An Improved Model for Calculation of Debt Specific Risk VaR with Tail Fitting

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1. **VaR of Debt Portfolios**
   - Debt Portfolio
   - Risk in Debt Portfolio
   - Total VaR

2. **MC DSR Framework**
   - Position SR Loss
   - Portfolio SR Loss
   - Portfolio Total VaR

3. **Distribution of Residuals**
   - Existing Distributions
   - Normal Kernel Distribution
   - Normal Kernel Distribution with Pareto Tails
A debt portfolio on the trading book is a portfolio consisting of the following instruments:

- Bond:
  - Corporate bonds
  - Agency bonds
  - Supranational bonds
  - Provincial/municipal bonds
  - Banker’s acceptance
  - Non-domestic sovereign issues
  - etc.

- Single-name credit default swap (CDS)
Credit default swap

A CDS is an instrument which provides a protection against the risk of a default on a bond issued by a reference entity.

- The protection buyer periodically pays premium (CDS spread) of X bps in annual basis until the maturity or the occurrence of default of the reference entity, whichever is earlier;
- If the reference entity defaults, the price of bonds issued by the reference entity collapse. The CDS contract provides the issuance for the protection buyer:
  - The protection seller pays the par value of the bond;
  - The protection buyer delivers the bond.

The use of CDS

- Hedge: protect against the default of the reference entity of a bond;
- Speculation: bet on the health of the reference entity, may not hold any bond issued by the reference entity;
- Arbitrage: capital structure arbitrage, etc.
Risk in Debt Portfolio

- Risk embedded in positions (bond or CDS) of debt portfolio includes:
  - Market risk: change of PnL due to systematic risk factors that affect the overall performance of the financial markets, such as interest rates;
  - Specific risk: change of PnL to idiosyncratic risk factors which exclude credit events;
  - Migration/default risk: change of PnL due to the change of the credit rating of a bond or default on a bond;
  - Other risk: liquidity risk, counterparty credit risk (OTC trades), etc.

- Time horizon for different risks
  - Market risk and specific risk cover price volatility that would normally occur over a short period (e.g. 10 days);
  - Migration/default risk captures migration and default risk over a longer period (e.g. 1 year).

- Since Basel 2.5, banks are required to
  - Develop an Incremental Risk Charge (IRC) model to calculate capitals reserved for migration/default risk;
  - Modify the existing risk model to account for both market risk and specific risk.
A term “total risk” is proposed to cover market risk and specific risk:

Total risk = Market risk + Specific risk.

The value-at-risk (VaR) measure is used to measure total risk; VaR of a portfolio is defined by

$$\text{VaR}_\alpha(\mathcal{L}) = \inf \{ l \in \mathbb{R} : P[\mathcal{L} \leq l] \geq \alpha \} ,$$

- $\mathcal{L}$ is the portfolio loss;
- $\alpha$ is the predetermined confidence level, for total risk, $\alpha = 99%$;
- VaR can be interpreted as “We are $\alpha$ certain that we will not lose more than VaR$_\alpha$(\mathcal{L}) dollars in the considered time horizon.”
- Mathematically, VaR of $\mathcal{L}$ with confidence level $\alpha$ is the $\alpha$-quantile of $\mathcal{L}$.
Outline

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**Decomposition of Spread**

- Denote \( y \) and \( r \) as a bond’s yield and the corresponding benchmark government bond yield respectively, and the yield spread is \( y - r \);
- Let \( s \) be the spread of a CDS;
- The day-over-day change of the spread,

\[
z = \begin{cases} 
\Delta (y - r), & \text{for bond} \\
\Delta s, & \text{for CDS} 
\end{cases}
\]

can be decomposed by

\[
z = \beta \tilde{z} + \epsilon;
\]

- \( \tilde{z} \): the average spread change of bonds (or CDSs) which are within the same currency/sector/rating category and have similar tenor;
- \( \beta \): the sensitivity of the spread change of the bond (or the CDS) to the average spread change;
- \( \epsilon \): the idiosyncratic spread change (residual), of the bond (or the CDS);
- \( \tilde{z} \) and \( \epsilon \) are assumed to be independent;

