≪ Quantum Groups & Crystal Bases V ≫

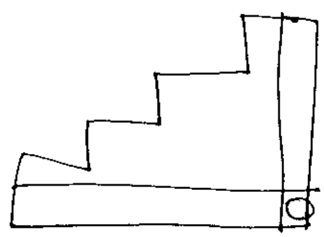
6/19/09
Ottawa

7. Young walls

$\mathfrak{g} = A_{n-1}^{(1)} = \widehat{\mathfrak{sl}}_n$, $\lambda = \Lambda_0$

$\boxed{\text{Def}}$ ① Y is a colored Young diagram on Λ_0

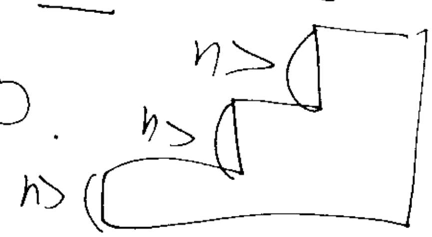
if $Y = (y_k)_{k \geq 0} =$



, $y_{k+1} \leq y_k \ \forall k \geq 0$.

② Y is an n-reduced colored YD on Λ_0.

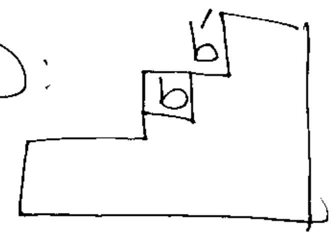
if $0 \leq y_k - y_{k+1} < n \ \forall k \geq 0$.



Notation; $\mathcal{Z}(\Lambda_0) = \{\text{colored YD's on } \Lambda_0\}$

$\mathcal{Y}(\Lambda_0) = \{\text{n-reduced} \quad " \quad \}$

$\boxed{\text{Def}}$ (1) $\boxed{b}$ is an <u>removable</u> <u>i-block</u> if it can be removed to get a colored YD:



(2) $\underline{b'|}$ is an <u>admissible</u> <u>i-slot</u> if one can add an i-block to get a colored YD:

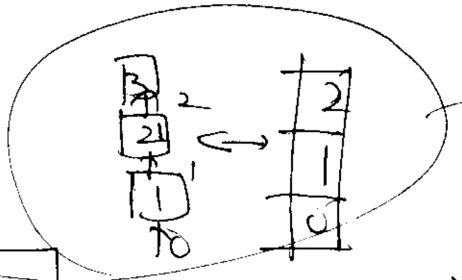
Assign $+$ to an admissible i-block (the column if its top)
$-$ " removable i-block

Cancel $(+,-)$-pair $\rightsquigarrow (- \; - \cdots - \; + \; + \; + \; +)$
$*$ ☆

Define $\tilde{e}_i Y = Y \nearrow \boxed{i}$ at $*$
$\tilde{f}_i Y = Y \swarrow \boxed{i}$ at ☆

$\mathrm{wt}\, Y = \Lambda_0 - \sum_{i \in I} k_i \alpha_i$, $k_i = \#$ of i-blocks in Y

$\varepsilon_i(Y) = \#$ of $-$'s, $\varphi_i(Y) = \#$ of $+$'s.



ideal pr ③

Thm (Misra–Miwa 91)

① $\mathcal{Z}(\Lambda_0)$ is a $U_q(A^{(1)}_{n-1})$-crystal

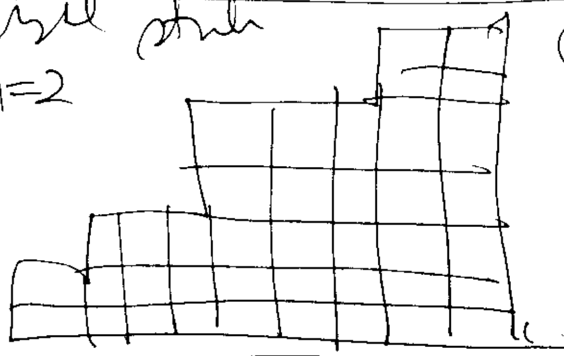
② $\mathcal{Y}(\Lambda_0) = \mathcal{C}(\emptyset) \cong B(\Lambda_0)$.

