≪Quantum Groups & Crystal Bases III≫

6/18/09
Ottawa

4. Crystal Bases

$M=\bigoplus_{\lambda\in P} M_\lambda \in \mathcal{O}_{int}$, $u\in M_\lambda$

Fix $i\in I$: $\exists !$ expression

$$u=\sum_{k\geq 0} f_i^{(k)} u_k, \quad e_i u_k=0, \quad f_i^{(k)}=f_i^k/[k]_i!$$

$\rightsquigarrow$ i-string decomp of u

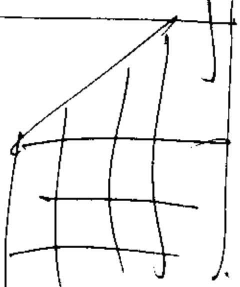
(idea: sl_2-decomp + loc. nilp.)

Define the Kashiwara operators by

$$\tilde{e}_i u=\sum_{k\geq 1} f_i^{(k-1)} u_k, \quad \tilde{f}_i u=\sum_{k\geq 0} f_i^{(k+1)} u_k .$$

(Exercise): U_i-sl-module decomp

Uniqueness: Apply $e_i^{(k)}$



~~Def A crystal basis~~

$$A_0 = \{ f/g \in \mathbb{C}(q) \mid f, g \in \mathbb{C}[q],\ g(0) \neq 0 \}$$

$\boxed{\text{Def}}$ A crystal basis of M is a pair (L, B), where

i) L = free A_0-submodule of M s.t. $M = \mathbb{C}(q) \otimes_{A_0} L$

ii) B is a $\mathbb{C}$-basis of $L/qL \cong \mathbb{C} \otimes_{A_0} L$

iii) $L = \bigoplus_{\lambda \in P} L_\lambda$, where $L_\lambda = L \cap M_\lambda$

iv) $B = \coprod_{\lambda \in P} B_\lambda$, where $B_\lambda = B \cap (L_\lambda / q L_\lambda)$

v) $\tilde{e}_i L \subset L$, $\tilde{f}_i L \subset L$ $\forall i \in I$

vi) $\tilde{e}_i B \subset B \cup \{0\}$, $\tilde{f}_i B \subset B \cup \{0\}$ $\forall i \in I$

vii) $\forall b, b' \in B$, $\forall i \in I$, $\tilde{f}_i b = b' \Leftrightarrow b = \tilde{e}_i b'$.

$$M \supset L \longrightarrow L/qL \supseteq B, \quad G(B)$$

Define $b \xrightarrow{i} b' \iff \tilde{f}_i b = b'$

$\rightsquigarrow$ (B, arrows) : crystal (graph) of M.

$\boxed{\text{Rmk}}$ (1) Almost all combinatorial features of M are reflected in B :

(e.g.) $\dim_{\mathbb{Q}(q)} M_\lambda = \operatorname{rk}_{A_0} L_\lambda = \dim_{\mathbb{C}} L_\lambda / qL_\lambda = \# B_\lambda$

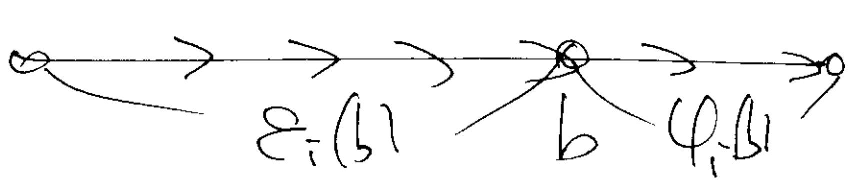
$\rightsquigarrow$ One can compute ch M!

(2) Moreover, the crystals behave very nicely w.r.t. taking the tensor product.