- The risk due to \( \beta \tilde{z} \) is captured in market risk model;
- The specific risk model examine the risk due to residuals \( \epsilon \).
Let $\mathcal{P}$ be a position’s daily PnL, then a position’s daily loss due to the idiosyncratic risk factor can be approximated by

- Delta approximation (first-order)

$\mathcal{L} \approx - \frac{\partial \mathcal{P}}{\partial \epsilon} \bigg|_{\epsilon=0} \cdot \epsilon = -\delta \sigma \bar{\epsilon},$

where $\delta = \frac{\partial \mathcal{P}}{\partial \epsilon} \bigg|_{\epsilon=0}$, $\sigma$ is the standard deviation of $\epsilon$, and $\bar{\epsilon}$ is the normalized residual;

- Closed-form distribution for portfolio loss under certain assumption on the distribution of residuals;
- Linear loss approximation, risk beyond the first order is ignored;

- Delta-Gamma approximation (second-order)

$\mathcal{L} \approx - \left( \frac{\partial \mathcal{P}}{\partial \epsilon} \bigg|_{\epsilon=0} \cdot \epsilon + \frac{1}{2} \frac{\partial^2 \mathcal{P}}{\partial \epsilon^2} \bigg|_{\epsilon=0} \cdot \epsilon^2 \right) = -\delta \sigma \bar{\epsilon} - \frac{1}{2} \gamma \sigma^2 \bar{\epsilon}^2,$

where $\gamma = \frac{\partial^2 \mathcal{P}}{\partial \epsilon^2} \bigg|_{\epsilon=0}$;

- No closed-form distribution for portfolio loss, Monte Carlo simulation is needed;
- Second order approximation, more accurate.
Most risk engines are capable to provide sensitivity of positions to shocks applied to credit spreads;

- Risk number, $csPV_{m_j}, j = 1, \ldots, J$: the position PnL if an absolute shock of $m_j$ bp is applied to the credit spread;

- $\delta$ and $\gamma$ can be estimated by linear squares regression:
Estimation of Delta and Gamma (Cont’d)

- $\delta$ in Delta approximation can be estimated by:

\[
\min_{\delta} \sum_{j=1}^{J} (csPV_{m_j} - \delta m_j)^2
\]

- Solution is

\[
\delta = (X'X)^{-1}X'Y = \frac{\sum_{j=1}^{J} m_j csPV_{m_j}}{\sum_{j=1}^{J} m_j^2},
\]

where $X = [m_1, \ldots, m_j]'$ and $Y = [csPV_{m_1}, \ldots, csPV_{m_j}]'$.

- $\delta$ and $\gamma$ in Delta-Gamma approximation can be estimated by:

\[
\min_{\delta, \gamma} \sum_{j=1}^{J} \left( csPV_{m_j} - \left( \delta m_j + \frac{1}{2} \gamma m_j^2 \right) \right)^2
\]

- Solution is

\[
\begin{bmatrix}
\delta \\
\gamma
\end{bmatrix} = (X'X)^{-1}X'Y,
\]

where $X = \begin{bmatrix} m_1 & \frac{1}{2} m_1 \\ \vdots & \vdots \\ m_J & \frac{1}{2} m_J \end{bmatrix}$, and $Y = \begin{bmatrix} csPV_{m_1} \\ \vdots \\ csPV_{m_J} \end{bmatrix}$. 
Positions in a debt portfolio have
- Different issuer/reference entity;
- Different tenor/maturity;

Positions with different issuer/reference entities have different marginal residual distribution;

Positions with same issuer/reference entity but different tenor/maturity may have a very different marginal residual distribution as well;

However, it is not practical to model every position’s marginal residual distribution:
- Missing data;
- Too computationally intense;
Tenors/maturities can be mapped to “proxy tenors”

