Nonlinear Price Impact and Portfolio Choice

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Fields Institute, Toronto, January 28\textsuperscript{th}, 2015
• Motivation:
   Optimal Rebalancing and Execution.

• Model:
   Nonlinear Price Impact.
   Constant investment opportunities and risk aversion.

• Results:
   Optimal policy and welfare. Implications.
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Price Impact and Market Frictions

- Classical theory: no price impact. Same price for any quantity bought or sold. Merton (1969) and many others.
- Bid-ask spread: constant (proportional) “impact”. Price depends only on sign of trade. Constantinides (1985), Davis and Norman (1990), and extensions.
- Price linear in trading rate. Asymmetric information equilibria (Kyle, 1985), (Back, 1992). Quadratic transaction costs (Garleanu and Pedersen, 2013)
- Literature on nonlinear impact focuses on optimal execution. Portfolio choice?
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Portfolio Choice with Frictions

- With constant investment opportunities and constant relative risk aversion:
  - Classical theory: hold portfolio weights constant at Merton target.
  - Proportional bid-ask spreads: hold portfolio weight within buy and sell boundaries (no-trade region).
  - Linear impact: trading rate proportional to distance from target.
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- **Inputs**
  - Price exogenous. Geometric Brownian Motion.
  - Constant relative risk aversion and long horizon.
  - Nonlinear price impact: trading rate one-percent higher means impact $\alpha$-percent higher.

- **Outputs**
  - Optimal trading policy and welfare.
  - High liquidity asymptotics.
  - Linear impact and bid-ask spreads as extreme cases.

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Market

- Brownian Motion \((W_t)_{t \geq 0}\) with natural filtration \((\mathcal{F}_t)_{t \geq 0}\).
- *Best quoted* price of risky asset. Price for an infinitesimal trade.

\[
\frac{dS_t}{S_t} = \mu dt + \sigma dW_t
\]

- Trade \(\Delta \theta\) shares over time interval \(\Delta t\). Order filled at price

\[
\tilde{S}_t(\Delta \theta) := S_t \left(1 + \lambda \left| \frac{\Delta \theta t S_t}{\Delta t X_t} \right|^\alpha \sgn(\dot{\theta}) \right)
\]

where \(X_t\) is investor's wealth.

- \(\lambda\) measures illiquidity. \(1/\lambda\) market depth. Like Kyle's (1985) lambda.
- Price worse for larger quantity \(\left| \Delta \theta \right|\) or shorter execution time \(\Delta t\).
  Price linear in quantity, inversely proportional to execution time.
- Impact of dollar trade \(S_t \Delta \theta\) declines as large investor's wealth increases.
- Makes model scale-invariant.
  Doubling wealth, and all subsequent trades, doubles final payoff exactly.
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Alternatives?

- **Alternatives**: quantities $\Delta \theta$, or share turnover $\Delta \theta/\theta$. Consequences?

  - **Quantities ($\Delta \theta$):**

    $$\tilde{S}_t(\Delta \theta) := S_t + \frac{\Delta \theta}{\Delta t}$$

  - Price impact independent of price. Not invariant to stock splits!
  - Suitable for short horizons (liquidation) or mean-variance criteria.
  - **Share turnover:**
    Stationary measure of trading volume (Lo and Wang, 2000). Observable.

    $$\tilde{S}_t(\Delta \theta) := S_t \left( 1 + \frac{\Delta \theta}{\theta_t \Delta t} \right)$$

  - Problematic. Infinite price impact with cash position.
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Wealth and Portfolio

- Continuous time: cash position

\[ dC_t = -S_t \left(1 + \lambda \left| \frac{\dot{\theta} S_t}{X_t} \right|^\alpha \text{sgn}(\dot{\theta}) \right) d\theta_t = - \left( \frac{S_t \dot{\theta}_t}{X_t} + \lambda \left| \frac{\dot{\theta}_t S_t}{X_t} \right|^{1+\alpha} \right) X_t dt \]

- Trading volume as wealth turnover \( u_t := \frac{\dot{\theta}_t S_t}{X_t} \).
  Amount traded in unit of time, as fraction of wealth.

