Lecture 5

global weyl module

$\lambda \in P^+$ $\quad W(\lambda) \ni w_\lambda$ $\quad (n^+ \otimes \mathbb{C}[t,t^{-1}])\, w_\lambda = 0$

$Lg$ - mod $\quad h w_\lambda = \lambda(h) w_\lambda$

$(Lg \oplus \mathbb{C}d)$ - mod

$(x_\alpha^-)^{\lambda(h_\alpha)+1} \cdot w_\lambda = 0$

$d w_\lambda = 0.$

$W(\lambda)$ left $L(g)$ - module also a ~~w~~ right

$L\mathfrak{h}$ - module.

$W(\lambda) = \mathcal{U}(Lg)\, w_\lambda.$

$(u w_\lambda, h \otimes f) = u\, (h \otimes f) w_\lambda.$

$\{ u \in U(L\mathfrak{h}) : w_\lambda u = u w_\lambda = 0 \} = \mathrm{Ann}_{(L\mathfrak{h})}\, w_\lambda$

ideal in $(L\mathfrak{h})$ $\quad \underline{\mathcal{U}(L\mathfrak{h})} \simeq \mathbb{C}[\cancel{P_0, \ldots,} P_n]$

~~$\simeq \mathbb{C}[P_{i,r}]/I$~~

$\mathrm{Ann}_{L\mathfrak{h}}\, w_\lambda$

$\mathrm{Ann}\, L_{\mathfrak{h}}(w_\lambda) = \mathbb{C}^{L\mathfrak{h}}[P_{i,j} : \; i = 1, \ldots, n;\; j \geq \lambda(h_i) + 1 ]$

$A_\lambda$ ~~$\cong$~~ $\mathbb{C}[P_{1,1}, \ldots, P_{1,\lambda(h_1)}, P_{2,1}, \ldots, P_{2,\lambda(h_2)}, \ldots, P_{n,1}, \ldots, P_{n,\lambda(h_n)}]$

$$P_{i,r} = \exp - \sum_{s \geq 1} \left( \frac{h_i \otimes t^s}{s} \right) u^s$$

local weyl modules

$$W(\lambda) \otimes_{A_\lambda} A_\lambda / \mathfrak{M}.$$

ex: $\underline{g} = sl_2$ $\quad W(\underline{m}, \underline{a})$

$\underline{m} \in (\mathbb{Z}^+)^r = (\mathbb{Z}_+)^r, \quad a \in (\mathbb{C}^*)^r$

$m = m_1, \dots m_r, \quad a = a_1 \dots a_r$

$$\pi_{\underline{m}, \underline{a}} = (1 - a_1 u)^{m_1} \cdots (1 - a_r u)^{m_r}$$

$\mathfrak{M} \to P_s$ = coeff of $u^s$ in above polynomial.

$W(\lambda) \otimes_{A_\lambda}$ is a flat functor from categ of left $A_\lambda$-mod $\longrightarrow$ categ of f.d. Lg-modules

same as say dim local weyl modules is ind. of point chosen same as CP conj/thm non-semisimple categories

applications: block decomp.

$\mathcal{C} = \bigoplus \mathcal{C}_\chi$

$V = \bigoplus V_\chi$, $0 \to V_1 \to V \to V_2 \to 0$

$V_1 \in V_{\chi_1}$, $V_2 \in V_{\chi_2}$ $V = V_1 \oplus V_2$ -

parametrize blocks - BGG categ done via center for simple dim alg

$\mathcal{F}$ = f.d. rep of $L\mathfrak{g}$ full subcateg of

block decomp: ~~usi~~ given by using local weyl modules - play an essential role

- C Moura, Untwisted case Senesi (twisted)

"graded local weight modules" - loop weyl modules

- blocks

$$\text{global weyl} \nearrow W(\lambda) \longrightarrow \text{local weyl. } W(\underline{\lambda}, \underline{a})$$

$$\searrow P(\lambda)\ ? \qquad \text{irr} \downarrow V(\underline{\lambda}, \underline{a})$$

even for
lots of problems – ~~identity~~ blocks and modules are very big, so finding projective is still hard – ongoing process to find right subcateg. etc BGG duality.

– so alternative approach – explain it.
approach was motivated by results from quantum affine algebra

<u>gen comments:</u> ~~both~~ quantum affine alg & affine alg have far too many irr. reps: indecomposable reps.

$W(\lambda, a) \longrightarrow V(\lambda, a)$ affine case
small piece.

lots of quotients of $W(\lambda, a)$
or in quantum aff case lots of irr reps
challenge: find interesting families of

reps and see where they live.

irreps of quantum affine alg: Kirillov–Reshetikhin modules – mathematical physics, solvable lattice models, [HKOTY], context of crystal bases.

Remark: fd reps of quantum aff alg do not in gen. have crystal basis

Kashiwara: conj that only the KR mod & their tensor prod. will have crystal bases.