<table>
<thead>
<tr>
<th>Position tenor</th>
<th>Proxy tenor</th>
</tr>
</thead>
<tbody>
<tr>
<td>[0yr, 1yr)</td>
<td>1yr</td>
</tr>
<tr>
<td>[1yr, 3yr)</td>
<td>2yr</td>
</tr>
<tr>
<td>[4yr, 9yr)</td>
<td>5yr</td>
</tr>
<tr>
<td>[9yr, 15yr)</td>
<td>10yr</td>
</tr>
<tr>
<td>[15yr, +∞)</td>
<td>20yr</td>
</tr>
</tbody>
</table>

Positions are grouped into categories $K_{n,m,j}$, $n = 1, \ldots, N$, $m = 1, \ldots, 5$ and $j = 1, 2$, where

- $K_{n,m,1} = \{k| \text{position} \, k \, \text{is a bond with the} \, m \text{th proxy tenor and the} \, n \text{th issuer} \}$,
- $K_{n,m,2} = \{k| \text{position} \, k \, \text{is a CDS with the} \, m \text{th proxy tenor and the} \, n \text{th reference entity} \}$.

Positions within the same subset share a common residual.
The $h$-day portfolio loss is computed by

- **Delta approximation:**
  \[\mathcal{L} \approx - \sum_{n=1}^{N} \sum_{m=1}^{5} \sum_{j=1}^{2} \left( \sqrt{h} \tilde{\delta}_{n,m,j} \right) \tilde{\epsilon}_{n,m,j}\]
  where $\tilde{\delta}_{n,m,j} = \sum_{k \in K_{n,m,j}} \delta_k \sigma_k$;

- **Delta-Gamma approximation**
  \[\mathcal{L} \approx - \sum_{n=1}^{N} \sum_{m=1}^{5} \sum_{j=1}^{2} \left( \left( \sqrt{h} \tilde{\delta}_{n,m,j} \right) \tilde{\epsilon}_{n,m,j} + \frac{1}{2} \left( h \tilde{\gamma}_{n,m,j} \right) \tilde{\epsilon}_{n,m,j}^2 \right)\]
  where $\tilde{\gamma}_{n,m,j} = \sum_{k \in K_{n,m,j}} \gamma_k \sigma_k^2$. 
Portfolio Total VaR

- The total VaR with the confidence level $\alpha$ is defined by
  \[
  \text{VaR}_\alpha (\mathcal{L}_{TR}) := \inf \{ q \in \mathbb{R} : \mathbb{P} [\mathcal{L}_{TR} \leq q] \geq \alpha \};
  \]

- The total portfolio loss can be approximated by the summation of the loss calculated in the market risk model and the SR loss:
  \[
  \mathcal{L}_{TR} \approx \mathcal{L}_{MR} + \mathcal{L}_{SR};
  \]

- A scenario-based market risk model usually generates $I$ scenario losses:
  \[
  \mathcal{L}_{MR}^{(1)}, \ldots, \mathcal{L}_{MR}^{(I)};
  \]

- Assuming that shocks on residuals are independent with shocks on other market risk factors, $\mathcal{L}_{MR}$ are independent with $\mathcal{L}_{SR}$. Hence,
  \[
  \mathbb{P} [\mathcal{L}_{TR} \leq q] = \mathbb{P} [\mathcal{L}_{MR} + \mathcal{L}_{SR} \leq q] \\
  = \mathbb{E} [\mathbb{P} [\mathcal{L}_{MR} + \mathcal{L}_{SR} \leq q | \mathcal{L}_{MR}]] \\
  \approx \frac{1}{I} \sum_{i=1}^{I} \mathbb{P} [\mathcal{L}_{SR} \leq q - \mathcal{L}_{MR}^{(i)}];
  \]

- Rest of the problem: model the distribution of residuals to calculate
  \[
  \mathbb{P} [\mathcal{L}_{SR} \leq q - \mathcal{L}_{MR}^{(i)}].
  \]
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In practice, normalized residuals $\bar{\epsilon}$ are usually modeled by the multi-variate student’s t distribution with DoF $\nu$;

If the Delta approximation is applied:

- The portfolio SR loss, $\mathcal{L}_{SR}$, is a linear combination of random variables subject to multi-variate student’s t distribution;
- Consequently, $\sqrt{\frac{\nu}{\nu-2}} \frac{\mathcal{L}_{SR}}{\sqrt{h\sigma_P}}$ is a uni-variate student t distribution with the same degree of freedom $\nu$;

- $\sigma_P = \sqrt{h} \sqrt{\delta^T \rho \delta}$ is the $h$-day portfolio SR PnL volatility;
- $\rho$ is the correlation matrix of $\bar{\epsilon}$;

- The probability $\Pr(\mathcal{L}_{SR} \leq q - \mathcal{L}^{(i)}_{MR})$ can be computed analytically:

$$\Pr(\mathcal{L}_{SR} \leq q - \mathcal{L}^{(i)}_{MR}) = t_{\nu} \left( \frac{\sqrt{\frac{\nu}{\nu-2}} \cdot (q - \mathcal{L}^{(i)}_{MR})}{\sqrt{h\sigma_P}} \right);$$

- No closed-form solution for the Delta-Gamma approximation.
The marginal student’s t distribution may not be close to the distribution of some bond/CDS residuals:
Empirical Distribution

- Given historical data of normalized residuals, $\bar{\epsilon}_{n,m,j}^{(1)}, \ldots, \bar{\epsilon}_{n,m,j}^{(U)}$, we can compute the empirical CDF for $\bar{\epsilon}_{n,m,j}$:

$$\tilde{F}_{n,m,j}(x) = \frac{1}{U} \sum_{u=1}^{U} I\{\bar{\epsilon}_{n,m,j}^{(u)} \leq x\},$$

where $I_A$ is an indicator variable

$$I_A = \begin{cases} 1, & \text{if } A \text{ is true}, \\ 0, & \text{otherwise}. \end{cases}$$

- We could use the empirical distribution implied by $\bar{\epsilon}_{n,m,j}^{(1)}, \ldots, \bar{\epsilon}_{n,m,j}^{(U)}$ as the marginal distribution of $\bar{\epsilon}_{n,m,j}$, BUT

- The empirical distribution is not continuous, which is not desirable from the aspect of sampling;
- Sampling from an empirical distribution implied by $U$ unique observations generates at most $U$ unique samples.
Given a series of observations $x^{(1)}, \ldots, x^{(U)}$, the kernel density estimator can be used to estimate the unknown density:

$$\hat{f}_h(x) = \frac{1}{Uh} \sum_{u=1}^{U} K \left( \frac{x - x^{(u)}}{h} \right);$$

- $K(\cdot)$ is the kernel function, which determines the shape of the density;
- $h$ is the bandwidth or smoothing constant, which determines the smoothness of the density.
Normal Kernel Distribution (NK)

- Normal kernel:
  \[ K(x) = \phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}; \]

- Normal kernel CDF estimator for the marginal distribution of normalized residuals:
  \[ \hat{F}_{n,m,j}(x) = \frac{1}{Uh} \sum_{i=1}^{U} \Phi \left( \frac{x - \hat{e}_{n,m,j}^{(i)}}{h} \right), \]

  where \( \Phi(\cdot) \) is the CDF of the standard normal distribution;
**Bandwidth Selection**

- The bandwidth $h$ determines the quality of the estimation
  - Larger bandwidth: less variance but more bias;
  - Smaller bandwidth: less bias but more variance;

- The bandwidth $h$ determines the smoothness of the density:
  - Larger bandwidth: smoother density estimation;

- A rule of thumb for normal kernel: “Silverman’s rule of thumb”
  
  \[ h = \left( \frac{4 \hat{\sigma}^5}{3U} \right)^{1/5} \approx 1.06\hat{\sigma}U^{-1/5}, \]

  where $\hat{\sigma}$ is the sample standard deviation.
The normal kernel with the bandwidth by Silverman’s rule of thumb usually generates

- Well-suited estimates for densities in the middle portion of the distribution;
- Under-smoothed, high variance tails;

To better estimate the tails of the distribution, the generalized Pareto (GP) distribution can be used to model the distribution of exceedances of residuals over pre-determined thresholds;