- Dynamics for wealth \( X_t := \theta_t S_t + C_t \) and risky portfolio weight \( Y_t := \frac{\theta_t S_t}{X_t} \).

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- Illiquidity...

- ...reduces portfolio return \(-\lambda u_t^{1+\alpha}\).
  Turnover effect quadratic: quantities times price impact.

- ...increases risky weight \(\lambda Y_t u_t^{1+\alpha}\).
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  \[ dC_t = -S_t \left( 1 + \lambda \left| \frac{\dot{\theta} S_t}{X_t} \right|^\alpha \text{sgn}(\dot{\theta}) \right) d\theta_t = - \left( \frac{S_t \dot{\theta}_t}{X_t} + \lambda \left| \frac{\dot{\theta} S_t}{X_t} \right|^{1+\alpha} \right) X_t dt \]

- Trading volume as wealth turnover \( u_t := \frac{\dot{\theta} S_t}{X_t} \). Amount traded in unit of time, as fraction of wealth.

- Dynamics for wealth \( X_t := \theta_t S_t + C_t \) and risky portfolio weight \( Y_t := \frac{\theta_t S_t}{X_t} \)

\[
\frac{dX_t}{X_t} = Y_t(\mu dt + \sigma dW_t) - \lambda |u_t|^{1+\alpha} dt \\

\frac{dY_t}{Y_t} = (Y_t(1 - Y_t)(\mu - Y_t \sigma^2) + (u_t + \lambda Y_t |u_t|^{1+\alpha}))dt + \sigma Y_t(1 - Y_t)dW_t
\]

- Illiquidity...

- ...reduces portfolio return \((-\lambda u_t^{1+\alpha})\). Turnover effect quadratic: quantities times price impact.

- ...increases risky weight \((\lambda Y_t u_t^{1+\alpha})\). Buy: pay more cash. Sell: get less. Turnover effect linear in risky weight \(Y_t\). Vanishes for cash position.
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Admissible Strategies

Definition

Admissible strategy: process \((u_t)_{t \geq 0}\), adapted to \(\mathcal{F}_t\), such that system

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has unique solution satisfying \(X_t \geq 0\) a.s. for all \(t \geq 0\).

- Contrast to models without frictions or with transaction costs:
  control variable is not risky weight \(Y_t\), but its “rate of change” \(u_t\).

- Portfolio weight \(Y_t\) is now a *state variable*.

- Illiquid vs. perfectly liquid market.
  Steering a ship vs. driving a race car.

- Frictionless solution \(Y_t = \frac{\mu}{\gamma \sigma^2}\) unfeasible. A still ship in stormy sea.
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Objective

- Investor with relative risk aversion $\gamma$.
- Maximize equivalent safe rate, i.e., power utility over long horizon:
  \[
  \max_u \lim_{T \to \infty} \frac{1}{T} \log E \left[ X_T^{1-\gamma} \right]^{\frac{1}{1-\gamma}}
  \]
- Tradeoff between speed and impact.
- Optimal policy and welfare.
- Implied trading volume.
- Dependence on parameters.
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If $\frac{\mu}{\gamma\sigma^2} \in (0, 1)$, then the optimal wealth turnover and equivalent safe rate are:

$$\hat{u}(y) = \left| \frac{q(y)}{(\alpha+1)\lambda(1-yq(y))} \right|^{1/\alpha} \text{sgn}(q(y))$$

$$\text{EsR}_\gamma(\hat{u}) = \beta$$

where $\beta \in (0, \frac{\mu^2}{2\gamma\sigma^2})$ and $q : [0, 1] \mapsto \mathbb{R}$ are the unique pair that solves the ODE

$$-\hat{\beta} + \mu y - \gamma \frac{\sigma^2}{2} y^2 + y(1-y)(\mu - \gamma\sigma^2 y)q$$

$$+ \frac{\alpha}{(\alpha + 1)^{1+1/\alpha}} \frac{|q|^{\frac{\alpha+1}{\alpha}}}{(1-yq)^{1/\alpha}} \lambda^{-1/\alpha} + \frac{\sigma^2}{2} y^2 (1-y)^2 (q' + (1-\gamma)q^2) = 0$$