$Y \longleftrightarrow \mathbb{P}$

☆

(Example)

① Crystal struct $n=2$

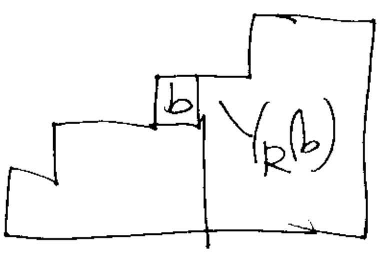


② $\mathcal{Y}(\Lambda_0)$: 3-reduced YD's

③ $\longleftarrow \mathbb{P} = (P_k)_{k\geq 0}$

P_k = top

Let $Y =$

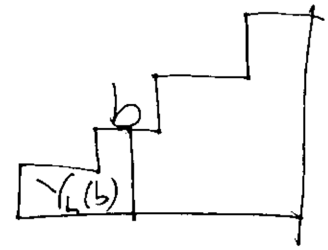


, b : removable i-block.

Set $Y_R(b)$ = "right of b"

$$R_i(b) = \varphi_i(Y_R(b)) - \varepsilon_i(Y_R(b))$$

Define
$$e_i Y = \sum_{b:\ \text{removable } i\text{-block}} q^{-R_i(b)} (Y \nearrow b)$$

b : admissible i-slot in Y = 

Set

$Y_L(b)$ = "left of b", $L_i(b) = \varphi_i(Y_L(b)) - \varepsilon_i(Y_L(b))$.

Defn

$$f_i Y = \sum_{b:\text{ admissible } i\text{-slot}} q^{L_i(b)} (Y \swarrow b)$$

Clearly, $q^h Y = q^{\langle h, \mathrm{wt}\, Y \rangle} Y$.

Thm (Hayashi, Misra-Miwa)

① $\mathcal{F}(\Lambda) = \bigoplus_{Z \in \mathcal{Z}(\Lambda_0)} \mathbb{Q}(q) Z$ becomes a $U_q(A^{(1)}_{n-1})$-module in the category $\mathcal{O}_{int}$.

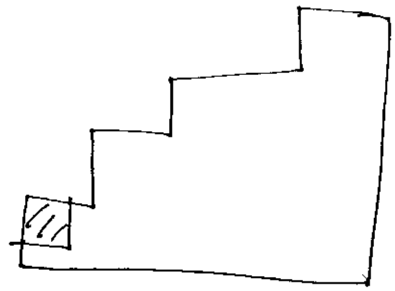
② $U_q(A^{(1)}_{n-1})\, \phi \cong V(\Lambda_0)$.

In particular,

Q: How to construct $G(\Lambda)$?

LLT algorithm

Y : n-reduced



Let b_1 be 1st removable block of color, say, i_1.

If $Y_1 = Y \nearrow b_1$ is reduced, stop ; $Y^{(1)} = Y \nearrow b_1 = Y_1$

If not, remove the next removable i_1-block b_1',

and set $Y_2 = Y_1 \nearrow b_1'$: If Y_2 is reduced, stop

& set $Y^{(1)} = Y_2$: if not, remove the next removable

i_1- block b_1'', ..., $Y^{(1)} = Y_{k_1}$: reduced.

Let b_2 be the 1st removable block of $Y^{(1)}$ of color,

say, i_2 : do the same procedure to get

$Y^{(2)} = Y^{(1)}_{k_2}$; ..., $Y^{(r)} = Y^{(r-1)}_{k_r} = \emptyset$

Set $A(Y) = f_{i_1}^{(k_1)} \cdots f_{i_r}^{(k_r)} \emptyset$.

Define a total order $Y_0 > Y_1 > \cdots > Y_t$ on $Z(\Lambda_0)$ refining the dominance order $\trianglerighteq$.