The density of the GP distribution with shape parameter $\xi$, scale parameter $\sigma$ and location parameter $\mu$, is

$$
g (x | \xi, \sigma, \mu) = \begin{cases} 
\frac{1}{\sigma} \left( 1 + \frac{\xi(x-\mu)}{\sigma} \right)^{-\left(1 + 1/\xi \right)} & \text{for } x \geq \mu \text{ when } \xi > 0, \text{ or for } \mu \leq x \leq \mu - \sigma/\xi \text{ when } \xi < 0, \\
\frac{1}{\sigma} e^{-\frac{x-\mu}{\sigma}} & \text{for } x \geq \mu \text{ when } \xi = 0, \\
0 & \text{otherwise},
\end{cases}
$$

- $\xi$: shape parameter;
- $\sigma$: scale parameter;
- $\mu$: location parameter;

The CDF of GP distribution is

$$
G (x | \xi, \sigma, \mu) = \begin{cases} 
1 - \left( 1 + \frac{\xi(x-\mu)}{\sigma} \right)^{-1/\xi} & \text{for } x \geq \mu \text{ when } \xi > 0, \text{ or for } \mu \leq x \leq \mu - \sigma/\xi \text{ when } \xi < 0, \\
1 - e^{-\frac{x-\mu}{\sigma}} & \text{for } x \geq \mu \text{ when } \xi = 0, \\
0 & \text{for } x < \mu, \\
1 & \text{otherwise}.
\end{cases}
$$
Pareto Distribution (Cont’d)

- Capable to fit a wide variety of fat-tailed data
Pareto Tails

Upper tail:
- Select an upper tail threshold $\hat{\alpha}$, e.g. $\hat{\alpha} = 90\%$;
- Calculate the $\hat{\alpha}$ quantile of the normal kernel distribution, $\hat{Q}_{n,m,j}$;
- Calculate upper exceedances, $i_{n,m,j}^{(i)}$, by:
  \[
  i_{n,m,j}^{(i)} = \bar{\epsilon}_{n,m,j}^{(i)} - \hat{Q}_{n,m,j}, \text{ for } i \in \hat{S}_{n,m,j} = \left\{ i | \bar{\epsilon}_{n,m,j}^{(i)} > \hat{Q}_{n,m,j} \right\} ;
  \]
- Choose a proper GP distribution, $GP\left(\hat{\xi}_{n,m,j}, \hat{\sigma}_{n,m,j}, 0\right)$, to fit $i_{n,m,j}^{(i)}$ by the maximum likelihood estimation (MLE):
  \[
  \left(\hat{\xi}_{n,m,j}, \hat{\sigma}_{n,m,j}\right) := \arg \max_{\xi, \sigma} \sum_{i \in \hat{S}_{n,m,j}} \ln g \left( i_{n,m,j}^{(i)} | \xi, \sigma, 0 \right) ;
  \]
- The optimization problem can be solved by Nelder–Mead method;

Lower tail:
- Select a lower tail threshold $\underline{\alpha}$, e.g. $\underline{\alpha} = 10\%$;
- Calculate the $\underline{\alpha}$ quantile of the normal kernel distribution, $\underline{Q}_{n,m,j}$;
- Calculate lower exceedances, $\underline{i}_{n,m,j}^{(i)}$, by:
  \[
  \underline{i}_{n,m,j}^{(i)} = \underline{Q}_{n,m,j} - \bar{\epsilon}_{n,m,j}^{(i)}, \text{ for } i \in \underline{S}_{n,m,j} = \left\{ i | \bar{\epsilon}_{n,m,j}^{(i)} < \underline{Q}_{n,m,j} \right\} ;
  \]
- Choose a proper GP distribution, $GP\left(\underline{\xi}_{n,m,j}, \underline{\sigma}_{n,m,j}, 0\right)$, to fit $\underline{i}_{n,m,j}^{(i)}$ by MLE:
  \[
  \left(\underline{\xi}_{n,m,j}, \underline{\sigma}_{n,m,j}\right) := \arg \max_{\xi, \sigma} \sum_{i \in \underline{S}_{n,m,j}} \ln g \left( \underline{i}_{n,m,j}^{(i)} | \xi, \sigma, 0 \right) .
  \]
Combining the normal kernel distribution and the Pareto tails enable us to model the distribution of residuals by the following semi-parametric model:

\[
\bar{\epsilon}_{n,m,j} = (\hat{Q}_{n,m,j} - \mathcal{Y}) \cdot \mathbb{I}\{\mathcal{X} \in (-\infty, \hat{Q}_{n,m,j})\} \\
+ \mathcal{X} \cdot \mathbb{I}\{\mathcal{X} \in [\hat{Q}_{n,m,j}, \hat{Q}_{n,m,j}]\} \\
+ (\hat{Q}_{n,m,j} + \mathcal{Z}) \cdot \mathbb{I}\{\mathcal{X} \in (\hat{Q}_{n,m,j}, +\infty)\},
\]

- $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ are mutually independent;
- $\mathcal{X}$ is subject to the normal kernel distribution with CDF $\hat{F}_{n,m,j}(x)$;
- $\mathcal{Y}$ follows GP distribution with CDF $G(x | \hat{\xi}_{n,m,j}, \hat{\sigma}_{n,m,j}, 0)$;
- $\mathcal{Z}$ follows GP distribution with CDF $G(x | \hat{\xi}_{n,m,j}, \hat{\sigma}_{n,m,j}, 0)$.

The CDF of the NKPT distribution is a piecewise function:

\[
F_{n,m,j}(x) = \begin{cases} 
\hat{\alpha} \left(1 - G\left(\hat{Q}_{n,m,j} - x \middle| \hat{\xi}_{n,m,j}, \hat{\sigma}_{n,m,j}, 0\right)\right), & x \in (-\infty, \hat{Q}_{n,m,j}), \\
\hat{F}_{n,m,j}(x), & x \in [\hat{Q}_{n,m,j}, \hat{Q}_{n,m,j}], \\
\hat{\alpha} + (1 - \hat{\alpha}) G\left(x - \hat{Q}_{n,m,j} \middle| \hat{\xi}_{n,m,j}, \hat{\sigma}_{n,m,j}, 0\right), & x \in (\hat{Q}_{n,m,j}, +\infty).
\end{cases}
\]
COMPARISON OF MARGINAL DISTRIBUTIONS

AMERICAN INTL GROUP INC, NY: 1-yr Bond
Empirical vs NK

AMERICAN INTL GROUP INC, NY: 1-yr Bond
Empirical vs NKPT

AMERICAN INTL GROUP INC, NY: 1-yr Bond
Empirical vs T4
COMPARISON OF MARGINAL DISTRIBUTIONS (CONT’D)

![Graphs showing the distribution of residuals for American Intl Group Inc., NY: 1-yr Bond](image)
**JOINT DISTRIBUTION: COPULA**

- Copula is usually used to construct the joint distribution from marginal distributions;
- A copula is defined as a distribution on the unit cube $[0, 1]^N$:
  \[ C(u_1, u_2, \ldots, u_N) = \mathbb{P} [U_1 \leq u_1, U_2 \leq u_2, \ldots, U_N \leq u_N]; \]
  
  - E.g. given a correlation matrix $\rho$, the student’s t copula with 4 DoF can be written as
    \[ C_\rho^t (u_1, u_2, \ldots, u_N) = t_4^\rho (t_4^{-1} (u_1), t_4^{-1} (u_2), \ldots, t_4^{-1} (u_N)); \]