$q(0) = \lambda \frac{1}{(\alpha+1)^{1+1/\alpha}} (\alpha + 1)^{\frac{1}{\alpha+1}} \left( \frac{\alpha+1}{\alpha} \hat{\beta} \right)^{\frac{\alpha}{\alpha+1}}$, $\frac{\alpha}{(\alpha+1)^{1+1/\alpha}} \frac{|q(1)|^{\frac{\alpha+1}{\alpha}}}{(1-q(1))^{1/\alpha}} \lambda^{-1/\alpha} = \hat{\beta} - \mu + \gamma \frac{\sigma^2}{2}$

- License to solve an ODE of Abel type. Function $q$ and scalar $\beta$ not explicit.
- Asymptotic expansion for $\lambda$ near zero?
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### Theorem

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- Asymptotic expansion for $\lambda$ near zero?
Trading Rate \((\mu = 8\%, \sigma = 16\%, \lambda = 0.1\%, \gamma = 5)\)

Trading rate (vertical) against current risky weight (horizontal) for \(\alpha = 1/3\) (red) and \(\alpha = 2/3\) (blue). Dashed lines are no-trade boundaries \((\alpha = 0)\).
**Asymptotics**

**Theorem**

$c_\alpha$ and $s_\alpha$ unique pair that solves

$$s'(z) = z^2 - c - \alpha(\alpha + 1)^{-\left(1+1/\alpha\right)}|s(z)|^{1+1/\alpha}$$

$$\lim_{z \to \pm \infty} \frac{|s_\alpha(z)|}{|z|^{\frac{2\alpha}{\alpha+1}}} = (\alpha + 1)^{\alpha - \frac{\alpha}{\alpha+1}}$$

Set $l_\alpha := \left[\left(\frac{\sigma^2}{2}\right)^3 \gamma \bar{Y}^4 (1 - \bar{Y})^4\right]^{\frac{\alpha+1}{\alpha+3}}$, $A_\alpha = \left(\frac{2l_\alpha}{\gamma \sigma^2}\right)^{1/2}$, $B_\alpha = l_\alpha^{-\frac{\alpha}{\alpha+1}}$.

Asymptotic optimal strategy and welfare:

$$\hat{u}(y) = -\left|\frac{s_\alpha\left(\lambda^{-\frac{1}{\alpha+3}}(y - \bar{Y})/A_\alpha\right)}{B_\alpha(\alpha + 1)}\right|^{1/\alpha} \text{sgn}(y - \bar{Y})$$

$$\text{EsR}_\gamma(\hat{u}) = \frac{\mu^2}{2\gamma \sigma^2} - c_\alpha l_\alpha \lambda^{\frac{2}{\alpha+3}} + o(\lambda^{\frac{2}{\alpha+3}})$$

- Implications?
Asymptotics

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Set \( l_{\alpha} := \left[ \left( \frac{\sigma^2}{2} \right)^3 \gamma \bar{Y}^4 (1 - \bar{Y})^4 \right]^{\frac{\alpha + 1}{\alpha + 3}} \), \( A_{\alpha} = \left( \frac{2l_{\alpha}}{\gamma \sigma^2} \right)^{1/2} \), \( B_{\alpha} = l_{\alpha}^{-\frac{\alpha}{\alpha + 1}} \).

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\[ \text{EsR}_{\gamma}(\hat{u}) = \frac{\mu^2}{2\gamma \sigma^2} - c_{\alpha}l_{\alpha}\lambda^{2/\alpha+3} + o\left(\lambda^{2/\alpha+3}\right) \]

• Implications?
Trading Policy

- Trade towards $\bar{Y}$. Buy for $y < \bar{Y}$, sell for $y > \bar{Y}$.
- Trade faster if market deeper. Higher volume in more liquid markets.
- Trade slower than with linear impact near target. Faster away from target. With linear impact trading rate proportional to displacement $|y - \bar{Y}|$.
- As $\alpha \downarrow 0$, trading rate:
  vanishes inside no-trade region
  explodes to $\pm \infty$ outside region.
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Long-term weight ($\mu = 8\%, \, \sigma = 16\%, \, \gamma = 5$)

Density (vertical) of the long-term density of rescaled risky weight $Z^0$ (horizontal) for $\alpha = 1/3$ (red) and $\alpha = 2/3$ (blue). Dashed line is uniform density ($\alpha \to 0$).
Universal Constant $c_\alpha$

$c_\alpha$ (vertical) against $\alpha$ (horizontal).
Linear Impact ($\alpha = 1$)

- Solution to

$$s'(z) = z^2 - c - \alpha(\alpha + 1)^{-\frac{1}{\alpha}}|s(z)|^{1+1/\alpha}$$

is $c_1 = 2$ and $s_1(z) = -2z$.

