Kirillov–Reshetikhin crystals for nonexceptional types

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University of Ottawa, July 2, 2009
Outline

Affine crystals

KR crystals

Perfectness

Affine Schubert calculus
<table>
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<th>Goals</th>
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<tr>
<td>1. Report on recent progress on KR crystals for nonexceptional types</td>
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<tr>
<td>2. Ground work for Brant Jones’ talk (after this one!)</td>
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<td>3. Relation to affine Schubert calculus</td>
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</table>
Progress on Kirillov-Reshetikhin crystals ...

- **Existence of KR crystals**
  - Existence of KR crystals for nonexceptional types
    → joint with Masato Okado (arXiv:0706.2224)

- **Combinatorial models for KR crystals**
  - Types $D_n^{(1)}, B_n^{(1)}, A_{2n-1}^{(2)}$
    → AS (arXiv:0704.2046)
  - Types $C_n^{(1)}, A_{2n}^{(2)}, D_{n+1}^{(2)}$
    → joint with Ghislain Fourier and Masato Okado (arXiv:0810.5067)
  - Type $E_6^{(1)}, ...$
    → joint with Brant Jones

- **Perfectness**
  - Perfectness of all nonexceptional KR crystals
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- **Perfectness**
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... and relation to affine Schubert calculus

- **Symmetric functions and geometry:**
  - \( k \)-Schur functions, affine Stanley symmetric functions
    → joint with Thomas Lam and Mark Shimozono for type \( C \)
    (arXiv:0710.2720)
  - \( K \)-theory of the affine Grassmannian, stable affine Grothendieck polynomials
    → joint with Thomas Lam and Mark Shimozono
    (arXiv:0901.1506)
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Motivation

**g Lie algebra/Kac–Moody Lie algebra**

- **Crystal bases** are combinatorial bases for $U_q(g)$ as $q \to 0$
- **Affine finite crystals:**
  - appear in 1d sums of exactly solvable lattice models
  - path realization of integrable highest weight $U_q(g)$-modules
  - fermionic formulas, generalized Kostka polynomials, symmetric functions
  - fusion/quantum cohomology structure constants
- Irreducible **finite-dimensional affine $U_q(g)$-modules** classified by Chari-Pressley via Drinfeld polynomials
- HKOTY conjectured that the Kirillov-Reshetikhin modules $W_{r,s}^{r,s}$ have crystal bases
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Crystal graph
Axiomatic Crystals

A $U_q(g)$-crystal is a nonempty set $B$ with maps

$$
\text{wt: } B \to P
$$

$$
e_i, f_i: B \to B \cup \{\emptyset\} \quad \text{for all } i \in I
$$

satisfying

$$
f_i(b) = b' \iff e_i(b') = b \quad \text{if } b, b' \in B
$$

$$
\text{wt}(f_i(b)) = \text{wt}(b) - \alpha_i \quad \text{if } f_i(b) \in B
$$

$$
\langle h_i, \text{wt}(b) \rangle = \varphi_i(b) - \varepsilon_i(b)
$$

Write $\bullet \overset{i}{\longrightarrow} \bullet$ for $b' = f_i(b)$
**Definition**

$B, B'$ crystals

$B \otimes B'$ is $B \times B'$ as sets with

$$\text{wt}(b \otimes b') = \text{wt}(b) + \text{wt}(b')$$

$$f_i(b \otimes b') = \begin{cases} f_i(b) \otimes b' & \text{if } \varepsilon_i(b) \geq \varphi_i(b') \\ b \otimes f_i(b') & \text{otherwise} \end{cases}$$
Tensor products

Definition

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\end{cases}
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Existence of Kirillov-Reshetikhin crystals

**Theorem (OS 07)**

The Kirillov-Reshetikhin crystals $B^{r,s}$ exist for nonexceptional types.