B : crystal of M, $b \in B$

Def $\varepsilon_i(b) = \max\{k \geq 0 \mid \tilde{e}_i^k b \neq 0\}$

$\varphi_i(b) = \max\{k \geq 0 \mid \tilde{f}_i^k b \neq 0\}$



(L_j, B_j) : crystal basis of M_j $(j=1,2)$

$L = L_1 \otimes_{A_0} L_2$, $B = B_1 \times B_2$

$\boxed{\text{Thm}}$ (Tensor product rule)

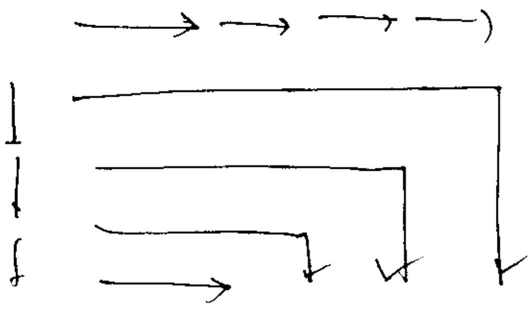
(L, B) is a crystal basis of $M_1 \otimes_{\mathbb{Q}(q)} M_2$,

where
$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes \tilde{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2) \end{cases}$$

$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2). \\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2) \end{cases}$$

<pf> Reduce to sl_2-case
Induction on $\dim M_2$.

(Example)



Clebsch-Gordan decomposition.

Def $b \in B$ is a maximal vector if $\tilde{e}_i b = 0$ $\forall i \in I$

Cor ~~① $B_1 \otimes B_2$:~~

① B_1, B_2: crystals

$b_1 \otimes b_2$ is a maximal vector in $B_1 \otimes B_2$

$\iff$ (b_1 is a maximal vector
$\varepsilon_i(b_2) \le \varphi_i(b_1) = \langle h_i, \mathrm{wt}\, b_1 \rangle$ $\forall i \in I$.

② $B_1, \dots, B_r$: crystals

$b_1 \otimes \cdots \otimes b_r$ is a maximal vector in $B_1 \otimes \cdots \otimes B_r$

$\iff$ $b_1 \otimes \cdots \otimes b_k$ is a maximal vector $\forall k = 1, \dots, r$.

<pf> (Easy exercise.)

$B_1, \dots, B_r$: crystals

$b_1 \otimes \cdots \otimes b_r \in B_1 \otimes \cdots \otimes B_r$

Write $-$'s & $+$'s under b_k
$\varepsilon_i(b_k)$ many, $\varphi_i(b_k)$ many

$$b_1 \quad \otimes \quad b_2 \quad \otimes \quad \cdots \otimes \quad b_r$$

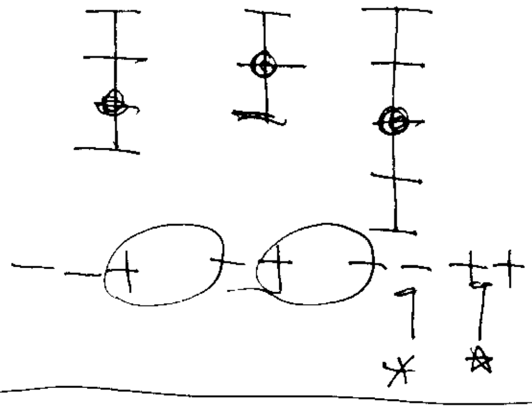
$$- - - + + + \qquad - - - + + + \qquad \cdots \qquad - - - + + +$$

Cancel $(+,-)$ pairs; $m_i(- - - + + + +)$

$$b_1 \otimes b_2 \otimes \cdots \otimes b_k \otimes \cdots \otimes b_l \otimes \cdots \otimes b_r$$

$$\Rightarrow \begin{cases} \tilde e_i(b_1 \otimes \cdots \otimes b_r) = b_1 \otimes \cdots \otimes \tilde e_i b_k \otimes \cdots \otimes b_r \\ \tilde f_i(b_1 \otimes \cdots \otimes b_r) = b_1 \otimes \cdots \otimes \cdots \otimes \tilde f_i b_l \otimes \cdots \otimes b_r \end{cases}$$

(Example) $b_1 \otimes b_2 \otimes b_3 \in B_1 \otimes B_2 \otimes B_3 \cong B(3) \otimes B(2) \otimes B(4)$



$\boxed{\text{Thm}}$ (Kashiwara 91)