- Optimal policy and welfare:

$$\hat{u}(y) = \sigma \sqrt{\frac{\gamma}{2\lambda}} (\bar{Y} - y) + O(1)$$

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Transaction Costs ($\alpha \downarrow 0$)

- Solution to

$$s'(z) = z^2 - c - \alpha (\alpha + 1)^{-1+1/\alpha} |s(z)|^{1+1/\alpha}$$

converges to $c_0 = (3/2)^{2/3}$ and

$$s_0(z) := \lim_{\alpha \to 0} s_\alpha(z) = \begin{cases} 1, & z \in (-\infty, -\sqrt{c_0}], \\ z^3/3 - c_0 z, & z \in (-\sqrt{c_0}, \sqrt{c_0}), \\ -1, & z \in [\sqrt{c_0}, +\infty). \end{cases}$$

- Optimal policy and welfare:

$$Y_\pm = \frac{\mu}{\gamma \sigma^2} \pm \left( \frac{3}{4\gamma} \bar{Y}^2 (1 - \bar{Y})^2 \right)^{1/3} \varepsilon^{1/3}$$

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- Compare to transaction cost model (Gerhold et al., 2014).
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Trading Volume and Welfare

• Expected Trading Volume

$$|ET| := \lim_{T \to \infty} \frac{1}{T} \int_0^T |\hat{u}_\lambda(Y_t)| dt = K_\alpha \left[ \left( \frac{\sigma^2}{2} \right)^3 \gamma \bar{Y}^4 (1 - \bar{Y})^4 \right]^{\frac{1}{\alpha+3}} \lambda^{-\frac{1}{\alpha+3}} + o(\lambda^{-\frac{1}{\alpha+3}})$$

• Define welfare loss as decrease in equivalent safe rate due to friction:

$$\text{LoS} = \frac{\mu^2}{2\gamma \sigma^2} - \text{EsR}_\gamma(\hat{u})$$

• Zero loss if no trading necessary, i.e. $\bar{Y} \in \{0, 1\}$.

• Universal relation:

$$\text{LoS} = N_\alpha \lambda |ET|^{1+\alpha}$$

where constant $N_\alpha$ depends only on $\alpha$.

• Linear effect with transaction costs (price, not quantity).

Superlinear effect with liquidity (price times quantity).
Trading Volume and Welfare

• Expected Trading Volume

\[ |ET| := \lim_{T \to \infty} \frac{1}{T} \int_0^T |\hat{u}_\lambda(Y_t)| \, dt = K_\alpha \left[ \left( \frac{\sigma^2}{2} \right)^3 \gamma \bar{Y}^4 (1 - \bar{Y})^4 \right]^{\frac{1}{\alpha + 3}} \lambda^{-\frac{1}{\alpha + 3}} + o\left(\lambda^{-\frac{1}{\alpha + 3}}\right) \]

• Define welfare loss as decrease in equivalent safe rate due to friction:

\[ \text{LoS} = \frac{\mu^2}{2\gamma \sigma^2} - \text{EsR}_\gamma(\hat{u}) \]

• Zero loss if no trading necessary, i.e. \( \bar{Y} \in \{0, 1\} \).

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Neither a Borrower nor a Shorter Be

Theorem

If $\frac{\mu}{\gamma \sigma^2} \leq 0$, then $Y_t = 0$ and $\hat{u} = 0$ for all $t$ optimal. Equivalent safe rate zero.

