Kirillov–Reshetikhin crystals for nonexceptional types

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Outline

Goals

1. Report on recent progress on KR crystals for nonexceptional types
2. Ground work for Brant Jones' talk (after this one!)
3. Relation to affine Schubert calculus

Progress on Kirillov-Reshetikhin crystals ...

- **Existence of KR crystals**
  - Existence of KR crystals for nonexceptional types
    → joint with Masato Okado (arXiv:0706.2224)
- **Combinatorial models for KR crystals**
  - Types $D_n^{(1)}$, $B_n^{(1)}$, $A_{2n-1}^{(2)}$
    → AS (arXiv:0704.2046)
  - Types $C_n^{(1)}$, $A_{2n}^{(2)}$, $D_{n+1}^{(2)}$
    → joint with Ghislain Fourier and Masato Okado (arXiv:0810.5067)
  - Type $E_6^{(1)}, \ldots$
    → joint with Brant Jones
- **Perfectness**
  - Perfectness of all nonexceptional KR crystals
    → joint with Ghislain Fourier and Masato Okado (arXiv:0811.1604)

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... and relation to affine Schubert calculus

- **Symmetric functions and geometry:**
  - k-Schur functions, affine Stanley symmetric functions
    $\longrightarrow$ joint with Thomas Lam and Mark Shimozono for type C (arXiv:0710.2720)
  - K-theory of the affine Grassmannian, stable affine Grothendieck polynomials
    $\longrightarrow$ joint with Thomas Lam and Mark Shimozono (arXiv:0901.1506)

Outline

Affine crystals

KR crystals

Perfectness

Affine Schubert calculus

Motivation

$\mathfrak{g}$ Lie algebra/Kac–Moody Lie algebra

- Crystal bases are combinatorial bases for $U_q(\mathfrak{g})$ as $q \to 0$
- Affine finite crystals:
  - appear in 1d sums of exactly solvable lattice models
  - path realization of integrable highest weight $U_q(\mathfrak{g})$-modules
  - fermionic formulas, generalized Kostka polynomials, symmetric functions
  - fusion/quantum cohomology structure constants
- Irreducible finite-dimensional affine $U_q(\mathfrak{g})$-modules classified by Chari-Pressley via Drinfeld polynomials
- HKOTY conjectured that the Kirillov-Reshetikhin modules $W^{r,s}$ have crystal bases

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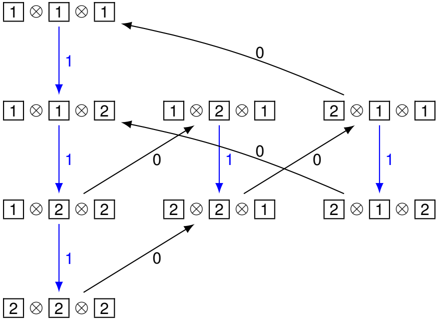
Crystal graph



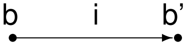
Axiomatic Crystals

A $U_q(\mathfrak{g})$-crystal is a nonempty set B with maps

$$\mathrm{wt}\colon B \to P$$
$$e_i, f_i\colon B \to B \cup \{\emptyset\} \quad \text{for all } i \in I$$

satisfying

$$f_i(b) = b' \Leftrightarrow e_i(b') = b \qquad \text{if } b, b' \in B$$
$$\mathrm{wt}(f_i(b)) = \mathrm{wt}(b) - \alpha_i \qquad \text{if } f_i(b) \in B$$
$$\langle h_i\,, \mathrm{wt}(b)\rangle = \varphi_i(b) - \varepsilon_i(b)$$

Write 

for $b' = f_i(b)$

Tensor products

Definition

B, B' crystals

$B \otimes B'$ is $B \times B'$ as sets with

$$\mathrm{wt}(b \otimes b') = \mathrm{wt}(b) + \mathrm{wt}(b')$$

$$f_i(b \otimes b') = \begin{cases} f_i(b) \otimes b' & \text{if } \varepsilon_i(b) \geq \varphi_i(b') \\ b \otimes f_i(b') & \text{otherwise} \end{cases}$$

$$\underbrace{\overset{b}{- - -}}_{\varphi_i(b)} \otimes \underbrace{\overset{b'}{+ + + +}}_{\varepsilon_i(b')}$$

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Outline

Existence of Kirillov-Reshetikhin crystals

Theorem (OS 07)

The Kirillov-Reshetikhin crystals $B^{r,s}$ exist for nonexceptional types.

