①

$\hat{\mathfrak{g}}$ - affine alg $\hat{\mathfrak{g}} = L(\mathfrak{g}) \oplus \mathbb{C}c \oplus \mathbb{C}d$.

$L(\mathfrak{g}) = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$. $\hat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d$

V weight mod. $V = \bigoplus_{\mu \in \hat{\mathfrak{h}}^*} V_\mu$

$a \in \mathbb{C}$ $V^a = \bigoplus_{\substack{\mu \in \hat{\mathfrak{h}}^* \\ \mu(c) = a}} V_\mu$.

$V^a \subseteq V$ $\hat{\mathfrak{g}}$- submod $V = \bigoplus_{a \in \mathbb{C}} V^a$

So can assume $V = V^a$.

integrable modules: let $\{x_i^\pm, h_i\} \cup \{d\}$ $i = 0, \ldots, n$ be the Chevalley generators of $\hat{\mathfrak{g}}$.

$\mathfrak{g}$ is gen. $\{x_i^\pm, h_i\}_{i=1,\ldots,n}$, $h_0 = c - h_\theta$

V is integrable if $\forall i = 0, \ldots, n$ V is a direct sum of f.d. modules to the sl_2-sub of $\hat{\mathfrak{g}}$ spanned by $\{x_i^\pm, h_i\}$

sub. & quot. of integ modules are integrable.

<u>Lemma</u>: V be integrable $\hat{\mathfrak{g}}$-module and ~~$V = V^a$ for~~ s. assume $V = V^a$. Then $a \in \mathbb{Z}$.

<u>Pf</u>: consider the subalg ~~$\mu \in \mathfrak{h}'$~~

~~$\mathfrak{sl}'$~~ $(x_0^{\pm}, x_0, h_0)$. Then since

~~$\mathcal{U}$~~(V is a sum of $\mathfrak{sl}_2$ - fd. [illegible]

$V_{\mu} \neq 0 \Rightarrow \mu(h_0) = \mu(c - h_{\theta}) \in \mathbb{Z}$

On the other hand $\mu(h_i) \in \mathbb{Z}$ $\forall i$

$\Rightarrow \mu(c) \in \mathbb{Z}$.

Say a module has level k if $V = V^k$.

– positive level / level zero / negative level.

<u>Recall</u>: that for $\lambda \in \hat{P}^+$ we defined

$V(\lambda) \in \hat{\mathcal{O}}$ and saw that it was

integrable. $\lambda \in \hat{P}^+ \Rightarrow \lambda(c) \in \mathbb{Z}_+$

$\lambda(c) \geq 0$.

$\times$ example of level zero modules ← the quot $\hat{\mathfrak{g}}/\mathfrak{g}$
$\leftrightarrow$ examples of modules for $L\mathfrak{g} \oplus \mathbb{C}d$. $[x\otimes t^n, y \otimes t^m] = [x,y] \otimes t^{n+m}$

is the adjoint rep of $\hat{\mathfrak{g}}$ on itself

$d(x\otimes t^n) = n\, x \otimes t^n$

$[c, \hat{\mathfrak{g}}] = 0$

$\hat{\mathfrak{g}}$ is clearly reducible.

$0 \subseteq \mathbb{C}c \subseteq L\mathfrak{g} \oplus \mathbb{C}c \subseteq \hat{\mathfrak{g}}$

$\hat{\mathfrak{g}} / (L\mathfrak{g} \oplus \mathbb{C}c) \cong \mathbb{C}d.$ ← trivial $\hat{\mathfrak{g}}$-module

composition series for $\hat{\mathfrak{g}}$

and $\hat{\mathfrak{g}}$ is indecomposable.

$\underline{L\mathfrak{g} \oplus \mathbb{C}c} / \mathbb{C}c \cong L\mathfrak{g}$ as vs.

$(x\otimes t^n, \lambda) \to x \otimes t^n.$

$$(x\otimes t^n)(y \otimes t^m) = [x,y] \otimes t^{n+m} + n\delta_{n,-m}(k(x,y)c)$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \| \; 0$

~~indec.~~

example generalizes as follows:

for $a \in \mathbb{C}^*$ let $ev_a : Lg \to g$

$x \otimes f \to f(a)x$.

V rep of $\underline{g}$ $\quad ev_a^* V \underset{vs.}{\simeq} V$ rep of Lg

$(x \otimes f) v = f(a) x v$.

V is irr $\to ev_a^* V$ ~~is~~ is irr.

V is ~~fd~~ $\Rightarrow ev_a^* V$ integ

$ev_a^* V \otimes \mathbb{C}[t, t^{-1}] = L(V)$ this

is an $(g \otimes \mathbb{C}d)$ module. (and hence $\hat{g}$-module).