$L = A_0$-span of colored YD's.

$$\Longrightarrow \begin{cases} A(Y) = Y + \sum_{Z \lneq Y} A_{Y,Z}(q)\, Z \\ \overline{A(Y)} = A(Y) \end{cases} \quad \text{(Exercise)}$$

But $\underbrace{A(Y) \not\equiv Y \bmod qL}_{\text{in general}}$,

Let $Y = Y_0 > Y_1 > \cdots > Y_t$ be the n-reduced colored YD's on Λ_0 of $\mathrm{wt}(Y)$.

$\boxed{\text{Lemma}}$ $G(Y_t) = A(Y_t)$

Suppose we have computed $G(Y_{k+1}), \cdots, G(Y_t)$.

1) If $A_{Y_k, Y_{k+1}}(q) = \sum_{i=-r}^{r'} a_i q^i$, then set $\gamma_{k+1}(q) = \sum_{i=1}^{r} a_{-i}(q^i + q^{-i}) + a_0$.

2) If the coeff of Y_s $(s>k+1)$ in

$$A(Y_k)-\sum_{p=k+1}^{s-1}\gamma_p(q)G(Y_p) \text{ is } \sum_{i=-r}^{r'} a_i q^i, \text{ then}$$

set $\gamma_s(q)=\sum_{i=1}^{r} a_{-i}(q^i+\bar{q}^i)+a_0$.

$\rightarrow$ We obtain $\gamma_{k+1}(q), \gamma_{k+2}(q), \cdots, \gamma_t(q)$ s.t.

$$\overline{\gamma_s(q)}=\gamma_s(\bar{q})=\gamma_s(q) \quad \forall s\geq k+1.$$

Set $G(Y_k)=A(Y_k)-\gamma_{k+1}(q)G(Y_{k+1})-\cdots-\gamma_t(q)G(Y_t)$.

$$\Longrightarrow \begin{cases} \overline{G(Y_k)}=G(Y_k) \\ G(Y_k)\equiv Y_k \mod qL \end{cases}$$

(LLT)

Thm Continuing this procedure, we obtain

$$G(Y)=Y+\sum_{Z<Y}K_{Y,Z}(q)Z.$$

Note: $H_n(\xi)$, $\xi^n=1$, Hecke alg.

Z: $Y(D)\Rightarrow$ Specht module S^Z, indecomposable.

Thm $\{\text{irreducible } H_N(\xi)\text{-modules}\}$
$\cong \{ D^Y \mid Y : n\text{-reduced YD's with } N \text{ blocks} \}$

LLT conj = Ariki's Thm (1996)

$$K_{Y,Z}(1) = [S^Z : D^Y]$$

To summarize, we have

$U_q(A^{(1)}_{n-1})$	$H_N(\xi)$, $\xi^n = 1$
Z : YD	S^Z
Y : n-reduced	D^Y
$y(\Lambda_0) \cong B(\Lambda_0)$	$\bigoplus_{N=0}^{\infty} K_0(H_N(\xi))$
$F(\Lambda_0)$, LLT	$K_{Y,Z}(1) = [S^Z : D^Y]$

Actually, Ariki ~~correspond~~ proved:

$$\bigoplus_{N=0}^{\infty} K_0(H_N(\xi)) \rightsquigarrow B(\Lambda_0)$$

$$\bigoplus_{N=0}^{\infty} K_0(H_N^{\lambda}(\xi)) \rightsquigarrow B(\lambda)$$

$$\bigoplus_{N=0}^{\infty} K_0(\widehat{H}_N(\xi)) \rightsquigarrow B(\infty)$$

Q: What can we say for other quantum affine algebra?

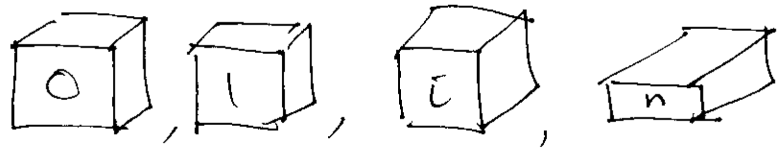
Another motivation: can we find a ~~realization~~ combinatorial realization of $B(\lambda)$ for other quantum affine algebras?

→ Combinatorics of Young walls.