Proof uses results on characters by Nakajima and Hernandez.

Combinatorial models for these crystals can be constructed using the classical decompositions

$$B^{r,s} \cong \bigoplus_{\Lambda} B(\Lambda)$$

and the automorphism σ (i special node $\sigma(i) = 0$)

$$f_0 = \sigma^{-1} \circ f_i \circ \sigma$$

$$e_0 = \sigma^{-1} \circ e_i \circ \sigma$$

or using the virtual crystal construction

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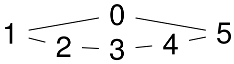
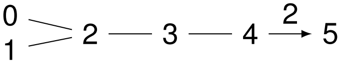
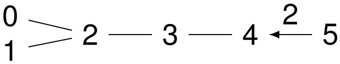
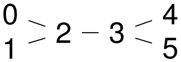
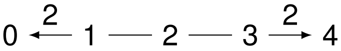
Dynkin diagrams

Type	Dynkin diagram
$A_5^{(1)}$	
$B_5^{(1)}$	
$A_9^{(2)}$	
$D_5^{(1)}$	
$C_5^{(1)}$	
$D_5^{(2)}$	
$A_{10}^{(2)}$	

Type $A_{n-1}^{(1)}$

KMN[2] proved existence of crystals $B^{r,s}$ for Kirillov-Reshetikhin modules $W^{r,s}$

$$B^{r,s} \cong B(s^r) \quad \text{as } \{1, 2, \ldots, n-1\}\text{-crystal}$$

automorphism σ

Promotion operator pr uniquely defined by Shimozono

$$\begin{array}{ccc} B^{r,s} & \xrightarrow{\mathrm{pr}} & B^{r,s} \\ \downarrow f_a & & \downarrow f_{a+1} \\ B^{r,s} & \xrightarrow[\mathrm{pr}]{} & B^{r,s} \end{array}$$

$$\langle h_{a+1}\,,\, \mathrm{wt}(\mathrm{pr}(b))\rangle = \langle h_a\,,\, \mathrm{wt}(b)\rangle$$

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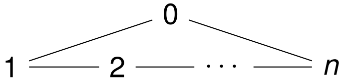

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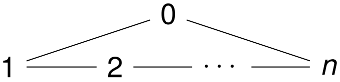

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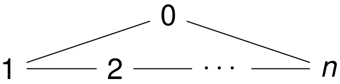

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Then $\quad e_0 = \mathrm{pr}^{-1} \circ e_1 \circ \mathrm{pr} \quad f_0 = \mathrm{pr}^{-1} \circ f_1 \circ \mathrm{pr}$

Promotion for type A_{n-1}

Classical crystal: $B(s^r)$ set of Young tableaux of shape (s^r) over alphabet $\{1, 2, \ldots, n\}$

Promotion:

- Remove righmost n, play jeu de taquin and repeat.
- Increase all entries by one and place 1's in the empty spaces.

