Weak Reflection Principle and Static Hedging of Barrier Options

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Static vs Dynamic

- In Finance, hedging is a process of offsetting the risks arising from holding a financial instrument by trading (buying and selling) other instruments.

- When the market is complete, the price of a derivative contract can always be replicated by dynamically trading the underlying asset $S$.
  - Such dynamic hedging strategies have certain drawbacks, in particular, due to the presence of transaction costs.

- Whether the market is complete or not, there sometimes exists a static portfolio of simpler (liquid) derivatives, such that the value of the portfolio matches the value of the target (exotic) derivative at any time before the barrier is hit.
Barrier Options: Up-and-Out Put

- Consider the problem of
  - static hedging of
  - barrier options.

- For the sake of transparency, we focus on the Up-and-Out Put (UOP) option:
  - An UOP written on the underlying process $S$ is issued with a maturity date $T > 0$, a strike price $K > 0$, and a flat upper barrier $U > K$.
  - At expiry, it pays off:

$$I_{\{\sup_{t \in [0, T]} S_t < U\}} \cdot (K - S_T)^+$$
When it exists, a **static hedging** strategy of a barrier option is characterized by a function

$$G : [0, \infty) \to \mathbb{R},$$

such that a European option with payoff $G(S_T)$ at $T$ **has the same value as the target barrier option** up to and including the time when the barrier is hit.

One can, then, **hedge the barrier option** by

1. opening a **long position** in a **European option with payoff** $G$ and
2. trading it at **zero cost** for the corresponding ”vanilla” option when/if the underlying hits the barrier.
Example: Static Hedge in Black’s Model

- Consider the **Black’s model** where the risk-neutral process for the underlying $S$ is given by a geometric Brownian motion:

$$dS_t = S_t \sigma dW_t,$$

with $S_0 < U$ and $\sigma \in \mathbb{R}$.

- **Carr-Bowie (1994)** show that static hedge of an UOP in such model is given by:

$$G(S) = (K - S)^+ - \frac{K}{U} \left(S - \frac{U^2}{K}\right)^+$$

- Hence an UOP can be replicated exactly by being **long one put** struck at $K$ and **short** $\frac{K}{U}$ **calls** struck at $\frac{U^2}{K}$.

- The exact same hedge works in a generalization of the Black model where $\sigma$ is an **unknown** stochastic process independent of $W$. 
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Static Hedge of an UOP in Black’s Model

$$G(S) = (K - S)^+ - \frac{K}{U} \left( S - \frac{U^2}{K} \right)^+$$

Figure: Static Hedge payoff (blue) and the boundary (red). $K^* = \frac{U^2}{K}$
Related results

- **Explicit exact model-dependent static hedge**


- **Robust static hedging with beliefs**: *N.-Obłój* (in progress). We use the exact model-dependent static hedges as building blocks to construct sub- and super replicating strategies that work in classes of models.

- **Optimization-based approach** to find approximate static hedge: *Sachs, Maruhn, Giese, Sircar, Avelaneda*.
Exact static hedge in diffusion models

In *Carr-N. (2011)*, we provide **exact static hedging** strategies for barrier options in the following class of models.

- **Pricing of contingent claims is linear**: it is done by taking expectations of discounted payoffs under some pricing measure.

- **Interest rate** $r$ is constant.

- Under the pricing measure, the underlying $S$ follows a **time-homogeneous diffusion**:

\[ dS_t = \mu(S_t)dt + \sigma(S_t)dB_t \]

- We make some regularity assumptions on $\mu$ and $\sigma$. In particular, our results hold for all models where $\sigma(S)/S$ is bounded away from zero, and $\mu(S)/S$ and $\sigma(S)/S$ have limits at the boundary points, and are bounded along with their first three derivatives.
Static Hedge of an UOP

Function $G$ has to be of the form $G(S) = (K - S)^+ - g(S)$, where:

- $g(S) = 0$ for $S < U$,

and the price of a European option with payoff $g$ is equal to the price of a put with strike $K$ and the same maturity, whenever the underlying hits the barrier.

Figure: Static hedge payoff $G$ (blue) and the barrier (red).
"Mirror" Image

Find \( g \), s.t. it has support in \((U, \infty)\) and

\[
\mathbb{E} \left[ h(S_\tau) \mid S_0 = U \right] = \mathbb{E} \left[ g(S_\tau) \mid S_0 = U \right], \quad \text{for all } \tau > 0
\]

Figure: Price functions of the options with payoffs \( h \) (blue) and \( g \) (green), along the barrier \( S = U \) (red)
Problem formulation