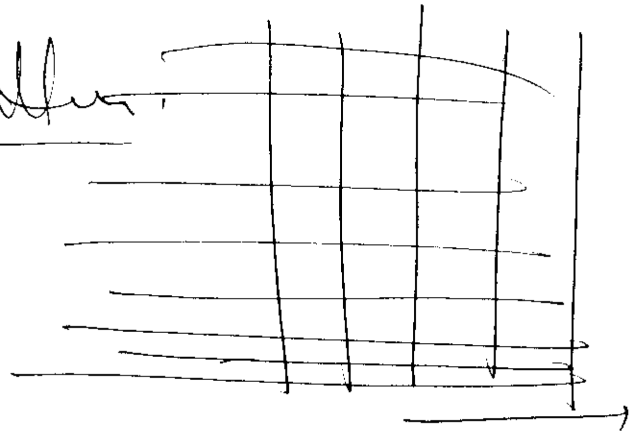
Idea: LEGO + Tetris

Let $\mathfrak{g} = B_n^{(1)}$, $\lambda = \Lambda_0, \Lambda_1$, or Λ_n

blocks:

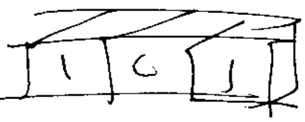


pattern:

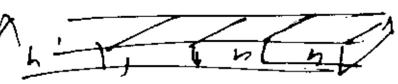


ground state wall

$\lambda = \Lambda_0$:



$\lambda = \Lambda_n$:

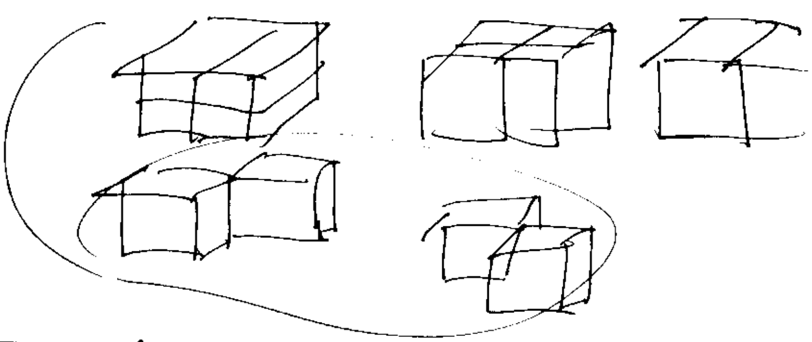


gravity

Def $Y = (y_k)_{k\geq 0}$ is a Young wall if i) pattern ii) gravity iii) $\exists$ no full columns of equal ht.

only if

(Example) not allowed



allowed

gravity

$\boxed{\text{Def}}$ ① δ-column = a cycle of path

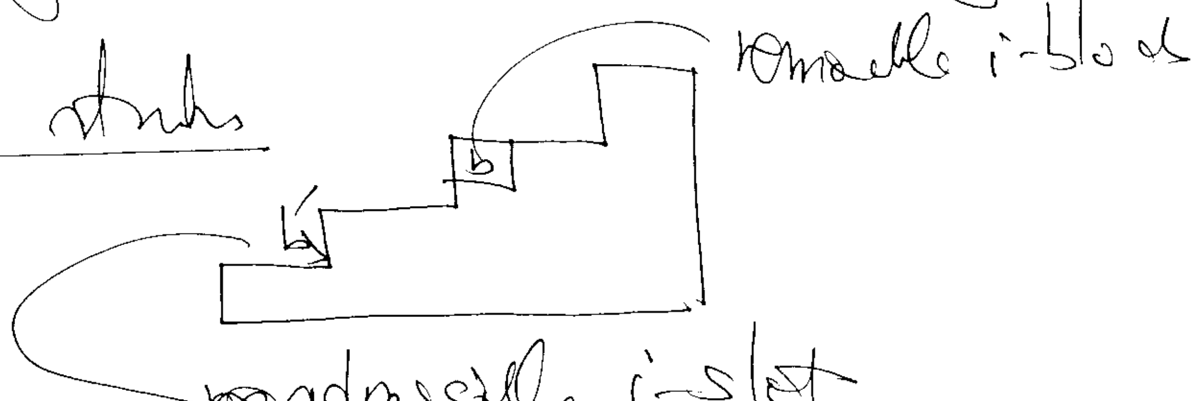
② removable δ-column if it can be removed to get a YW.