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2	3	3
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Types $B_n^{(1)}$, $D_n^{(1)}$, $A_{2n-1}^{(2)}$

$$B^{r,s} \cong V^{r,s} \cong \bigoplus_{\Lambda} B(\Lambda) \qquad \text{as } \{1,2,\ldots,n\}\text{-crystal}$$

where Λ is obtained from $s\Lambda_r$ by removing $\begin{array}{|c|}\hline \\ \hline \\ \hline\end{array}$

Dynkin diagram automorphism σ interchanging 0 and 1

$$f_0 = \sigma \circ f_1 \circ \sigma$$
$$e_0 = \sigma \circ e_1 \circ \sigma$$

Theorem (OS 07)

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Classical decomposition

By construction

$$V^{r,s} \cong \bigoplus_{\Lambda} B(\Lambda)$$

as $X_n = D_n, B_n, C_n$ crystals

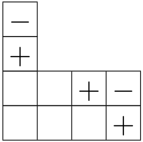
$\Rightarrow$ crystal arrows f_i, e_i are fixed for $i = 1, 2, \ldots, n$ using Kashiwara-Nakashima tableaux

Definition of σ

$X_n \to X_{n-1}$ branching

$$B_{X_n}(\Lambda) \cong \bigoplus_{\substack{\pm \text{ diagrams } P \\ \text{outer}(P) = \Lambda}} B_{X_{n-1}}(\text{inner}(P))$$

$\pm$ diagrams



$\lambda \subset \mu \subset \Lambda$

inner shape (λ) outer shape (Λ)

Λ/μ horizontal strip filled with $-$

μ/λ horizontal strip filled with $+$

Definition of σ

X_{n-1} highest weight vectors
are in bijection with $\pm$ diagrams via Φ

$\Phi :$

$-$			
$+$			
		$+$	$-$
			$+$

$\mapsto$

$\bar{4}$			
4			
2	3	3	$\bar{1}$
1	1	2	2

$\vec{a} = (1, 2, \quad 1, 2, 3, 4, 5, 6, 4, \quad 1, 2, 3, 4, 5, 6, 4, 3, 2)$

$\Phi(P) = f_{\vec{a}}$

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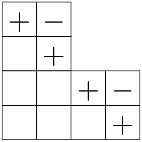
Definition of σ

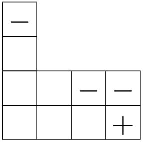
σ on $\pm$ diagrams

P $\pm$ diagram of shape Λ/λ

columns of height h in λ

$$\begin{aligned} h \not\equiv r \bmod 2: &\quad \text{interchange number of} \\ &\quad + \text{ and } - \text{ above } \lambda \\ h \equiv r \bmod 2: &\quad \text{interchange number of} \\ &\quad \mp \text{ and empty above } \lambda \end{aligned}$$

$P =$ 

$\mathfrak{S}(P) =$ 

$r \geq 6$
$s = 5$

Definition of σ

σ on tableaux

$b \in V^{r,s}$

$$e_{\vec{\mathbf{a}}} := e_{a_1} \cdots e_{a_\ell} \quad \text{such that } e_{\vec{\mathbf{a}}}(b) \text{ is } X_{n-1} \text{ highest weight vector}$$

$$f_{\overleftarrow{\mathbf{a}}} := f_{a_\ell} \cdots f_{a_1}$$

Then

$$\sigma(b) = f_{\overleftarrow{\mathbf{a}}} \circ \Phi \circ \mathfrak{S} \circ \Phi^{-1} \circ e_{\vec{\mathbf{a}}}(b)$$

Example

$V^{4,5}$ of type $D_6^{(1)}$

$$b = \begin{array}{|c|c|cc} \hline \bar{4} & \bar{4} & & \\ \hline 3 & 4 & & \\ \hline 2 & 3 & \multicolumn{1}{|c|}{\bar{1}} & \multicolumn{1}{c|}{\bar{1}} \\ \hline 1 & 1 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{3} \\ \hline \end{array} \xrightarrow{e_4 e_6 e_5 e_4 e_2} \begin{array}{|c|c|cc} \hline 4 & \bar{4} & & \\ \hline 3 & 4 & & \\ \hline 2 & 3 & \multicolumn{1}{|c|}{\bar{1}} & \multicolumn{1}{c|}{\bar{1}} \\ \hline 1 & 1 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{2} \\ \hline \end{array}$$