$(x \otimes f)(v \otimes g) = f(a) x v \otimes fg$, $\quad d(v \otimes f) = v \otimes t \frac{d}{dt} f$

<u>ex</u>: V is irr. $\Rightarrow L(V)$ is an

V is fd $\Rightarrow L(V)$ is integ

irr. level zero module for $\hat{g}$.

generalize this construction still further

$\underline{a} = (a_1, \ldots, a_r) \in (\mathbb{C}^*)^r$

$ev_{\underline{a}} : Lg \to \bigoplus_r g = \underline{g}(r)$

semisimple Lie alg

$ev_{\underline{a}}(x \otimes f) = (f(a_1)x, \ldots, f(a_r)x).$

ex: $ev_{\underline{a}}$ is surjective iff $\underline{a}$ has distinct co-ordinates.

$\Rightarrow$ ~~V_n~~ irr. rep of $\mathfrak{g}(\ell)$, $\underline{a}$ has dist co-or. $\Rightarrow$ $ev_{\underline{a}}^* V$ is an irr. rep of $\mathfrak{g}(\sigma)$.

$L(\underline{a}) = ev_{\underline{a}}^* V \otimes \mathbb{C}[t, t^{-1}]$ define an action of $L(\mathfrak{g})$ on it as before.

But now: $L(e_{\underline{a}}^* V)$ is not always reducible but it is comp reducible.

$sl_2 \otimes \mathbb{C}[t, t^{-1}]$. $\mathbb{C}^2$ natural rep.

$r = 2$ $\quad sl_2 \oplus sl_2 \to \mathbb{C}^2 \otimes \mathbb{C}^2$

irr. rep of it

$sl_2 \, \mathbb{C}[t, t^{-1}] \to sl_2 \oplus sl_2 \to \text{end}(\mathbb{C}^2 \otimes \mathbb{C}^2)$

$x\otimes f \to (f(a_1)x, f(a_2)x)(v_1\otimes v_2)$

$f(a_1)xv_1\otimes v_2 + x\otimes f(a_2)v_2.$

Loop - mΩ

$(x\otimes t^s)(v_1\otimes v_2\otimes t^r)$

$= a_1^s\, xv_1\otimes v_2\otimes t^{r+s} + a_2^s\, v_1\otimes xv_2\otimes t^{r+s}$

$a_1 = -a_2, = 1 \quad \mathbb{C}^2\otimes\mathbb{C}^2. \quad v_+\otimes v_+$

check that $v_+\otimes v_+$ generates

a proper submodule

$v_+\otimes v_+\otimes t^r$

$v_+\otimes v_-\otimes t^r \quad v_-\otimes v_+\otimes t^r$

$v_-\otimes v_-\otimes t^r$

$h\otimes t^s \quad v_+\otimes v_+\otimes 1 \longrightarrow v_+\otimes v_+\otimes t^s$

$x\otimes t^s \quad + (-1)^s v_+\otimes v_+\otimes t^s$

$y\otimes t^s \quad v_+\otimes v_+\otimes t^{2s}$ can't pick up odd powers!

Thm: [C 86, CP 87]

Suppose V is an irr. integ $\hat{\mathfrak{g}}$-mod

$\dim V_\mu < \infty \quad \forall \mu \in \hat{\mathfrak{h}}^*$. Then

(i) level $V > 0$ then $V \simeq V(\lambda)$, $\lambda \in \hat{P}^+$

(ii) level $V < 0$ then $V \simeq V(-\lambda)$ $\lambda \in \hat{P}^+$

(iii) level $V = 0$ then $V \simeq$ one of

the loop-mod $\to L(ev_{\underline{a}}^* \underline{V})$ for some choice

of $\underline{a}$ and $\underline{V}$.

notice there is a tight conn between

irr. integ mod for $L\mathfrak{g} \oplus \mathbb{C} \to$ irr f.d. mod for $L\mathfrak{g}$

integ mod for $L\mathfrak{g} \oplus \mathbb{C} \leftrightarrow \begin{Bmatrix} \text{fd mod for } L\mathfrak{g} \\ \text{integ mod for } L\mathfrak{g} \end{Bmatrix}$

so now we understand irr. int. mod

where do we go from here

~~irr~~: $V(\lambda)$, $\lambda \in \hat{P}^+$ — deeper analysis

of structure & irr. ~~and this~~ is

~~$\forall$ any integ. mod. of +ve level, $\dim V_\mu < \infty$~~

~~then $V \in \hat{\mathcal{O}}$~~.

? V integ. $\dim V_\mu < \infty$ $\forall \mu \in \hat{\mathfrak{h}}^*$

then $V \in \hat{\mathcal{O}} \Rightarrow V \simeq \bigoplus$ irr. $V(\lambda)$.