③ Y is reduced if $\exists$ no removable δ-column

Notation: $Z(\lambda) = \{$ Young walls on $\lambda \}$

$\mathcal{Y}(\lambda) = \{$ reduced YW's on $\lambda \}$

Crystal structure



assign $-$, $--$ to removable i-block

$+$, $++$ to admissible i-slot

$-+$ to removable/admissible i-slot

and $(+,-)$-pairs

$\longrightarrow (- - - - \underset{p}{-} \underset{r}{+} + + + +)$

$\tilde{e}_i Y = Y \nearrow \boxed{i}$ at $*$

$\tilde{f}_i Y = Y \swarrow \boxed{i}$ at $\bigstar$

$\mathrm{wt}\, Y = \lambda - \sum_{i \in I} k_i \alpha_i$, $k_i = \#$ of i-blocks

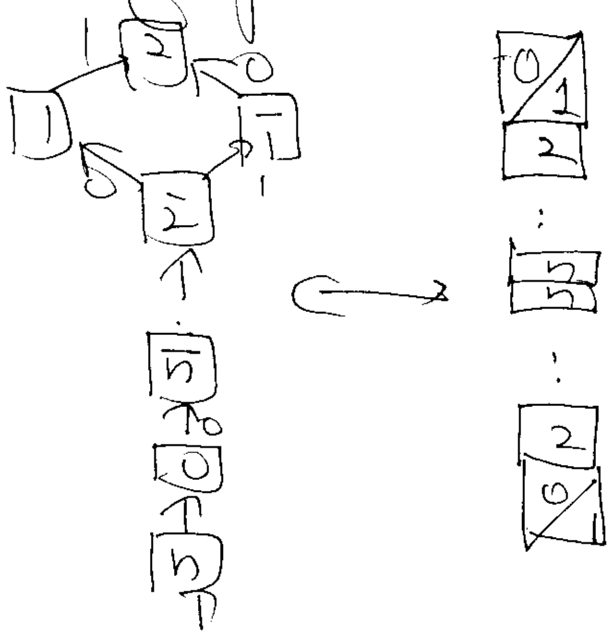
$\varepsilon_i(Y) = \#$ of $-$'s, $\varphi_i(Y) = \#$ of $+$'s

Thm (K2003) $\mathfrak{g}$: classical affine alg

① $Z(\lambda)$ is a $U_q(\mathfrak{g})$-crystal

② $\mathcal{Y}(\lambda) = \mathcal{C}(\emptyset) \cong B(\lambda)$.

⟨Idea of pf⟩



(type I) $e_i Y =$ $\qquad f_i Y =$

(type II) $e_i Y = \sum q_i^{-R_i(b)} (Y \nearrow b)$

$\qquad q_i^{-1} (1 - (-q_i^2)^{N+1}) Y \nearrow \boxed{1'}$

$f_i Y = \sum q_i^{L_i(b)} (Y \swarrow b) = q_i^{-1} (1 - (q_i^2)^{N+1}) Y \swarrow \boxed{1'}$

(type III) $\begin{cases} e_i Y = \sum \\ f_i Y = \sum \end{cases}$

$\boxed{\text{Thm}}$ (K & J.-H Kwon (2006))

1. $\mathcal{F}(\lambda)$ is a $U_q(\mathcal{G})$-module in $\mathcal{O}_{int}$
2. $\forall Y \in \mathcal{Y}(\lambda)$, $\exists$ an algorithm for computing

$$G(Y) = Y + \sum_{Z < Y} K_{Y,Z}(q) Z .$$

Rmk ① LHS is complete; i.e. RHS = ?
② Hyperbolic? Fixed sp? LLT-Ariki theory?
③ Young wall model for $\mathbb{B} = B^{ad}$?
④ LLT-Ariki theory for $\mathbb{B} = B^{ad}$?