$V(\lambda) = U_q(g) v_\lambda \quad \lambda \in P^+$

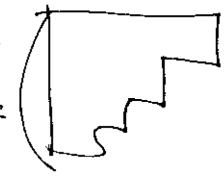
$L(\lambda) = A_0$-submodule of $V(\lambda)$ generated by $\tilde f_{i_1} \cdots \tilde f_{i_r} v_\lambda$ $\left(\begin{smallmatrix} r \geq 0 \\ i_k \in I \end{smallmatrix}\right)$

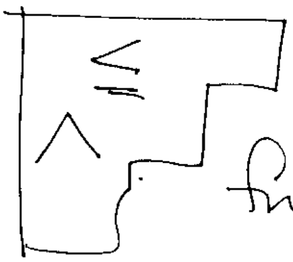
$B(\lambda) = \{\tilde f_{i_1} \cdots \tilde f_{i_r} v_\lambda + qL(\lambda)\} \setminus \{0\}$

$\Rightarrow (L(\lambda), B(\lambda))$ is a (unique) crystal basis of $V(\lambda)$.

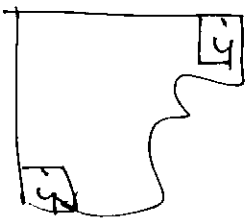
Problem How to realize $B(\lambda)$?

(Example) $\mathfrak{g} = \mathfrak{gl}_n$, $\lambda \in P^+$ — partition with at most n rows

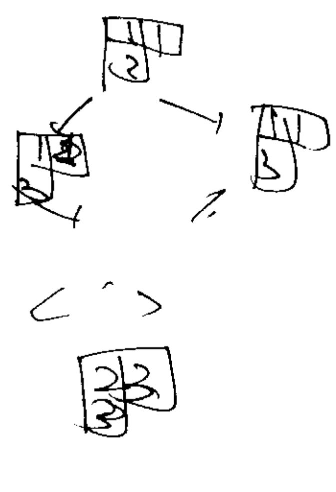
$= \quad n \geq$ 

T is a <u>semistandard tableau</u> of shape λ $\iff$ $T =$  with entries from $\{1, 2, \cdots, n\}$

$\Longrightarrow B(\lambda) \cong \{\text{semistandard tableaux of shape } \lambda\}$,

where  $\xrightarrow{\text{Far-Eastern reading}} \boxed{i_1} \otimes \boxed{i_2} \otimes \cdots \otimes \boxed{i_N} \in \mathbb{B}^{\otimes N}$,

$\mathbb{B}: \boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \cdots \to \boxed{n-1} \xrightarrow{n-1} \boxed{n}$

(Example) $n = 3$, $\lambda = \boxed{}\boxed{}$ over $\boxed{}$ $\Longrightarrow$ 

$u \in U_q^-(\mathfrak{g}) = U_q^-$; $i \in I$

$\Rightarrow \exists!\ u', u'' \in U_q^-$ s.t

$$e_i u = \frac{K_i u'' - K_i^{-1} u'}{q_i - q_i^{-1}} \quad \left(\text{induction on } |\alpha|,\ |u| = |\alpha|\right)$$

Defn $e_i''(u) = u''$, $e_i'(u) = u'$.

(Exercise) $\exists!$ i-string decomposition of u :

$$u = \sum_{k \geq 0} f_i^{(k)} u_k, \text{ where } e_i' u_k = 0.$$

Define the Kashiwara operators by $\tilde{e}_i u = \sum_{k \geq 1} f_i^{(k-1)} u_k$, $\tilde{f}_i u = \sum_{k \geq 0} f_i^{(k+1)} u_k$.

$\boxed{\text{Def}}$ A crystal basis of $U_q^-(\mathfrak{g})$ is a pair (L, B), where

① i) L = free A_0-lattice of $U_q^-(\mathfrak{g})$

ii) B is a $\mathbb{C}$-basis of L/qL

iii) $L = \bigoplus_{\alpha \in Q_+} L_{-\alpha}$, $L_{-\alpha} = L \cap (U_q^-)_{-\alpha}$

iv) $B = \coprod_{\alpha \in Q_+} B_{-\alpha}$, $B_{-\alpha} = B \cap L_{-\alpha} / qL_{-\alpha}$

v) $\tilde{e}_i L \subset L$, $\tilde{f}_i L \subset L$

vi) $\tilde{e}_i B \subset B \cup \{0\}$, $\tilde{f}_i B \subset B$

vii) $\forall b, b' \in B$, $\forall i \in I$, $\tilde{f}_i b = b' \Leftrightarrow b = \tilde{e}_i b'$.

