $88.75M

- **Pay**
  - All receipts ($94.68M) to Senior
  - Equity receives nothing

- **Yield**
  - Senior only appreciates 5.2\%, or L-30bp
  - Equity is worthless
Use CreditMetrics to evaluate the likelihood of each scenario

- **Individual default probabilities**
  - 1.2% for each asset, consistent with Ba rating

- **Correlations**
  - Assume a homogeneous portfolio; all pairs are the same
  - What level of correlation?

<table>
<thead>
<tr>
<th>Asset corr.</th>
<th>Low</th>
<th>Med</th>
<th>Hi</th>
</tr>
</thead>
<tbody>
<tr>
<td>Joint def. prob</td>
<td>1.4bp</td>
<td>4.6bp</td>
<td>16.7bp</td>
</tr>
</tbody>
</table>

- **Simulation gives probabilities for scenarios**

<table>
<thead>
<tr>
<th># defaults</th>
<th>0</th>
<th>1</th>
<th>...</th>
<th>14</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>probability</td>
<td>57.0%</td>
<td>21.6%</td>
<td>...</td>
<td>3.8bp</td>
<td>2.6bp</td>
</tr>
<tr>
<td>cum prob</td>
<td>57.0%</td>
<td>78.6%</td>
<td>...</td>
<td>99.93%</td>
<td>99.95%</td>
</tr>
</tbody>
</table>
Putting the probabilities together with cashflows gives risk and return information

- **Senior**
  - probability that L+80bp is not paid -- 7.2bp
  - conditional probability that some principal is not repaid, given that some interest is missed -- 4.5%

- **Equity**
  - mean return -- L+280bp
  - standard deviation -- 14.0%
  - probability of positive (L+1080bp or L+239bp) return -- 78.6%
  - probability of losing more than 50% -- 78bp

Not very meaningful!
Can now examine losses under stressed default rates

- **Senior notes**

- **Equity**

![Graphs showing default probabilities and conditional probabilities of missed interest and lost principal for different interest rates and bond spreads.](image-url)