**Proof** uses results on characters by Nakajima and Hernandez. Combinatorial models for these crystals can be constructed using the classical decompositions

$$B^{r,s} \cong \bigoplus_{\Lambda} B(\Lambda)$$

and the automorphism $\sigma$ ($i$ special node $\sigma(i) = 0$)

$$f_0 = \sigma^{-1} \circ f_i \circ \sigma$$

$$e_0 = \sigma^{-1} \circ e_i \circ \sigma$$

or using the virtual crystal construction.
Existence of Kirillov-Reshetikhin crystals

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# Dynkin diagrams

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<th>Type</th>
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<tr>
<td>$A_5^{(1)}$</td>
<td><img src="$A_5%5E%7B(1)%7D$" alt="Dynkin diagram" /></td>
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<td><img src="$B_5%5E%7B(1)%7D$" alt="Dynkin diagram" /></td>
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<tr>
<td>$A_9^{(2)}$</td>
<td><img src="$A_9%5E%7B(2)%7D$" alt="Dynkin diagram" /></td>
</tr>
<tr>
<td>$D_5^{(1)}$</td>
<td><img src="$D_5%5E%7B(1)%7D$" alt="Dynkin diagram" /></td>
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</tr>
<tr>
<td>$A_{10}^{(2)}$</td>
<td><img src="$A_%7B10%7D%5E%7B(2)%7D$" alt="Dynkin diagram" /></td>
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Type $A_{n-1}^{(1)}$

$\text{KMN}^2$ proved existence of crystals $B^{r,s}$ for Kirillov-Reshetikhin modules $W^{r,s}$

$$B^{r,s} \cong B(s^r) \quad \text{as } \{1, 2, \ldots, n-1\}\text{-crystal}$$

Promotion operator $pr$ uniquely defined by Shimozono

$$\langle h_{a+1}, \text{wt}(pr(b)) \rangle = \langle h_a, \text{wt}(b) \rangle$$
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Affine crystals

Type $A^{(1)}_{n-1}$

KMN proved existence of crystals $B^{r,s}$ for Kirillov-Reshetikhin modules $W^{r,s}$

$$B^{r,s} \cong B(s^r)$$ as $\{1, 2, \ldots, n-1\}$-crystal

Promotion operator $\text{pr}$ uniquely defined by Shimozono

$$\langle h_{a+1}, \text{wt(pr(b))} \rangle = \langle h_a, \text{wt(b)} \rangle$$

Then $e_0 = \text{pr}^{-1} \circ e_1 \circ \text{pr}$ $f_0 = \text{pr}^{-1} \circ f_1 \circ \text{pr}$
Promotion for type $A_{n-1}$

Classical crystal: $B(s^r)$ set of Young tableaux of shape $(s^r)$ over alphabet \{1, 2, \ldots, n\}

Promotion:
- Remove rightmost $n$, play jeu de taquin and repeat.
- Increase all entries by one and place 1’s in the empty spaces.
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**Example**

```
3 4 4
2 3 3
1 2 2
```
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```
3 4 ●
2 3 3
1 2 2
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Example

\[
\begin{array}{ccc}
3 & \bullet & 4 \\
2 & 3 & 3 \\
1 & 2 & 2 \\
\end{array}
\]
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**Classical crystal:** $B(s^r)$ set of **Young tableaux** of shape $(s^r)$ over alphabet $\{1, 2, \ldots, n\}$

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**Classical crystal:** $B(s^r)$ set of *Young tableaux* of shape $(s^r)$ over alphabet $\{1, 2, \ldots, n\}$

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**Example**

```
  3 3 ⬤
  2 2 3
  ⬤ 1 2
```
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\begin{array}{ccc}
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\bullet & 1 & \bullet \\
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2 2 2
• • 1
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Types $B_n^{(1)}, D_n^{(1)}, A_{2n-1}^{(2)}$

$B^{r,s}_r \cong V^{r,s}_r \cong \bigoplus_{\Lambda} B(\Lambda)$ as $\{1, 2, \ldots, n\}$-crystal

where $\Lambda$ is obtained from $s\Lambda_r$ by removing $\square$

Dynkin diagram automorphism $\sigma$ interchanging 0 and 1

$f_0 = \sigma \circ f_1 \circ \sigma$

$e_0 = \sigma \circ e_1 \circ \sigma$

Theorem (OS 07)

$V^{r,s}_r \cong B^{r,s}_r$ as a $\{0, 1, \ldots, n\}$-crystal
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Classical decomposition

By construction

\[ V^{r,s} \simeq \bigoplus_{\Lambda} B(\Lambda) \]

as \( X_n = D_n, B_n, C_n \) crystals

\( \Rightarrow \) crystal arrows \( f_i, e_i \) are fixed for \( i = 1, 2, \ldots, n \) using Kashiwara-Nakashima tableaux
Definition of $\sigma$

$X_n \rightarrow X_{n-1}$ branching

$$B_{X_n}(\Lambda) \cong \bigoplus_{\pm \text{ diagrams } P, \text{ outer}(P) = \Lambda} B_{X_{n-1}}(\text{inner}(P))$$

