Lecture 4.

Continue our study of level 0-modules

Recall that we are trying to imitate ideas from BGG / modular rep theory

- verma mod/Weyl mod
- irr mod
- projectives

$\mathcal{I}$ - categ of integrable level zero modules of $L(g)$. $\hat{g} = \mathbb{C}c \oplus Lg \oplus \mathbb{C}d$.

$\hat{g}/\mathbb{C}c \cong Lg \oplus \mathbb{C}d$

recall construction of irr. modules in $\mathcal{I}$ discussed yesterday. $\underline{a} \in (\mathbb{C}^*)^r$ $\underline{a} = (a_1, .., a_r)$ distinct co-ord.

surj hom $ev_{\underline{a}} : L(g) \to \underbrace{g \oplus .. \oplus g}_{r\text{ times}}$

$x \otimes f \to (f(a_1)x, .., f(a_r)x)$

Take any f.d. irrep of $g \oplus .. \oplus g \to V(\lambda_1) \otimes .. \otimes V(\lambda_r)$ $\lambda_i \in P^+$

$ev_{\underline{a}}^*(V(\lambda_1) \otimes \cdots \otimes V(\lambda_r))$ irr $L(g)$-mod

$$\lambda = \sum_{i=1}^{n} \lambda_i \in P^+$$

$$\underline{v_{\lambda}} = v_{\lambda_1} \otimes v_{\lambda_2} \otimes \cdots \otimes v_{\lambda_r}.$$

$$ev_{\underline{a}}^{*}\left(V(\lambda_1)\otimes \cdots \otimes V(\lambda_r)\right) = \mathcal{U}((L\mathfrak{g}))\underline{v_{\lambda}}$$

rel^{n}s: $\alpha \in R^+$, $(x_\alpha \otimes f)\underline{v_\lambda} = f(a_1)\, x_\alpha v_{\lambda_1} \otimes \cdots \otimes v_{\lambda_r} + \cdots + f(a_r)\, v_{\lambda_1} \otimes \cdots \otimes x_\alpha v_{\lambda_r} = 0$

$$h\underline{v_\lambda} = \lambda(h)\, \underline{v_\lambda}$$

$\Rightarrow$ $ev_a^{*}(V(\lambda_1)\otimes \cdots \otimes V(\lambda_r))$ is a quotient of a module generated by v_λ satisfying rel.

$$L(n^+).\ v_\lambda = 0 \qquad h v_\lambda = \lambda(h) v_\lambda.$$

Such a mod is not in our categ $\mathcal{I}$.

impose add.^{n}al rel^n $(x_\alpha^-)^{\lambda(h_\alpha)+1} v_\lambda = 0$

$\lambda \in P^+$

defn: The global weyl module $W(\lambda)$ is the $\mathcal{U}(L\mathfrak{g})$-module generated by w_λ with rel^{n}s $L(n^+)\, w_\lambda = 0$ $\quad h w_\lambda = \lambda(h) w_\lambda$

$(x_\alpha^-)^{\lambda(h_\alpha)+1} w_\lambda = 0.$

Prop: $W(\lambda) \in \mathcal{I} \ \forall \lambda \in P^+$

$W(\lambda)$ is ~~very~~ infinite dimensional - essentially we have imposed no condt^{n}s on $h \otimes t^s$ $h \in \mathfrak{h}, s \in \mathbb{Z}_+$

? What condt^{n}s do get imposed by the existing rel^{n}s. ~~- come back to that late~~

- come back to that - but for now lets discuss local or fin. dim. Weyl modules,
- reasonable to discuss. Since we know that $W(\lambda)$ have finite dim. quotients

$ev_{\underline{a}}^* (V(\lambda_1) \otimes . \otimes V(\lambda_r))$

$(h \otimes f)(v_{\lambda_1} \otimes .. \otimes v_{\lambda_r}) = \sum \lambda_i(h) f(a_i)(v_{\lambda_1} \otimes \otimes v_{\lambda_r})$

defn: given $\underline{a} \in (\mathbb{C}^\times)^r$, $\underline{\lambda} \in (P^+)^{(r)}$ $\sum \lambda_i = \lambda$

$\underline{\lambda} = (\lambda_1 \cdots \lambda_r)$

let $W(\underline{a}, \underline{\lambda})$ be quotient of $W(\lambda)$

by add'l rel^n $(h\otimes f - \sum \lambda_i(h_i) f(a_i)) w_\lambda = 0$

Thm: [CP] (i) $\dim W(\underline{a}, \underline{\lambda})$ is finite.