- Consider a stochastic process $X = (X_t)_{t \geq 0}$, started from zero: $X_0 = 0$.
- Introduce
  - $\Omega_1$ – the set of regular functions (e.g. continuous, with at most exponential growth) with support in $(-\infty, 0)$;
  - $\Omega_2$ – the set of regular functions with support in $(0, \infty)$.
- **Problem:** find a mapping $R : \Omega_1 \rightarrow \Omega_2$, such that, for any $f \in \Omega_1$:

$$
\mathbb{E}[f(X_t)] = \mathbb{E}[Rf(X_t)],
$$

for all $t \geq 0$. 
**Weak Reflection Principle**

- If there exists a mapping $S : \mathbb{R} \to \mathbb{R}$, which maps $(-\infty, 0)$ into $(0, \infty)$, and such that the process $X$ is **symmetric** with respect to this mapping:

$$\text{Law}(S(X_t), \ t \geq 0) = \text{Law}(X_t, \ t \geq 0),$$

then, the reflection $R$ is easy to construct:

$$Rf = f \circ S$$

- For example, **Brownian motion** $B$ is symmetric with respect to zero:

$$\text{Law}(-B_t)) = \text{Law}(B_t), \quad \forall t \geq 0,$$

and, therefore:

$$Rf(x) = f(-x)$$

- What we call a **Classical (Strong) Reflection Principle** arises as a combination of the **continuity** of $B$, its **strong Markov property**, and the above **symmetry**.
Applications

- The **Strong Reflection Principle** for Brownian motion is used
  - to compute the **joint distribution of Brownian motion** and its running maximum:
    \[ \mathbb{P}(B_T \leq K, \max_{t \in [0, T]} B_t \leq U) \]
  - or, more generally, solve the **Static Hedging problem** when the underlying is a Brownian motion (or any process symmetric with respect to the barrier).

- It turns out that the **Weak Reflection Principle** is enough to solve the above problems.

- We show how to extend this principle to a large class of Markov processes, which do not possess any strong symmetries!
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Weak Reflection Principle for time-homogeneous diffusions

To simplify the resulting expression, we assume that $\mu \equiv 0$.

$$dX_t = \sigma(X_t)dB_t$$

Then, the reflection mapping $R$ is given by

$$Rh(x) = \frac{2}{\pi i} \int_{\varepsilon-i\infty}^{\varepsilon+i\infty} \frac{z\psi_1(x, z)}{\partial_x\psi_1(0, z) - \partial_x\psi_2(0, z)} \int_{-\infty}^{0} \frac{\psi_1(s, z)}{\sigma^2(s)} h(s)ds \, dz,$$

where the functions $\psi^1(x, z)$ and $\psi^2(x, z)$ are the fundamental solutions of the associated Sturm-Liouville equation:

$$\frac{1}{2} \sigma^2(x) \psi_{xx}(x, z) - z^2 \psi(x, z) = 0$$
Solution to the Static Hedging problem

\((N.-\text{Carr, 2011})\)

- Recall that, in order to solve the static hedging problem, we only need to compute the mirror image of the put payoff.

\[ h(x) = (K - x)^+ \]

- Thus, the Static Hedge of an UOP option (with barrier \( U \) and strike \( K < U \)) is given by

\[ G(x) = (K - x)^+ - g(x), \]

where

\[ g(x) = \frac{1}{\pi i} \int_{\varepsilon-i\infty}^{\varepsilon+i\infty} \frac{\psi^1(x, z) \psi^1(K, z)}{\psi^1_x(U, z) - \psi^2_x(U, z)} \frac{dz}{z}, \]

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Weak Reflection Principle

Constant Elasticity of Variance: $\mu = 0, \sigma(S) = S^{1+\beta}$

Figure: The "mirror image" $g$ in the zero-drift CEV model with barrier $U = 1.2$ and strike $K = 0.5$: the case of $\beta = -0.5$, for small (left) and large (right) values of the argument.
Other CEV: Bachelier and Black-Scholes

Figure: The ”mirror image” $g$ in the zero-drift CEV model with barrier $U = 1.2$ and strike $K = 0.5$: the cases of $\beta = -1$ (left) and $\beta \approx 0$ (right).
Computation and extensions

\[ g(S) = \frac{1}{\pi i} \int_{\varepsilon - \infty}^{\varepsilon + \infty} \frac{\psi_1(S, z) \psi_1(K, z)}{\psi_1(U, z) - \psi_2(U, z)} \, dz, \]

- If \( \sigma(S) \) is piece-wise constant, the fundamental solutions \( \psi_1(S, z) \) and \( \psi_2(S, z) \) can be easily computed as linear combinations of exponentials, on each sub-interval in \( S \).
- This family of models is sufficient for all practical purposes.