$$\xrightarrow{\Phi^{-1}} \begin{array}{|c|c|cc} \hline + & - & & \\ \hline & + & & \\ \hline & & \multicolumn{1}{|c|}{-} & \multicolumn{1}{c|}{-} \\ \hline & & \multicolumn{1}{|c|}{} & \multicolumn{1}{c|}{+} \\ \hline \end{array} \xrightarrow{\mathfrak{S}} \begin{array}{|c|ccc} \hline - & & & \\ \hline & & & \\ \hline & \multicolumn{1}{|c|}{} & \multicolumn{1}{c|}{+} & \multicolumn{1}{c|}{-} \\ \hline & \multicolumn{1}{|c|}{} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{+} \\ \hline \end{array}$$

$$\xrightarrow{\Phi} \begin{array}{|c|ccc} \hline \bar{3} & & & \\ \hline 4 & & & \\ \hline 3 & \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{\bar{1}} \\ \hline 1 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{2} \\ \hline \end{array} \xrightarrow{f_2 f_4 f_5 f_6 f_4} \begin{array}{|c|ccc} \hline \bar{2} & & & \\ \hline \bar{4} & & & \\ \hline 3 & \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{\bar{1}} \\ \hline 1 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{2} \\ \hline \end{array} = \sigma(b)$$

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$$
b = \begin{array}{|c|c|cc}
\hline
\bar{4} & \bar{4} \\
\hline
3 & 4 \\
\hline
2 & 3 & \multicolumn{1}{|c|}{\bar{1}} & \multicolumn{1}{c|}{\bar{1}} \\
\hline
1 & 1 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{3} \\
\hline
\end{array}
\xrightarrow{e_4 e_6 e_5 e_4 e_2}
\begin{array}{|c|c|cc}
\hline
4 & \bar{4} \\
\hline
3 & 4 \\
\hline
2 & 3 & \multicolumn{1}{|c|}{\bar{1}} & \multicolumn{1}{c|}{\bar{1}} \\
\hline
1 & 1 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{2} \\
\hline
\end{array}
$$

$$
\xrightarrow{\Phi^{-1}}
\begin{array}{|c|c|cc}
\hline
+ & - \\
\hline
 & + \\
\hline
 & & \multicolumn{1}{|c|}{-} & \multicolumn{1}{c|}{-} \\
\hline
 & & \multicolumn{1}{|c|}{} & \multicolumn{1}{c|}{+} \\
\hline
\end{array}
\xrightarrow{\mathfrak{S}}
\begin{array}{|c|c|c|c|}
\hline
- \\
\cline{1-1}
 \\
\hline
 & & + & - \\
\hline
 & & & + \\
\hline
\end{array}
$$

$$
\xrightarrow{\Phi}
\begin{array}{|c|ccc}
\hline
\bar{3} \\
\hline
4 \\
\hline
3 & \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{\bar{1}} \\
\hline
1 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{2} \\
\hline
\end{array}
\xrightarrow{f_2 f_4 f_5 f_6 f_4}
\begin{array}{|c|ccc}
\hline
\bar{2} \\
\hline
\bar{4} \\
\hline
3 & \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{\bar{1}} \\
\hline
1 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{2} \\
\hline
\end{array}
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$$

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$$
b = \begin{array}{cccc}
\bar{4} & \bar{4} & & \\
3 & 4 & & \\
2 & 3 & \bar{1} & \bar{1} \\
1 & 1 & 2 & 3
\end{array}
\xrightarrow{e_4 e_6 e_5 e_4 e_2}
\begin{array}{cccc}
4 & \bar{4} & & \\
3 & 4 & & \\
2 & 3 & \bar{1} & \bar{1} \\
1 & 1 & 2 & 2
\end{array}
$$