$\boxed{\text{Thm}}$ (Kashiwara 91)

$L(\infty) = A_0$-submodule of $U_q^-(\mathfrak{g})$ spanned by $\tilde{f}_{i_1} \cdots \tilde{f}_{i_r} \mathbb{1}$ $(r \geq 0,\ i_k \in I)$,

$B(\infty) = \{\tilde{f}_{i_1} \cdots \tilde{f}_{i_r} \mathbb{1} + qL(\infty)\}$

$\Rightarrow (L(\infty), B(\infty))$ is a (unique) crystal base of $U_q^-(\mathfrak{g})$

$\boxed{\text{Problem}}$ How to realize $B(\infty)$?

Motivated by $B(\lambda)$, $B(\infty)$, tensor product rule, we introduce the notion of abstract crystals

$\boxed{\text{Def}}$ An abstract $U_q(\mathfrak{g})$-crystal or $\mathfrak{g}$-crystal

is a set B together with the maps $wt : B \to P$, $\tilde{e}_i, \tilde{f}_i : B \to B \cup \{0\}$, $\varepsilon_i, \varphi_i : B \to \mathbb{Z} \cup \{-\infty\}$

satisfying:

i) $wt(\tilde{e}_i b) = wt\, b + \alpha_i$ if $\tilde{e}_i b \neq 0$

$wt(\tilde{f}_i b) = wt\, b - \alpha_i$ if $\tilde{f}_i b \neq 0$

ii) $\varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) - 1$, $\varphi_i(\tilde{e}_i b) = \varphi_i(b) + 1$ if $\tilde{e}_i b \neq 0$

$\varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) + 1$, $\varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1$ if $\tilde{f}_i b \neq 0$

iii) $\varphi_i(b) = \varepsilon_i(b) + \langle h_i, wt\, b\rangle$

iv) $\forall b, b' \in B$, $\tilde{f}_i b = b' \Leftrightarrow b = \tilde{e}_i b'$.

v) $\varphi_i(b) = -\infty \Rightarrow \tilde{e}_i b = \tilde{f}_i b = 0$.

(Example) ① $B(\lambda)$, $B(\infty)$ are abstract crystals

$\varphi_i(b) = \varepsilon_i(b) + \langle h_i, wt\, b\rangle$

$wt\, b = -(\alpha_{i_1} + \cdots + \alpha_{i_r})$

② $T_\lambda = \{t_\lambda\}$, $wt(t_\lambda) = \lambda$, $\varepsilon_i(t_\lambda) = \varphi_i(t_\lambda) = -\infty$,
$\tilde{e}_i t_\lambda = \tilde{f}_i t_\lambda = 0$

③ $C = \{c\}$, $wt(c) = 0$, $\varepsilon_i(c) = \varphi_i(c) = 0$,
$\tilde{e}_i c = \tilde{f}_i c = 0$

④ $B_i = \{ b_i(-n) \mid n \geq 0 \}$

$wt\, b_i(-n) = -n\alpha_i$, $\varepsilon_i(b_i(-n)) = n$, $\varphi_i(b_i(-n)) = -n$,

$\varepsilon_j b_i(-n) = \varphi_j(b_i(-n)) = -\infty \quad j \neq i$

$\tilde{e}_i b_i(-n) = b_i(-n+1)$, $\tilde{f}_i b_i(-n) = b_i(-n-1)$

$\tilde{e}_j b_i(-n) = \tilde{f}_j b_i(-n) = 0 \quad j \neq i$

B_i : <u>elementary crystal</u>

$\boxed{\text{Def}}$ $\psi : B_1 \to B_2$ is a <u>crystal morphism</u>

iff i) $wt\, \psi(b) = wt\, b$, $\varepsilon_i \psi(b) = \varepsilon_i(b)$, $\varphi_i(\psi(b)) = \varphi_i(b)$

ii) $b \in B_1$, $\tilde{f}_i b \in B_1 \Rightarrow \psi(\tilde{f}_i b) = \tilde{f}_i \psi(b)$.