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• If Merton investor shorts, keep all wealth in safe asset, but do not short.
• If Merton investor levers, keep all wealth in risky asset, but do not lever.
• Portfolio choice for a risk-neutral investor!
• Corner solutions. But without constraints?
• Intuition: the constraint is that wealth must stay positive.
• Positive wealth does not preclude borrowing with block trading, as in frictionless models and with transaction costs.
• Block trading unfeasible with price impact proportional to turnover. Even in the limit.
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Control Argument

- Value function $v$ depends on (1) current wealth $X_t$, (2) current risky weight $Y_t$, and (3) calendar time $t$.

$$
dv(t, X_t, Y_t) = v_t dt + v_x dX_t + v_y dY_t + \frac{v_{xx}}{2} d\langle X \rangle_t + \frac{v_{yy}}{2} d\langle Y \rangle_t + v_{xy} d\langle X, Y \rangle_t
$$

$$
= v_t dt + v_x (\mu X_t Y_t - \lambda X_t |u_t|^{\alpha+1}) dt + v_x X_t Y_t \sigma dW_t
+ v_y (Y_t (1 - Y_t)(\mu - Y_t \sigma^2) + u_t + \lambda Y_t |u_t|^{\alpha+1}) dt + v_y Y_t (1 - Y_t) \sigma dW_t
+ \left( \frac{\sigma^2}{2} v_{xx} X_t^2 Y_t^2 + \frac{\sigma^2}{2} v_{yy} Y_t^2 (1 - Y_t)^2 + \sigma^2 v_{xy} X_t Y_t^2 (1 - Y_t) \right) dt,
$$

- Maximize drift over $u$, and set result equal to zero:

$$
v_t + y (1-y) (\mu - \sigma^2 y) v_y + \mu x y v_x + \frac{\sigma^2 y^2}{2} (x^2 v_{xx} + (1 - y)^2 v_{yy} + 2 x (1 - y) v_{xy})
+ \max_u (-\lambda x |u|^{\alpha+1} v_x + v_y (u + \lambda y |u|^{\alpha+1})) = 0.
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Homogeneity and Long-Run

- Homogeneity in wealth \( v(t, x, y) = x^{1-\gamma}v(t, 1, y) \).
- Guess long-term growth at equivalent safe rate \( \beta \), to be found.
- Substitution \( v(t, x, y) = \frac{x^{1-\gamma}}{1-\gamma} e^{(1-\gamma)\beta(T-t)+\int^y q(z)dz} \) reduces HJB equation

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-\beta + \mu y - \gamma \frac{\sigma^2}{2} y^2 + qy(1 - y)(\mu - \gamma \sigma^2 y) + \frac{\sigma^2}{2} y^2(1 - y)^2(q' + (1 - \gamma)q^2)
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- \( \beta = \frac{\mu^2}{2\gamma \sigma^2} \), \( q = 0 \), \( y = \frac{\mu}{\gamma \sigma^2} \) corresponds to Merton solution.
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Asymptotics away from Target

- Guess that $q(y) \to 0$ as $\lambda \downarrow 0$. Limit equation:
\[
\frac{\gamma \sigma^2}{2} (\bar{Y} - y)^2 = \lim_{\lambda \to 0} \frac{\alpha}{\alpha + 1} (\alpha + 1)^{-1/\alpha} |q|^{\alpha+1} \alpha \lambda^{-1/\alpha}.
\]

- Expand equivalent safe rate as $\beta = \frac{\mu^2}{2\gamma \sigma^2} - c(\lambda)$

- Function $c$ represents welfare impact of illiquidity.

- Expand function as $q(y) = q^{(1)}(y)\lambda^{1/2} + o(\lambda^{1/2})$.

- Plug expansion in HJB equation
\[
-\beta + \mu y - \gamma \frac{\sigma^2}{2} y^2 + y(1-y)(\mu - \gamma \sigma^2 y)q + \frac{q^2}{4\lambda(1-yq)} + \frac{\sigma^2}{2} y^2 (1-y)^2 (q' + (1-\gamma)q^2) =
\]

- which suggests asymptotic approximation
\[
q^{(1)}(y) = \lambda \frac{1}{\alpha+1} (\alpha + 1) \frac{1}{\alpha+1} \left( \frac{\alpha + 1}{\alpha} \frac{\gamma \sigma^2}{2} \right)^{\frac{\alpha}{\alpha+1}} |\bar{Y} - y|^{\frac{2\alpha}{\alpha+1}} \text{sgn}(\bar{Y} - y).
\]