$$
\xrightarrow{\Phi^{-1}}
\begin{array}{cccc}
+ & - & & \\
 & + & & \\
 & & - & - \\
 & & & +
\end{array}
\xrightarrow{\mathfrak{S}}
\begin{array}{cccc}
- & & & \\
 & & & \\
 & & + & - \\
 & & & +
\end{array}
$$

$$
\xrightarrow{\Phi}
\begin{array}{cccc}
\bar{3} & & & \\
4 & & & \\
3 & 3 & 3 & \bar{1} \\
1 & 2 & 2 & 2
\end{array}
\xrightarrow{f_2 f_4 f_5 f_6 f_4}
\begin{array}{cccc}
\bar{2} & & & \\
\bar{4} & & & \\
3 & 3 & 3 & \bar{1} \\
1 & 2 & 2 & 2
\end{array}
= \sigma(b)
$$

Type $C_n^{(1)}$

$$B^{r,s} \cong V^{r,s} \cong \bigoplus_{\Lambda} B(\Lambda) \qquad \text{as } \{1,2,\ldots,n\}\text{-crystal}$$

where Λ is obtained from $s\Lambda_r$ by removing $\square\square$

Virtual crystal: ambient crystal $\hat{V}^{r,s} = B^{r,s}$ of type $A_{2n+1}^{(2)}$

Definition

$V^{r,s}$ is the subset of $b \in \hat{V}^{r,s}$ such that $\sigma(b) = b$ such that

$$e_i = \begin{cases} \hat{e}_0 \hat{e}_1 & \text{for } i = 0 \\ \hat{e}_{i+1} & \text{for } 1 \le i \le n \end{cases}$$

Theorem (FOS 08)

$V^{r,s} \cong B^{r,s}$ as a $\{0,1,\ldots,n\}$-crystal

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Types $D_{n+1}^{(2)}$, $A_{2n}^{(2)}$

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Virtual crystal: ambient crystal $\hat{V}^{r,s} = B^{r,2s}$ of type $C_n^{(1)}$

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$V^{r,s} \subset \hat{V}^{r,s}$ is given by $e_i = \hat{e}_i^{m_i}$ where

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Outline

Perfectness of KR crystals

Conjecture (HKOTT)

The KR crystal $B^{r,s}$ is perfect if and only if $\frac{s}{c_r}$ is an integer.
If $B^{r,s}$ is perfect, its level is $\frac{s}{c_r}$.

	$(c_1, \ldots, c_n)$
$B_n^{(1)}$	$(1, \ldots, 1, 2)$
$C_n^{(1)}$	$(2, \ldots, 2, 1)$
other nonexceptional	$(1, \ldots, 1)$

Theorem (FOS 08)

If $\mathfrak{g}$ is of nonexceptional type, the Conjecture is true.

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Definition of perfectness

$P = \oplus_{i \in I} \mathbb{Z}\Lambda_i$ weight lattice of $\mathfrak{g}$, P^+ set of dominant weights.
$P_\ell^+ = \{\Lambda \in P^+ \mid \mathrm{lev}(\Lambda) = \ell\}$ level ℓ dominant weights

$$\varepsilon(b) = \sum_{i \in I} \varepsilon_i(b)\Lambda_i \quad \text{and} \quad \varphi(b) = \sum_{i \in I} \varphi_i(b)\Lambda_i$$

Definition

The crystal $\mathcal{B}$ is perfect of level ℓ if:

1. $\mathcal{B} \cong$ crystal graph of a finite-dimensional $U_q'(\mathfrak{g})$-module.
2. $\mathcal{B} \otimes \mathcal{B}$ is connected.
3. $\exists\, \lambda \in P_0$ such that $\mathrm{wt}(\mathcal{B}) \subset \lambda + \sum_{i \in I_0} \mathbb{Z}_{\le 0}\alpha_i$ and $\exists$ unique element in $\mathcal{B}$ of classical weight λ.
4. $\forall\, b \in \mathcal{B}$, $\mathrm{lev}(\varepsilon(b)) \ge \ell$.
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Example: $B^{2,2}$ of type $C_3^{(1)}$

$$B^{2,2} \cong B(2\Lambda_2) \oplus B(2\Lambda_1) \oplus B(0).$$

Bijection $\varepsilon : B^{2,2}_{\min} \to P_1^+$ given by:

b	$\varepsilon(b) = \varphi(b)$
$\emptyset$	Λ_0
$\begin{array}{\|c\|c\|}\hline 1 & \bar{1} \\ \hline\end{array}$	Λ_1
$\begin{array}{\|c\|c\|}\hline 2 & \bar{1} \\ \hline 1 & \bar{2} \\ \hline\end{array}$	Λ_2
$\begin{array}{\|c\|c\|}\hline 3 & \bar{2} \\ \hline 2 & \bar{3} \\ \hline\end{array}$	Λ_3

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Example: $B^{2,1}$ of type $C_3^{(1)}$

$$B^{2,1} \cong B(\Lambda_2).$$

$B^{2,1}$ is not perfect. ε is not a bijection:

b	$\varepsilon(b)$
$\begin{array}{\|c\|}\hline 2 \\ \hline 1 \\ \hline \end{array}$	Λ_0
$\begin{array}{\|c\|}\hline 3 \\ \hline 2 \\ \hline \end{array}$ $\begin{array}{\|c\|}\hline \bar{2} \\ \hline 2 \\ \hline \end{array}$	Λ_1
$\begin{array}{\|c\|}\hline \bar{3} \\ \hline 3 \\ \hline \end{array}$ $\begin{array}{\|c\|}\hline \bar{1} \\ \hline \bar{2} \\ \hline \end{array}$	Λ_2
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$\begin{array}{\|c\|}\hline 2 \\ \hline 1 \\ \hline\end{array}$		Λ_0
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$\begin{array}{\|c\|}\hline \bar{3} \\ \hline 3 \\ \hline\end{array}$	$\begin{array}{\|c\|}\hline \bar{1} \\ \hline \bar{2} \\ \hline\end{array}$	Λ_2
$\begin{array}{\|c\|}\hline \bar{2} \\ \hline \bar{3} \\ \hline\end{array}$		Λ_3

Kyoto path model

$B(\Lambda)$ highest weight infinite-dimensional crystal of type $\mathfrak{g}$
$u_\Lambda \in B(\Lambda)$ highest weight vector

Theorem (KMN[2])

$\Lambda \in P_s^+$
$B^{r_1,s}, B^{r_2,s}, \ldots$ *perfect of level-s*

$$\Phi : B(\Lambda) \cong \cdots \otimes B^{r_2,s} \otimes B^{r_1,s} \otimes B(\widetilde{\Lambda})$$

$\mathcal{B}$ perfect
$\mathcal{B}_{\min} = \{b \in \mathcal{B} \mid \mathrm{lev}(\varepsilon(b)) = s\}$
$\varepsilon, \varphi : \mathcal{B}_{\min} \to P_s^+$ are bijections
Induced automorphism $\tau = \varphi \circ \varepsilon^{-1}$ on P_s^+
Ground state $\Phi(u_\Lambda) = \cdots \otimes b_{\tau^2(\Lambda)} \otimes b_{\tau(\Lambda)} \otimes b_\Lambda$

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$B^{r_1,s}, B^{r_2,s}, \ldots$ *perfect of level-s*

$$\Phi : B(\Lambda) \cong \cdots \otimes B^{r_2,s} \otimes B^{r_1,s} \otimes B(\widetilde{\Lambda})$$

$\mathcal{B}$ perfect
$\mathcal{B}_{\min} = \{b \in \mathcal{B} \mid \mathrm{lev}(\varepsilon(b)) = s\}$
$\varepsilon, \varphi : \mathcal{B}_{\min} \to P_s^+$ are bijections
Induced automorphism $\tau = \varphi \circ \varepsilon^{-1}$ on P_s^+
Ground state $\Phi(u_\Lambda) = \cdots \otimes b_{\tau^2(\Lambda)} \otimes b_{\tau(\Lambda)} \otimes b_\Lambda$