(ii) $W(\underline{a}, \underline{\lambda}) \underset{Lg\text{-mod}}{\cong} W(a_1, \lambda_1) \otimes \ldots \otimes W(a_s, \lambda_s)$.

if a_i's are distinct

$W(\underline{a}, \underline{\lambda})$ local - weyl modules.

defn ~~defn~~ given $\lambda \in P^+$, $a \in \mathbb{C}^*$

$W(\lambda, a)$ is the Lg-mod generated by

w_λ with rel's $Ln^+ = 0$ $(h\otimes f - \lambda(h) f(a)) w_\lambda = 0$

$(x_i^-)^{\lambda(h_i)+1} = 0$

- why did we call these the weyl modules - the whole idea of defining these objects came from studying irr f.d. reps of quantum affine algebras. ~~q~~ $U_q(\hat{g}) \leftarrow$ ~~V~~ irr mod.

$/\mathbb{C}(q)$

analog of Kostant $\mathbb{Z}$-form. $U_A(\hat{g})$

$A = \mathbb{C}[q, q^{-1}]$

$$V_A \otimes_A \mathbb{C} \quad \leftarrow \text{f.d. mod for } U(\mathfrak{g})$$

Conj: $W(\lambda, a)$ — specialization $(q=1)$ of certain irr. f.d reps of quantum affine alg

these are also biggest in a certain sense — which is something they share with verma modules and $\mathrm{Ind}(\lambda)$ in mod. rep theory i.e. have

Universal properties.

defn: Say that $M \in \mathcal{I}$ is highest weight with hw λ if its generated by an element

m_λ s.t. $\quad L(n^+)\, m_\lambda = 0 \quad h m_\lambda = \lambda(h) m_\lambda$

<u>Lemma</u>: M is a weight mod $M = \bigoplus_{\mu \in \mathfrak{h}^*} M_\mu$

~~and~~ ~~is~~ a wt $\mu \leq \lambda - Q^+$

$W(\lambda)$ is a universal highest weight mod with h.w. λ.

<u>universal properties of $W(\lambda, a)$.</u>

- requires more work.

<u>defn</u>: M is a ℓ-highest weight mod if $\exists\ m_\lambda \in M$ s.t.

$h m_\lambda = \lambda(h) m_\lambda \qquad (h \otimes t^n) m_\lambda = d_n(h) m_\lambda$

$L n^+ m_\lambda = 0 \qquad\qquad d_n(h) \in \mathfrak{h}^*$

$(x_i^-)^{\lambda(h_i)+1} m_\lambda = 0$

$\boxed{\text{examples above } W(\lambda, a) \quad d_n(h) = \lambda(h)\, a^n}$

$W_{\mathfrak{g}}(\underline{\lambda}, \underline{a}) \qquad d_{\lambda}(h) = \sum \lambda_i(h) a_i^n$

very specific elements of $\mathfrak{g}^*$

<u>Thm</u>: [CP] Any ℓ- highest weight module in $\mathcal{I}$ is a quotient of $W(\underline{\lambda}, \underline{a})$ for some choice of $\underline{\lambda} \in (P^+)^n$, $a \in (\mathbb{C}^\times)^n$.

- way one proves this thm is the foll. one uses garland's result and its enough to do it for $L(sl_2)$

$x, y, h, \; xt^r$ std bas

$M \qquad \ell$- highest wt.

$\lambda = k. \qquad M$

$(xt^r) m_k = 0 \quad \forall r \in \mathbb{Z}_+$

$y^{k+1} . m_k = 0$

$\Rightarrow (xt)^{k+1} y^{(k+1)} m_\lambda = 0$

$sl_2, \; \mathfrak{g}$

$x^{k+1} y^{k+1} m_\lambda$

$=$

by using Garland's form

$\Rightarrow (x t)^{k+1} (y^{k+1}) = P_{k+1} m + \left(\sum \underbrace{}_{\cancel{y(t)}} - \underbrace{}_{L_n} \underbrace{}_{L_n^+}\right) m$

$\overset{\shortparallel}{0}$

P_{k+1} = coeff of u^{k+1} in

$$\exp\left(- \sum_{s \geq 1} \frac{(h \otimes t^s)\, u^s}{s}\right).$$

$\Rightarrow p_{k+1} \cdot m = 0 = p_s \; \forall s \geq k+1$

$p_0 = 1, \quad p_1 = h_1, \quad p_2 = \frac{-h_2}{2} + \frac{(h_1)^2}{2}$

$p_3 = \frac{-h_3}{3} + \frac{3 h_1 h_2}{2} - \frac{h_1^3}{3!}$

$\Rightarrow \mathbb{C}[p_1, p_2, \dots] \simeq \mathbb{C}[h \otimes t, h \otimes t^2, \dots h]$

ies

is a quot of

$L n^+ m_k = 0$

$h m_k = \lambda(h)$

$p_s = 0 \;\; \forall s \geq k+1.$

Scalars $p_1, \dots, p_k$

choose them arbitrarily.