$\pm$ diagrams

$p$ \[ \begin{array}{ccc} + & - & \lambda \subset \mu \subset \Lambda \\ - & + & \text{inner shape} \\ + & - & \text{outer shape} \\ + & + & \end{array} \\

\Lambda/\mu \text{ horizontal strip filled with } - \\
\mu/\lambda \text{ horizontal strip filled with } +$
**Definition of \( \sigma \)**

\( X_{n-1} \) highest weight vectors are in bijection with \( \pm \) diagrams via \( \Phi \)

\[ \Phi: \begin{array}{c|c|c|c|c} - & + & + & - \hline + & - & + \end{array} \rightarrow \begin{array}{c|c|c|c|c} 4 & 4 & 2 & 3 \hline 2 & 3 & 3 & 1 \hline 1 & 1 & 2 & 2 \end{array} \]

\( \vec{a} = (1, 2, 1, 2, 3, 4, 5, 6, 4, 1, 2, 3, 4, 5, 6, 4, 3, 2) \)

\[ \Phi(P) = f_{\vec{a}} \]
Definition of $\sigma$

$X_{n-1}$ highest weight vectors
are in bijection with $\pm$ diagrams via $\Phi$

$\Phi : \begin{array}{ccc}
- & + & - \\
& & + \\
& & \\
\end{array} \mapsto \begin{array}{cccc}
4 & 4 \\
2 & 3 & 3 & 1 \\
1 & 1 & 2 & 2 \\
\end{array}$

$\tilde{a} = (1, 2, 1, 2, 3, 4, 5, 6, 4, 1, 2, 3, 4, 5, 6, 4, 3, 2)$

$\Phi(P) = f_{\tilde{a}} \begin{array}{cccc}
4 & \\
3 & 2 \ 2 \ 2 \ 2 \\
1 & 1 & 1 & 1 \\
\end{array}$
**Definition of $\sigma$**

$\sigma$ on $\pm$ diagrams

$P \pm$ diagram of shape $\Lambda/\lambda$

columns of height $h$ in $\lambda$

$h \not\equiv r \mod 2$: interchange number of $+$ and $-$ above $\lambda$

$h \equiv r \mod 2$: interchange number of $\mp$ and empty above $\lambda$

$P = \begin{array}{ccc}
+ & - & + \\
+ & + & - \\
- & + & + \\
\end{array}$

$\mathcal{G}(P) = \begin{array}{ccc}
- & - & - \\
+ & + & + \\
\end{array}$

$r \geq 6$

$s = 5$
Definition of $\sigma$

$\sigma$ on tableaux

$b \in V^{r,s}$

$e_\rightarrow_a := e_{a_1} \cdots e_{a_\ell}$ such that $e_\rightarrow_a(b)$ is $X_{n-1}$ highest weight vector

$f_\leftarrow_a := f_{a_\ell} \cdots f_{a_1}$

Then

$\sigma(b) = f_\leftarrow_a \circ \Phi \circ \mathcal{S} \circ \Phi^{-1} \circ e_\rightarrow_a(b)$
Example

$V^{4,5}$ of type $D_{6}^{(1)}$

$\begin{array}{ccc}
4 & 4 & \\
3 & 4 & \\
2 & 3 & 1 & 1 \\
1 & 1 & 2 & 3 \\
\end{array}$

$\begin{array}{ccc}
4 & 4 & \\
3 & 4 & \\
2 & 3 & 1 & 1 \\
1 & 1 & 2 & 2 \\
\end{array}$

$\Phi^{-1} \rightarrow + - + - - +$

$\Phi \rightarrow 3 4$

$\begin{array}{ccc}
3 & 3 & 3 & 1 \\
3 & 3 & 3 & 1 \\
1 & 2 & 2 & 2 \\
\end{array}$

$e_{4} e_{6} e_{5} e_{4} e_{2} \rightarrow 2 4$

$\begin{array}{ccc}
\Phi^{-1} \rightarrow - & - & + \\
\Phi \rightarrow 3 3 3 1 \\
\end{array}$

$f_{2} f_{4} f_{5} f_{6} f_{4} \rightarrow 3 3 3 1$

$= \sigma(b)$
Example

$V^{4,5}$ of type $D_6^{(1)}$

$\Phi^{-1} \rightarrow$

$b = \begin{array}{cccc}
4 & 4 \\
3 & 4 \\
2 & 3 & 1 & 1 \\
1 & 1 & 2 & 3
\end{array}$