<u>zero level modules:</u> with f.d. wt space not semisimple

and describing simples is only describing

a small piece of the category

– how does one set about analyzing

this problem?

- look for inspiration in the methods used to study two famous non-semisimple categ in the rep theory of simple Lie algebras

- BGG category $\mathcal{O}$
- modular rep. theory - rep. theory for characteristic p.

BGG categ: $\mathcal{O} = \bigoplus \mathcal{O}_\chi$

$\mathfrak{g}$ simple. $\chi \in \mathrm{Hom}(Z(\mathfrak{g}), \mathbb{C})$.

- each $\mathcal{O}_\chi$ has finitely many simples

$\chi = \chi_\lambda \quad V(w(\lambda+\rho)-\rho)$

- Universal objects $M(w(\lambda+\rho)-\rho)$.
- projective objects $P(w(\lambda+\rho)-\rho)$

- BGG duality between

$[P(\lambda) : M(\mu)] \qquad [M(\mu) : V(\lambda)]$.

• $\mathcal{O}_\chi$ is equivalent to the categ of a module of a finite-dimensional alg with some sp properties — quasihereditary algebras (CPS).

<u>Modular Rep Theory:</u>

$\underline{g}$ simple Lie alg. $\{x_\alpha : \alpha \in R\}$ $\{h_i : i \in I\}$

$g_{\mathbb{Z}} = \mathbb{Z}$-span of chevalley basis.

$g_{\mathbb{Z}}$ is a $\mathbb{Z}$-subalg $[x_\alpha, x_\beta] = \underset{\mathbb{Z}}{\underset{\cap}{c_{\alpha\beta}}} x_{\alpha+\beta}$

generalized to arb. irr. rep. of $\underline{g}$

$\lambda \in P^+$ $V(\lambda)$ lattice in $V(\lambda)$ we mean a free $\mathbb{Z}$-module of rank $= \dim_{\mathbb{C}} V(\lambda)$

$V(\lambda)$ has a minimal lattice given by the Kostant $\mathbb{Z}$-form of $\mathcal{U}(\underline{g})$.

~~ma~~ Let $\mathcal{U}_{\mathbb{Z}}(\underline{g})$ be the $\mathbb{Z}$-subalg generated by $\frac{x_\alpha^n}{r!}$, $\frac{\cancel{x_\alpha}^s}{\cancel{s!}}$ $\alpha \in R$.

$x_\alpha^{(r)} = \frac{x_\alpha^n}{r!}$ ~ Chevalley group $\exp(\mathrm{ad}\, x_\alpha)$ $1 + \mathrm{ad}\, x_\alpha + \frac{\mathrm{ad}\, x_\alpha^2}{2!} + \cdots$

$\mathcal{U}_\mathbb{Z}(n^+)$ $x_\alpha^{(r)}$: $\alpha \in R^+$

$\mathcal{U}_\mathbb{Z}(n^-)$ $\cancel{x}_\alpha^{(r)}$ $\alpha \in R^+$

? $\mathcal{U}_\mathbb{Z}(\underline{h})$ Ans. not $h_i^{(r)}$

more subtle than that.

? $\mathcal{U}_\mathbb{Z}(h)$ subalg generated $\binom{h_i}{k}$ $i = 1, \ldots n$, $k \in \mathbb{Z}^+$

$$\binom{h_i}{k} = \frac{h_i(h_i - 1) \ldots (h_i - k + 1)}{k!}$$

Kost.

$\cancel{Al_2}$ $\mathcal{U}_\mathbb{Z}(\underline{g}) = \mathcal{U}_\mathbb{Z}(n^-)\, \mathcal{U}_\mathbb{Z}(h)\, \mathcal{U}_\mathbb{Z}(n^+)$

$\mathcal{U}_\mathbb{Z}(n^\pm)$ has basis of ordered monom

$x_{\beta_1}^{(s_1)} \ldots x_{\beta_N}^{(s_N)}$ $s_i \in \mathbb{Z}_+$

Crucial thing is to get hold of the elts $\binom{h}{k}$

$\underline{sl_2}$: $\quad y^{(r)} x^{(s)} \quad$ PBW form

$x^{(s)} y^{(r)} \in \mathcal{U}_{\mathbb{Z}}(sl_2)$ in wrong order though

so one needs to reorder

Lemma: $$x^{(s)} y^{(r)} = \sum_{k=0}^{\min(r,s)} y^{(r-k)} \binom{h}{k} x^{(s-k)}$$

$\mathcal{U}_{\mathbb{Z}}(\underline{g}) \otimes_{\mathbb{Z}} \mathbb{C} \cong \mathcal{U}(g)$

$v_\lambda \in V(\lambda) \qquad n^+ v_\lambda = 0 \qquad h v_\lambda = \lambda(h) v_\lambda$.