$$\Pi = 1 + b_1 u + \cdots + b_k u^k.$$

Let $a_1, \ldots, a_r$ be distinct roots of this poly

$$\Pi = (1 - a_1 u)^{m_1} \cdots (1 - a_r u)^{m_r}$$

ex:

$$W(\underline{m}, \underline{a}) \simeq W(m_1, a_1) \otimes \cdots \otimes W(m_r, a_r).$$

$\underline{a} = (a_1, \ldots a_r)$

$$(h \otimes f)(w_{m_1} \otimes \cdots \otimes w_{m_r})$$

$$\Sigma\, m_1 f(a_1) + \cdots + m_r f(a_r))$$

$\underline{m} = (m_1, \ldots m_r)$

so M is a quot of $W(\underline{m}, \underline{a})$.

– so we have all these mod defined

$$W(\underline{\lambda}, \underline{a}) \to ev_{\underline{a}}^{*}\, V(\lambda, a) \to 0$$

but are they different? if not then we have our goal, if they are same then we at least have generators for irr mod fortunately one knows from the rep theory of quantum aff. alg that weyl modules ~~being~~ ~~sp~~ecials are bigger than the irr

- irr mod in quantum affine case $q=1$

→ bigger than irr mod in affine case

weyl mod sit above $q=1$ limit of irr mod

Specific conjecture on the structure of weyl module.

Let $\omega_1, \ldots, \omega_n$ be fund wts of $\mathfrak{g}$

$$\frac{2(\omega_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \delta_{ij} \quad , \quad \lambda = \sum_{i=1}^{n} r_i \omega_i$$

$$W_q(\lambda, a) \underset{\mathfrak{g}\text{-mod}}{\cong} W(\omega_1, a_1)^{\otimes r_1} \otimes \cdots \otimes W(\omega_n, a)^{\otimes r_n}.$$

Th: So in principle if you know $W(\omega_i, a)$ for all fund. you know

$W(\lambda, a)$

$W(\omega_i, a)$ "known" if $\underline{g}$ is of classical type

exceptional types ?

$\underline{g} = sl_{n+1}$ $\quad W(\omega_i, a) \underset{g\text{-mod}}{\simeq} V(\omega_i)$

$\underline{sl_2}$: $\quad W(m, a) \simeq V(\omega) \otimes \cdots \otimes V(\omega)$

$$\simeq \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$$

$$\dim W(m, a) = 2^m.$$

$$\dim \begin{matrix} ev_a^* V(m) \\ \cancel{V(m, a)} \end{matrix} = (m+1)$$

~~$[W(m, a) : V(r)] = [\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2 : V(r)]$~~

$W(m, a)$ has comp series with comp factor $ev_a^* V(r)$

$$[W(m, a) : ev_a^* V(r)] = [\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2 : V(r)]$$

- $\mathfrak{g} = A, D, E$ weyl modules can actually be "identified" with Demazure modules in level 1 modules A - CL

ADE - I L

false for non-simply laced case.

weyl modules are bigger.

- application of weyl modules - one is always interested in block decomp.
- weyl modules play an ess. tool in describing blocks in the categ of fd reps of $L\mathfrak{g}$ [CM]

twisted [Senesi]

- blocks are very big - look for other ways to cut our study.

justify global / local weyl modules.

right action of $U(L\mathfrak{h})$ on $W(\lambda)$

$\underline{sl_2}$

$W(m) \times U(L_h) \to W(m)$

$(g w_m, \quad u) \to g u w_m$

$(g w_m \quad P_{r+1}) \to \quad 0 \quad r \geq m$

$W(m) \times \mathbb{C}[P_1, \ldots P_m] \to W(m)$

$W(m)_m \cong \mathbb{C}[P_1, \ldots P_m]$ inf. dim

$\begin{array}{ccc} W(m) & & \otimes \\ U(L_g) & \mathbb{C}[P_1, \ldots P_n] & \mathbb{C}[P_1 \ldots P_n] \end{array} \quad \mathbb{C}_{(a_1, \ldots a_r)}$

$\underline{\simeq} \quad W(\underline{m}, \underline{a})$

conj [CP] $\Rightarrow$ dim $W(\underline{m}, \underline{a})$ depend

thm on m and not the pt

ie. $W(m)$ free module fr $\mathbb{C}[P_1 \ldots P_m]$

– Fourier discuss this g.o.d.

of ~~as~~ co-ord ring of an alg variety