$\rightarrow e_4 e_6 e_5 e_4 e_2$

$\Phi \rightarrow$

$\begin{array}{cccc}
4 & 4 \\
3 & 4 \\
2 & 3 & 1 & 1 \\
1 & 1 & 2 & 2
\end{array}$

$\Phi^{-1} \rightarrow$

$\begin{array}{cccc}
+ & - \\
+ & + \\
- & - \\
+ & +
\end{array}$

$\sigma$

$\begin{array}{cccc}
+ & - \\
+ & +
\end{array}$

$f_2 f_4 f_5 f_6 f_4$

$\Rightarrow$

$\begin{array}{cccc}
2 & 4 \\
3 & 3 & 3 & 1 \\
1 & 2 & 2 & 2
\end{array}$

$\Rightarrow$

$\begin{array}{cccc}
3 & 3 & 3 & 1 \\
1 & 2 & 2 & 2
\end{array}$

$\Rightarrow = \sigma(b)$
Example

$V_{4,5}^{4,5}$ of type $D_6^{(1)}$

\[ b = \begin{array}{ccc}
4 & 4 & 4 \\
3 & 4 & 4 \\
2 & 3 & 1 & 1 \\
1 & 1 & 2 & 3
\end{array} \]

$\Phi^{-1} \rightarrow$

\[ \begin{array}{ccc}
+ & - & + \\
+ & - & + & +
\end{array} \]

$\rightarrow$

\[ \begin{array}{ccc}
\sigma & (b) & = \\
\end{array} \]

\[ \begin{array}{ccc}
3 & 3 & 3 & 1 \\
4 & 3 & 3 & 3 & 1 \\
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Type $C_n^{(1)}$

$$B^{r,s} \cong V^{r,s} \cong \bigoplus_{\Lambda} B(\Lambda)$$ as $\{1, 2, \ldots, n\}$-crystal

where $\Lambda$ is obtained from $s\Lambda_r$ by removing $\square$

Virtual crystal: ambient crystal $\hat{V}^{r,s} = B^{r,s}$ of type $A_{2n+1}^{(2)}$

**Definition**

$V^{r,s}$ is the subset of $b \in \hat{V}^{r,s}$ such that $\sigma(b) = b$ such that

$$e_i = \begin{cases} \hat{e}_0 \hat{e}_1 & \text{for } i = 0 \\ \hat{e}_{i+1} & \text{for } 1 \leq i \leq n \end{cases}$$

**Theorem (FOS 08)**

$V^{r,s} \cong B^{r,s}$ as a $\{0, 1, \ldots, n\}$-crystal
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$$B^{r,s} \cong V^{r,s} \cong \bigoplus_{\Lambda} B(\Lambda) \quad \text{as } \{1, 2, \ldots, n\}-\text{crystal}$$

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Outline

Affine crystals

KR crystals

Perfectness

Affine Schubert calculus
Perfectness of KR crystals

Conjecture (HKOTT)

The KR crystal $B^{r,s}$ is perfect if and only if $\frac{s}{cr}$ is an integer. If $B^{r,s}$ is perfect, its level is $\frac{s}{cr}$.

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Theorem (FOS 08)

If $\mathfrak{g}$ is of nonexceptional type, the Conjecture is true.
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**Definition of perfectness**

\[ P = \bigoplus_{i \in I} \mathbb{Z} \Lambda_i \] weight lattice of \( g \), \( P^+ \) set of dominant weights.

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The crystal \( B \) is perfect of level \( \ell \) if:

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$$B^{2,2} \cong B(2\Lambda_2) \oplus B(2\Lambda_1) \oplus B(0).$$

Bijection $\varepsilon : B^{2,2}_{\min} \to P_1^+$ given by:

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Kyoto path model

$B(\Lambda)$ highest weight infinite-dimensional crystal of type $\mathfrak{g}$
$u_\Lambda \in B(\Lambda)$ highest weight vector

**Theorem (KMN²)**

$\Lambda \in P_s^+$
$B^{r_1,s}, B^{r_2,s}, \ldots$ perfect of level-$s$

$\Phi : B(\Lambda) \cong \cdots \otimes B^{r_2,s} \otimes B^{r_1,s} \otimes B(\tilde{\Lambda})$