- Consider a random vector $[\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_N]'$ with continuous marginal distribution $F_i$:
  - Transform $\mathcal{X}_i$ to $\mathcal{Y}_i$ by
    \[ \mathcal{Y}_i = t_4^{-1} (F_i (\mathcal{X}_i)); \]
  - Assume $[\mathcal{Y}_1, \mathcal{Y}_2, \ldots, \mathcal{Y}_N]'$ follows a multi-variate T4 distribution with the correlation matrix $\rho$, then the joint distribution of $[\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_N]'$ can be written as
    \[ \mathbb{P} [\mathcal{X}_1 \leq x_1, \mathcal{X}_2 \leq x_2, \ldots, \mathcal{X}_N \leq x_N] = C_\rho^t (F_1 (x_1), F_2 (x_2), \ldots, F_N (x_N)); \]
  - The marginal distribution of $\mathcal{X}_i$ is preserved while defining a correlation structure of $[\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_N]'$ via the correlation structure of $[\mathcal{Y}_1, \mathcal{Y}_2, \ldots, \mathcal{Y}_N]'$. 
Normalized residuals, $\bar{\epsilon}_{n,m,j}$, are modeled by

$$\bar{\epsilon}_{n,m,j} = F^{-1}_{n,m,j} \left( t_4 \left( \omega_{n,m,j} \right) \right)$$

- The marginal distribution of $\bar{\epsilon}_{n,m,j}$ is the NKPT distribution with CDF $F_{n,m,j}(x)$;
- A student’s t copula with 4 DoF is used for the joint distribution:
  - The intermediate random vector $\omega$ follows a multi-variate T4 distribution with a correlation matrix $\varphi$;
  - $\varphi$ is the correlation matrix of normalized residuals $\bar{\epsilon}$.
- No analytical solution for the distribution of the portfolio SR loss;
- Instead, MC simulation is needed to calculate $P \left( \mathcal{L}_{SR} \leq q - \mathcal{L}^{(i)}_{MR} \right)$;
MC DSR with Student’s t Copula NKPT

- Sample \( \omega \) from the multi-variate T4 distribution with the correlation matrix \( \varphi \):
  \[ \omega^{(1)}, \ldots, \omega^{(K)}; \]
- Calculate \( \tilde{\epsilon}^{(1)}, \ldots, \tilde{\epsilon}^{(K)} \) by
  \[ \tilde{\epsilon}_{n,m,j}^{(k)} = F_{n,m,j}^{-1}(t_4(\omega_{n,m,j}^{(k)})); \]
- Compute portfolio SR losses by Delta approximation
  \[ \mathcal{L}_{SR}^{(k)} = - \sum_{n=1}^{N} \sum_{m=1}^{5} \sum_{j=1}^{2} \left( \sqrt{h_{n,m,j}} \tilde{\epsilon}_{n,m,j}^{(k)} \right); \]
  or Delta-Gamma approximation
  \[ \mathcal{L}_{SR}^{(k)} = - \sum_{n=1}^{N} \sum_{m=1}^{5} \sum_{j=1}^{2} \left( \left( \sqrt{h_{n,m,j}} \right) \tilde{\epsilon}_{n,m,j}^{(k)} + \frac{1}{2} \left( h_{n,m,j} \right) \left( \tilde{\epsilon}_{n,m,j}^{(k)} \right)^2 \right); \]
- Approximate \( \mathbb{P}\left( \mathcal{L}_{SR} \leq q - \mathcal{L}_{MR}^{(i)} \right) \) by
  \[ \mathbb{P}\left( \mathcal{L}_{SR} \leq q - \mathcal{L}_{MR}^{(i)} \right) \approx \frac{1}{K} \sum_{k=1}^{K} 1 \{ \mathcal{L}_{SR}^{(k)} \leq q - \mathcal{L}_{MR}^{(i)} \}. \]
### Testing Results: 99% TR VaR

<table>
<thead>
<tr>
<th>Model</th>
<th>Delta</th>
<th>Delta-Gamma</th>
</tr>
</thead>
<tbody>
<tr>
<td>T4</td>
<td>15,230,674.10</td>
<td>15,053,488.33</td>
</tr>
<tr>
<td>Copula NK</td>
<td>15,733,387.31</td>
<td>15,565,004.61</td>
</tr>
<tr>
<td>Copula NKPT</td>
<td>15,809,243.83</td>
<td>15,516,868.54</td>
</tr>
</tbody>
</table>

- Assumption of multivariate student’s t distribution underestimates the risk (about half million for the testing portfolio);
- Compared with the Delta approximation, the Delta-Gamma approximation lowers the VaR number.