- Derivative explodes at target $\bar{Y}$. Need different expansion.
Asymptotics away from Target

• Guess that $q(y) \to 0$ as $\lambda \downarrow 0$. Limit equation:

$$\frac{\gamma \sigma^2}{2} (\bar{Y} - y)^2 = \lim_{\lambda \to 0} \frac{\alpha}{\alpha + 1} (\alpha + 1)^{-1/\alpha} |q|^{\frac{\alpha + 1}{\alpha}} \lambda^{-1/\alpha}.$$ 

• Expand equivalent safe rate as $\beta = \frac{\mu^2}{2\gamma \sigma^2} - c(\lambda)$

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Asymptotics close to Target

- Zoom in around target weight $\bar{Y}$.

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Issues

- How to make argument rigorous?
- Heuristics yield ODE, but no boundary conditions!
- Relation between ODE and optimization problem?
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Verification

Lemma

Let $q$ solve the HJB equation, and define $Q(y) = \int_y^\infty q(z)dz$. There exists a probability $\hat{P}$, equivalent to $P$, such that the terminal wealth $X_T$ of any admissible strategy satisfies:

$$E[X_T^{1-\gamma}]^{\frac{1}{1-\gamma}} \leq e^{\beta T + Q(y)} E_{\hat{P}}[e^{-(1-\gamma)Q(Y_T)}]^{\frac{1}{1-\gamma}},$$

and equality holds for the optimal strategy.

- Solution of HJB equation yields asymptotic upper bound for any strategy.
- Upper bound reached for optimal strategy.
- Valid for any $\beta$, for corresponding $Q$.
- Idea: pick largest $\beta^*$ to make $Q$ disappear in the long run.
- A priori bounds:
  $$\beta^* < \frac{\mu^2}{2\gamma\sigma^2}$$  (frictionless solution)
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Existence

Theorem

Assume $0 < \frac{\mu}{\gamma \sigma^2} < 1$. There exists $\beta^*$ such that HJB equation has solution $q(y)$ with positive finite limit in 0 and negative finite limit in 1.

- for $\beta > 0$, there exists a unique solution $q_{0,\beta}(y)$ to HJB equation with positive finite limit in 0.
- for $\beta > \mu - \frac{\gamma \sigma^2}{2}$, there exists a unique solution $q_{1,\beta}(y)$ to HJB equation with negative finite limit in 1.
- there exists $\beta_u$ such that $q_{0,\beta_u}(y) > q_{1,\beta_u}(y)$ for some $y$;
- there exists $\beta_l$ such that $q_{0,\beta_l}(y) < q_{1,\beta_l}(y)$ for some $y$;
- by continuity and boundedness, there exists $\beta^* \in (\beta_l, \beta_u)$ such that $q_{0,\beta^*}(y) = q_{1,\beta^*}(y)$.
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- Boundary conditions are natural!
Existence

Theorem

Assume $0 < \frac{\mu}{\gamma \sigma^2} < 1$. There exists $\beta^*$ such that HJB equation has solution $q(y)$ with positive finite limit in 0 and negative finite limit in 1.

- for $\beta > 0$, there exists a unique solution $q_{0,\beta}(y)$ to HJB equation with positive finite limit in 0.
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Explosion with Leverage

**Lemma**

If $Y_t$ that satisfies $Y_0 \in (1, +\infty)$ and

$$dY_t = Y_t(1 - Y_t)(\mu dt - Y_t\sigma^2 dt + \sigma dW_t) + u_t dt + \lambda Y_t |u_t|^{1+\alpha} dt$$

explodes in finite time with positive probability.

**Lemma**

Let $\tau$ be the exploding time of $Y_t$. Then wealth $X_\tau = 0$ a.s on $\{\tau < +\infty\}$.

- Feller’s criterion for explosions.
- No strategy admissible if it begins with levered or negative position.
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Conclusion

- Finite market depth. Execution price power of wealth turnover.
- Large investor with constant relative risk aversion.
- Base price geometric Brownian Motion.
- Halfway between linear impact and bid-ask spreads.
- Trade towards frictionless portfolio.
- Do not lever an illiquid asset!
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