$B$ perfect

$B_{\text{min}} = \{ b \in B \mid \text{lev}(\varepsilon(b)) = s \}$
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Induced automorphism $\tau = \varphi \circ \varepsilon^{-1}$ on $P_s^+$

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**Level-$s$ adjoint KR crystals**

**Adjoint KR crystals:**
Take $r$ to correspond to highest root $\theta$.

Classical decomposition [Chari]:

$$B^{r,s} \cong \bigoplus_{0 \leq k \leq s} B(k\Lambda_r)$$

**Question:** Can we find level-$s$ KR crystals of all types?

**Answer:**
- Benkart et al. gave a uniform construction of level-1 perfect crystals for all types.
- Exceptional types:
  - Yamane type $G_2^{(1)}$
  - Kashiwara, Misra, Okada, Yamada type $D_4^{(3)}$
  - see Brant Jones' talk for level-$s$ type $E_6^{(1)}$, ...
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Schubert calculus

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- **Cohomology**: computations in cohomology ring of the Grassmannian $H^*(G/P)$ with $G = SL_n(\mathbb{C})$ and $P \subset G$ maximal parabolic 1950’s

- **Symmetric Functions**: cohomology ring of Grassmannian (with its natural Schubert basis) same as the algebra of symmetric functions (with Schur basis) 1950’s

- **Combinatorics**: multiplication of Schubert basis governed by Littlewood-Richardson rule 1970’s
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Affine Schubert calculus

Definition

- $G$ affine Kac–Moody group
- $P \subset G$ maximal parabolic subgroup
- $G/P$ affine Grassmannian $Gr$

Example: $\mathcal{K} = \mathbb{C}((t))$, $\mathcal{O} = \mathbb{C}[[t]]$
affine Grassmannian $Gr = SL_{k+1}(\mathcal{K})/SL_{k+1}(\mathcal{O})$

Theorem (Lam)

Schubert bases of $H_*(Gr)$ and $H^*(Gr)$ are given by $k$-Schur functions and affine Stanley symmetric functions of Lascoux, Lapointe, Morse

Structure constants include genus zero Gromov-Witten invariants or fusion coefficients
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Definition (nilHecke algebra)

The nilHecke algebra

- generators $A_1, \ldots, A_{n-1}$
- relations

$$A_i A_j = A_j A_i \quad \text{for } |i - j| \geq 2$$
$$A_i A_{i+1} A_i = A_{i+1} A_i A_{i+1}$$
$$A_i^2 = 0$$
Stanley symmetric functions for other types

• For each Weyl group $W$ one can construct a new nilHecke algebra by taking the associated graded $\mathbb{C}[W]$.

• Finding Stanley symmetric functions for each $W$ is equivalent to finding a particular commutative subalgebra of the nilHecke algebra.

Theorem (Lam; LSS 07)

Schubert bases of $H^*(\text{Gr})$ and $H^*(\text{Gr})$ are given by $k$-Schur functions and affine Stanley symmetric functions for type $A_n^{(1)}$ and $C_n^{(1)}$.

Theorem (LSS 09)

Schubert bases of $K^*(\text{Gr})$ and $K^*(\text{Gr})$ are given by K-$k$-Schur functions and affine stable Grothendieck polynomials.
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*Schubert bases of $H_*(Gr)$ and $H^*(Gr)$ are given by $k$-Schur functions and affine Stanley symmetric functions for type $A_n^{(1)}$ and $C_n^{(1)}$.***

**Theorem (LSS 09)**

*Schubert bases of $K_*(Gr)$ and $K^*(Gr)$ are given by $K$-$k$-Schur functions and affine stable Grothendieck polynomials.*
Relation to KR crystals

$k$-Schur functions

Structure coefficients

\[
s^{(k)}_{\lambda} s^{(k)}_{\mu} = \sum_{\nu} c^{k}_{\lambda,\mu} s^{(k)}_{\nu}
\]

Observation: (inspired by Postnikov and Stroppel/Korff)

- $s^{(k)}_{\lambda}$ evaluated at crystal operators acting on $B^{1,k}$ of type $A^{(1)}_{n-1}$ yields fusion coefficients
- $s^{(k)}_{\lambda}$ evaluated at crystal operators acting on $B^{n,1}$ of type $A^{(1)}_{n+k-1}$ yields quantum cohomology structure coefficients
Relation to KR crystals

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$s^{(k)}_{\lambda}$

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