$\boxed{\text{Def}}$ (I) $\psi : B_1 \to B_2$ crystal morphism

(1) ψ is a strict morphism if

$\psi \circ \tilde{e}_i = \tilde{e}_i \circ \psi$, $\psi \circ \tilde{f}_i = \tilde{f}_i \circ \psi$ $\forall i \in I$.

(We understand $\psi(0) = 0$.)

(2) ψ is an embedding if $\psi : B_1 \to B_2$ is injective.

We say that B_1 is a subcrystal of B_2.

B_1 is a full subcrystal of B_2 if ψ is a strict embedding. (~~$B_1 \cong$ conn comp of B~~)

($B_2 \cong B_1 \oplus (B_2 \setminus B_1)$)

~~Def~~ B_1, B_2 : crystals

Define

$B_1 \otimes B_2 = B_1 \times B_2$,

$\mathrm{wt}(b_1 \otimes b_2) = \mathrm{wt}\, b_1 + \mathrm{wt}\, b_2$

$\varepsilon_i(b_1 \otimes b_2) = \max\left(\varepsilon_i(b_1), \varepsilon_i(b_2) - \langle h_i, \mathrm{wt}\, b_1\rangle\right)$

$\varphi_i(b_1 \otimes b_2) = \max\left(\varphi_i(b_2), \varphi_i(b_1) + \langle h_i, \mathrm{wt}\, b_2\rangle\right)$

$$\tilde{e}_i(b_1\otimes b_2)=\begin{cases}\tilde{e}_i b_1\otimes b_2 & \varphi_i(b_1)\geq\varepsilon_i(b_2)\\ b_1\otimes\tilde{e}_i b_2 & \varphi_i(b_1)<\varepsilon_i(b_2)\end{cases}$$

$$\tilde{f}_i(b_1\otimes b_2)=\begin{cases}\tilde{f}_i b_1\otimes b_2 & \varphi_i(b_1)>\varepsilon_i(b_2)\\ b_1\otimes\tilde{f}_i b_2 & \varphi_i(b_1)\leq\varepsilon_i(b_2)\end{cases}$$

Prop $B_1\otimes B_2$ is an abstract crystal

Thm ① $\exists$ a map $\Psi_\lambda : B(\lambda)\to B(\infty)$ s.t

i) Ψ_λ is injective ii) $\Psi_\lambda(u_\lambda)=\mathbb{1}$

iii) $\Psi_\lambda(\tilde{f}_i b)=\tilde{f}_i\Psi_\lambda(b)$ where $\tilde{f}_i b\neq 0$

iv) $\Psi_\lambda(\tilde{e}_i b)=\tilde{e}_i\Psi_\lambda(b)$ $\forall b\in B(\lambda)$

v) $$\begin{cases}\mathrm{wt}\,\Psi_\lambda(b)=\mathrm{wt}\,b-\lambda\\ \varepsilon_i\Psi_\lambda(b)=\varepsilon_i(b) & \forall b\in B(\lambda)\end{cases}$$

Thm $\forall_{i \in I}$, $\exists!$ strict embedding
$$\Psi_i : B(\infty) \to B(\infty) \otimes B_i \quad \text{~~sending~~ s.t.}$$
$$\mathbb{1} \longmapsto \mathbb{1} \otimes b_i(0)$$

Recognition Thms

Thm B : crystal s.t.

i) $\mathrm{wt}\, B \subset -Q_+$

ii) $\exists\, b_0 \in B$ s.t. $\mathrm{wt}(b_0) = 0$

iii) $\forall b \neq b_0$, $\exists\, i \in I$ s.t. $\tilde{e}_i b \neq 0$ (connected)

iv) $\forall_{i \in I}$, $\exists$ strict embedding $\Psi_i : B \to B \otimes B_i$

$\Longrightarrow B \cong B(\infty)$, $b_0 \mapsto \mathbb{1}$

Thm $B(\lambda) \cong C(\mathbb{1} \otimes t_\lambda \otimes c) \hookrightarrow B(\infty) \otimes T_\lambda \otimes C$