$V_{\mathbb{Z}}(\lambda) = \mathcal{U}_{\mathbb{Z}}(g) v_\lambda \qquad$ minimal lattice in $V(\lambda)$

$V(\lambda) \simeq \mathbb{C} \otimes_{\mathbb{Z}} V_{\mathbb{Z}}(\lambda)$. $\quad \text{rank}_{\mathbb{Z}} V_{\mathbb{Z}}(\lambda) = \dim_{\mathbb{C}} V(\lambda)$

$\underline{\text{char } p}$. $\quad V_F(\lambda) \simeq F \otimes_{\mathbb{Z}} V_{\mathbb{Z}}(\lambda)$.

F char $F = p \qquad \dim_F V_F(\lambda) = \dim_{\mathbb{C}} V(\lambda)$.

however: $V_F(\lambda)$ is not necessarily
irreducible, and now it is called the Weyl module

character of $V_F(\lambda)$ is defined as follows

$\mu \in \mathfrak{h}^*$ $V_F(\lambda)_\mu = \left\{ v \in V_F(\lambda) : \binom{h}{k} v = \binom{\mu(h)}{k} v \right\}$

$$\mathrm{ch}\, V_F(\lambda) = \mathrm{ch}\, V_{\mathbb{C}}(\lambda)$$

$W_k(\lambda)$ are understood. at least char is known. $\dim W(\lambda) < \infty$

$W_F(\lambda)$ has a $\overset{!}{\circ}$ irr. quot. $V(\lambda)$

. $W_F(\lambda)$ has JH series

? what are the constituent of $W_F(\lambda)$

KL - type theory

mod rep — $W(\lambda)$, $V(\lambda)$

categ — $M(\lambda)$ $V(\lambda)$

next time we're going to talk about

a relation of weyl modules for affine alg.

but lets end todays lecture with the

$\mathbb{Z}$-form for affine Lie algebras.

garland: $\hat{R} = \{\alpha + n\delta : \alpha \in R, n \in \mathbb{Z}\}$ real roots

$\cup \{n\delta : n \in \mathbb{Z}, n \neq 0\}$ imag. root

$\mathcal{U}_{\mathbb{Z}}(\hat{g}) = \mathbb{Z}$- subalg generated by

$x^{(r)}_{\alpha + n\delta} \quad \alpha \in R, n \in \mathbb{Z},$

$\hat{R}^{+} \qquad \hat{R}^{-}$

$\mathcal{U}_{\mathbb{Z}}(\hat{g}) = \mathcal{U}_{\mathbb{Z}}(\hat{n}^-) \, U_{\mathbb{Z}}(h) \, \mathcal{U}_{\mathbb{Z}}(\hat{n}^+).$

missing pieces. because imag root are

somehow missing

• key pt. in garlante work is the foll. to get analog of $\binom{hi}{k}$

Lemma 4.5 of his paper involves in the case of $\hat{sl}_2$ rewriting

$\cancel{x_0^{(r)}}\ x \quad x_0^{+(r)} (x_1^+)^s$

$(yt)^{(r)} (x)^{(s)}$ horrible looking formula

• analyzing the formula

imaginary root vectors one needs in $U_{\mathbb{Z}}(\hat{g})$ are monomials in

$P_\alpha(u)$ = coeff of u^i in

$$\exp \sum - \frac{h_\alpha \otimes t^n}{n} u^n .$$

$V(\lambda)$, $\lambda \in \hat{P}^+$

• $U_{\mathbb{Z}}(\hat{g}) v_\lambda$ is a lattice in $V(\lambda)$

and $\mathcal{U}_{\mathbb{Z}}(\hat{g})v_\lambda \cap V(\lambda)_\mu$ is a lattice in $V(\lambda)_\mu$ of ~~dim~~ rank $=$ dim $V(\lambda)_\mu$.

<u>Ques:</u> $V_k(\lambda) = K \otimes_{\mathbb{Z}} V_{\mathbb{Z}}(\lambda)$ char $K = p$?

what are results on $V_k(\lambda)$ — tons of literature in s.s. case about these modules

in affine case [CJing] λ = basic level = level 1, $V_{\mathbb{Z}}(\lambda)$ remained irreducible

<u>Ques</u> g — extend. aff Lie alg
— multi loop alg.

what is $\mathcal{U}_{\mathbb{Z}}(g)$.

· $P_n(a)$ although they appear in the context of $\mathcal{U}_{\mathbb{Z}}(\hat{g})$ — they play a crucial role in understanding level 0 reps of $\hat{g}$