- The proposed static hedge also succeeds in all models that arise by running the time-homogeneous diffusion on an independent continuous stochastic clock.
- One can obtain a semi-robust extension of this static hedging strategy. More precisely, a strategy that succeeds in all models, as long as the market implied volatility stays within given bounds (beliefs about implied volatility are fulfilled).
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Robust static hedge with beliefs on implied volatility

Figure: Range of possible values of (beliefs on) implied volatility (blue), and the extremal implied volatility produced by a diffusion model (green)
Weak Reflection Principle for Lévy processes with one-sided jumps

- Note that, in principle, this method can be used any strong Markov process which does not jump across the barrier.

- The only problem is to establish the Weak Reflection of this process with respect to the barrier.

- In our ongoing work, we have developed the Weak Reflection principle for Lévy processes with one-sided jumps. The solution takes a similar form, in the sense that the reflection operator $R$ is given as an integral transform, with a kernel that can be computed through the characteristic exponent of the Lévy process $\psi$. For example, the image of $h(y) = 1_{\{y \leq K\}}$, for $K < 0$, is given by

$$Rh(y) = \frac{1}{2\pi i} \int_{\mathbb{R}^2} \frac{e^{\varepsilon y + iwy + iKz}}{z} \left( \frac{\psi'(w)}{\psi(w) - \psi(-iz) - \frac{1}{w + iz}} \right) dzdw$$
Beyond Finance

- The Weak Reflection Principle allows to compute the **joint distribution** of a time-homogeneous diffusion and its **running maximum**, through the marginal distribution of the process itself:

\[
P(X_T \leq K, \max_{t \in [0,T]} X_t \leq U) = \mathbb{E}(g^{K,U}(X_T)),
\]

where \( g^{K,U} \) is the mirror image of function \((K - .)^+\), with respect to the barrier \( U \).

- The **connection to PDE’s** yields various applications in **Physics** and **Biology**.
Summary

- I have presented a solution to the Static Hedging problem for barrier options.

- This solution provides *exact*, but *model-dependent*, hedge in all regular enough time-homogeneous diffusion models.

- In our ongoing work with J. Obloj, we develop the *semi-robust* hedges based on the above results.

- Static Hedging problem motivated the development of a new technique, the Weak Reflection Principle.

- We have developed the Weak Reflection principle to diffusion processes and one-sided Lévy processes.
The **Weak Reflection Principle** allows us to

- control the expected value of a function of the process,
- at any time when the process is at the barrier of a given domain,
- by changing the function outside of this domain.

Applications include **Finance, Physics, Biology, Computational Methods**.

Further extensions:

- Specific applications in Physics and Biology?
- More general domains?
- More general stochastic processes?
Non-existence result

- *Bardos-Douady-Fursikov (2004)* treat this problem for a general parabolic PDE, and prove the existence of approximate solutions $g_\varepsilon$, such that

$$\sup_{t \in [0, T]} \left| u^h(U, t) - u^{g_\varepsilon}(U, t) \right| < \varepsilon$$

- They show that an exact solution doesn’t exist in general...

- Their proof is *not constructive* - finding even an approximate solution is left as a separate problem.

- The example of non-existence relies heavily on the time-dependence of the coefficients in the corresponding PDE!
Appendix

Naive numerical approximation

Figure: Payoff function $g_\epsilon$ as a result of the naive least-square optimization approach
Figure: 2. Function $g$, for $\beta = -0.5$, $U = 1.2$, $K = 0.5$

Notice that there is a constant $K^* \geq U$, such that the support of $g$ is exactly $[K^*, \infty]$. 
Short-Maturity Behavior and Single-Strike Hedge

- Key observation: when time-to-maturity is small, only the values of $g$ around $K^*$ matter!

- Thus, for small maturities, the approximation of the payoff function $g$ with a scaled call payoff should perform well.

- We have:
  \[ \int_K^U \frac{dy}{\sigma(y)} = \int_U^{K^*} \frac{dy}{\sigma(y)}, \quad \eta = \sqrt{\frac{\sigma(K)}{\sigma(K^*)}}. \]

- Using the above, we can construct the single-strike sub- and superreplicating strategies: there exists $\delta > 0$, such that, whenever $S_t = U$,
  \[ [1 - \delta(T - t)] P_t(K) - \eta C_t(K^*) \leq 0 \leq [1 + \delta(T - t)] P_t(K) - \eta C_t(K^*) \]
Appendix

Function $g$: properties and numerical computation

- There exists a constant $K^* \geq U$, such that the support of $g$ is exactly $[K^*, \infty]$.

- Introduce the "signed geodesic distance":

$$Z(x) := \sqrt{2} \int_U^x \frac{dy}{\sigma(y)}$$

- Then $K^*$ is a solution of the equation

$$Z(K^*) + Z(K) = 0$$

- The function $g$ is "analytic with respect to the geodesic distance $Z$" in $(K^*, \infty)$:

$$g(x) = \sum_{k=1}^{\infty} c_k (Z(x) - Z(K^*))^k,$$

and the exists an algorithm for computing $c_k$'s.