Application Geometric construction of crystal bases

5. Global Bases

$A = \mathbb{C}[q, q^{-1}]$, A_0, A_∞

V : $\mathbb{C}(q)$-vector sp

V^A, V^{A_0}, V^{A_∞} : lattices of V

Set $E = V^A \cap V^{A_0} \cap V^{A_\infty}$: $\mathbb{C}$-vector sp

$\boxed{\text{Def}}$ $(V^A, V^{A_0}, V^{A_\infty})$ is a <u>balanced triple</u> for V if i) $A \otimes_{\mathbb{C}} E \cong V^A$, ii) $A_0 \otimes_{\mathbb{C}} E \cong V^{A_0}$, iii) $A_\infty \otimes_{\mathbb{C}} E \cong V^{A_\infty}$.

$\boxed{\text{Thm}}$ (TFAE)

1. $(V^A, V^{A_0}, V^{A_\infty})$: balanced triple
2. $E \xrightarrow{\sim} V^{A_0} / q V^{A_0}$
3. $E \xrightarrow{\sim} V^{A_\infty} / q^{-1} V^{A_\infty}$

$$E = V^{A_0} \cap V^{A_0} \cap V^{A_\infty} \underset{G}{\overset{\sim}{\rightleftarrows}} V^{A_0} / q V^{A_0}$$

B : basis of $V^{A_0} / q V^{A_0}$

$G(B) \overset{\text{def}}{=} \{ b\,G(b) \mid b \in B \}$

$\Longrightarrow$ $G(B)$ is an A-basis of V^A. (Exercise)

$\boxed{\text{Def}}$ ① B : <u>local basis</u> of V at $q = 0$

= crystal basis of V

② $G(B)$: <u>global basis</u> of V corr to B.

$\overline{} : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$, $e_i \mapsto e_i$, $f_i \mapsto f_i$, $q^h \mapsto q^{-h}$, $q \mapsto q^{-1}$

$\overline{} : V(\lambda) \to V(\lambda)$, $u \cdot v_\lambda \mapsto \overline{u} \cdot v_\lambda$

$U_A(\mathfrak{g})$ = A-subalg of $U_q(\mathfrak{g})$ generated by

$e_i^{(k)}, f_i^{(k)}, q^h, \left\{ \begin{matrix} K_i q_i^{-m} \\ m \end{matrix} \right\} = \frac{1}{[m]_i!} \prod_{k=1}^{m} \frac{K_i q_i^{n-m+1} - K_i^{-1} q_i^{-n+m-1}}{q_i - q_i^{-1}}$

$V(\lambda)^A = U_A(\mathfrak{g}) \cdot v_\lambda$

$L(\lambda)$ = A_0-submodule spanned by $\tilde{f}_{i_1} \cdots \tilde{f}_{i_r} v_\lambda$

$\overline{L(\lambda)} = \{ \overline{v} \mid v \in L(\lambda) \}$

Thm (Kas91) ① $(V(\lambda)^A, L(\lambda), \overline{L(\lambda)})$ is a balanced triple for $V(\lambda)$

② $\exists!$ A-basis $G(\lambda) = \{ G(b) \mid b \in B(\lambda) \}$ of $V(\lambda)^A$

s.t. i) $G(b) \equiv b \mod qL(\lambda)$

ii) $\overline{G(b)} = G(b) \quad \forall b \in B(\lambda)$

Problem How to ~~realize~~ construct $G(\lambda)$?

Similarly,

$\boxed{\text{Thm}}$ 1 $\left(U_A^-(\mathfrak{g}), L(\infty), \overline{L(\infty)}\right)$ is a balanced triple for $U_q^-(\mathfrak{g})$.

2 $\exists !$ A-basis $G(\infty) = \{G(b) \mid b \in B(\infty)\}$ of $U_A^-(\mathfrak{g})$ st

i) $G(b) \equiv b \mod qL(\infty)$

ii) $\overline{G(b)} = G(b) \quad \forall b \in B(\infty)$.

$\boxed{\text{Problem}}$ How to ~~realize~~ construct $G(\infty)$?