Kyoto path model

$B(\Lambda)$ highest weight infinite-dimensional crystal of type $\mathfrak{g}$
$u_\Lambda \in B(\Lambda)$ highest weight vector

Theorem (KMN[2])

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Level-s adjoint KR crystals

Adjoint KR crystals:
Take r to correspond to highest root θ.
Classical decomposition [Chari]:

$$B^{r,s} \cong \bigoplus_{0 \le k \le s} B(k\Lambda_r)$$

Question: Can we find level-s KR crystals of all types?

Answer:

- Benkart et al. gave a uniform construction of level-1 perfect crystals for all types
- Exceptional types:
  - Yamane type $G_2^{(1)}$
  - Kashiwara, Misra, Okada, Yamada type $D_4^{(3)}$
  - see Brant Jones' talk for level-s type $E_6^{(1)}, \ldots$

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Outline

Schubert calculus

- **Enumerative Geometry:** counting subspaces satisfying certain intersection conditions (Hilbert's 15th problem) Schubert, Pieri, Giambelli,... 1874
- **Cohomology:** computations in cohomology ring of the Grassmannian $H^*(G/P)$ with $G = SL_n(\mathbb{C})$ and $P \subset G$ maximal parabolic 1950's
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Affine Schubert calculus

Definition

G affine Kac–Moody group
$P \subset G$ maximal parabolic subgroup
G/P affine Grassmannian Gr

Example: $\mathcal{K} = \mathbb{C}((t))$, $\mathcal{O} = \mathbb{C}[[t]]$
affine Grassmannian $Gr = SL_{k+1}(\mathcal{K})/SL_{k+1}(\mathcal{O})$

Theorem (Lam)

Schubert bases of $H_*(Gr)$ and $H^*(Gr)$ are given by k-Schur functions and affine Stanley symmetric functions of Lascoux, Lapointe, Morse

Structure constants include genus zero Gromov-Witten invariants or fusion coefficients

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nilHecke algebra

Definition (nilHecke algebra)

The nilHecke algebra

- generators $A_1, \ldots, A_{n-1}$
- relations

$$A_i A_j = A_j A_i \qquad \text{for } |i-j| \geq 2$$
$$A_i A_{i+1} A_i = A_{i+1} A_i A_{i+1}$$
$$A_i^2 = 0$$

Stanley symmetric functions for other types

- For each Weyl group W one can construct a new nilHecke algebra by taking the associated graded $\mathbb{C}[W]$.
- Finding Stanley symmetric functions for each W is equivalent to finding a particular commutative subalgebra of the nilHecke algebra.

Theorem (Lam; LSS 07)

Schubert bases of $H_*(Gr)$ and $H^*(Gr)$ are given by k-Schur functions and affine Stanley symmetric functions for type $A_n^{(1)}$ and $C_n^{(1)}$.

Theorem (LSS 09)

Schubert bases of $K_*(Gr)$ and $K^*(Gr)$ are given by K-k-Schur functions and affine stable Grothendieck polynomials.

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Relation to KR crystals

k-Schur functions $s_\lambda^{(k)}$

Structure coefficients $s_\lambda^{(k)} s_\mu^{(k)} = \sum_\nu c_{\lambda\mu}^{k,\nu} s_\nu^{(k)}$

Observation: (inspired by Postnikov and Stroppel/Korff)

- s_λ evaluated at crystal operators acting on $B^{1,k}$ of type $A_{n-1}^{(1)}$ yields fusion coefficients
- s_λ evaluated at crystal operators acting on $B^{n,1}$ of type $A_{n+k-1}^{(1)}$ yields quantum cohomology structure coefficients

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