Some of this is now known – Okado, Schilling KR mod have crystal basis of classical type but not that these & ⊗ prod are the only ones that do

other literature T sys / Q syst Nakajima, Hernandez

$U_q(\mathfrak{g})$-mod decomp [C] for classical & exceptional cas

fermionic formula.

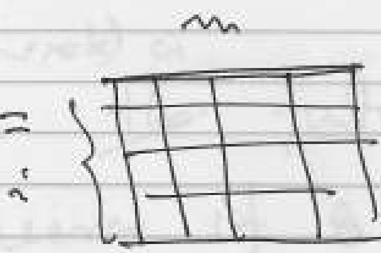
lets just talk (Gr?) about $q=1$ limit of KR modules. one knows one can do that [CP] and then one gets indecomp. fd module for $U(Lg)$ and in fact they are (proper) quotient of (local!) weyl modules

what are KR mod.. $\forall\, i \in I,\ m \in \mathbb{Z}_+$

$KR(m\omega_i)$ is a $L(g)$-mod with a prescribed $\underline{g}$-mod decomposition.

ex: $sl_{r+1}$ $\quad KR(m\omega_i) \simeq V(m\omega_i)$

ex: $B_n$. $m\omega_i =$



$i \times m$ rectangle.

KR. $\exists$ a fd module for $Lg$-mod whose $g$-char from $m\omega_i$ is given by dropping $2\times1$-dom. so that one has still a young diagram.

$$KR(m\omega_i) = V(m\omega_i) \oplus V((m-1)\omega_i + \omega_{i-2}) \oplus V((m-2)\omega_i + 2\omega_{i-2}) \oplus V((m-1)\omega_i + \omega_{i-2} + \omega_{i-4}). \oplus \cdots$$


Thm. [C] ~~$\mathfrak{g}$ classical.~~ Consider $\widetilde{KR}(m\omega_i, 1)$ local the quotient of the Weyl module $W(-m\omega_i, 1)$ by the single additional relation

$$(x_i^- \otimes t)\, w_{m\omega_i} = 0$$

(if it exists) a

then $KR(m\omega_i)$ is ~~the~~ quotient of $\widetilde{KR}(m\omega_i)$. Moreover if $\mathfrak{g}$ is classical $\widetilde{KR}(m\omega_i, 1)$ is the module predicted by KR.

Chari-Moura – twisted cases.

$\mathfrak{g}$ – classical

<u>Cor:</u> $K\ell(m\omega_i)$ satisfies the following

$$\left(\mathfrak{g} \otimes \frac{\mathbb{C}[t,t^{-1}]}{(t-1)^2}\right) K\ell(m\omega_i) = 0.$$

i.e. $K\ell(m\omega_i)$ is a module for the truncated loop alg

$\mathfrak{g} \oplus \mathfrak{g}(t-1) \qquad (t-1)^2 = 0$

– Takiff algebra.

replacing $t \to t-1$. one can then look

at rep $\mathfrak{g} \otimes \frac{\mathbb{C}[t]}{(t^2)} \to \mathfrak{g} \otimes \frac{\mathbb{C}(t)}{(t^2)}$.

approach that Greenstein & I took.

$\mathcal{I}$ – caty of fd rep.

$\mathcal{I}^m$ – caty of fd reps on which

$(\mathfrak{g} \otimes t^{n+1} \mathbb{C}[t]) V = 0$

$\mathcal{F}^0$ $(\mathfrak{g} \otimes t\,\mathbb{C}[t]) V = 0$ i.e. $\mathfrak{g}$-modules

pulled back via evaluation

ss $\mathcal{F}^0$ - simple.

$\mathcal{F}^1 \ni$ $(\mathfrak{g} \otimes t^2 \mathbb{C}[t]) V = 0$

KR $\in \mathcal{F}^1$ $\mathcal{F}^1$ interesting,

nice objects in it

- just away from being s.s.
- BGS - Koszulity, algebras are as nice as possible without being s.s.

CG: identified interesting subcategory of $\mathcal{F}^1$ using ideas of [CPS] s.t.

① they contain finitely many ir

(2) have enough proj $P_1 \ldots P_r$

$$\operatorname{End}_{\mathcal{F}} \left( \bigoplus_{i=1}^{r} P_i \right).$$

categ ~~of~~ ~~mod~~ left mod for this alg $\simeq$ fd ass sub categ of $\mathcal{F}_1$

$\Downarrow$

homological properties are well behaved

and this algebra is Koszul.

$\mathcal{F}_1 \to$ natural eq. to categ of left modules of a fd Koszul alg ~~gra~~

$\mathfrak{g} \oplus \mathfrak{g}_{a1}$

$\mathfrak{g} \ltimes V$ $\quad V$ rep of $\mathfrak{g}$

play the same game ~~de~~ undeformed inf. Hecke alg

~~$\mathcal{U}(\mathfrak{g} \ltimes V)$~~ again Khare, Ridenour – study Koszul for categ of rep of $\mathfrak{g} \ltimes V$