Aperiodic Tilings: Notions and Properties

Michael Baake & Uwe Grimm

Faculty of Mathematics
University of Bielefeld, Germany

Department of Mathematics and Statistics
The Open University, Milton Keynes, UK
Quasicrystals
Dan Shechtman

Wolf Prize in Physics 1999

Nobel Prize in Chemistry 2011
Periodic point sets

**Definition:** A (discrete) point set $\Lambda \subset \mathbb{R}^d$ is called *periodic*, when $t + \Lambda = \Lambda$ holds for some $t \neq 0$. It is called *crystallographic* when the group of periods, $\text{per}(\Lambda) = \{ t \in \mathbb{R}^d \mid t + \Lambda = \Lambda \}$, is a lattice.

**Crystallographic restriction:** If $(t, M)$ is a Euclidean motion that maps a crystallographic point set $\Lambda \subset \mathbb{R}^d$ onto itself, the characteristic polynomial of $M$ has integer coefficients only.

In particular, for $d \in \{2, 3\}$, the possible rotation symmetries have order 1, 2, 3, 4 or 6.
Non-periodic point sets

Definition: A discrete point set $\Lambda \subset \mathbb{R}^d$ is called non-crystallographic when $\text{per}(\Lambda)$ is not a lattice, and non-periodic when $\text{per}(\Lambda) = \{0\}$.

Examples: $\mathbb{Z} \setminus \{0\}$
$(\mathbb{Z} \setminus \{0\}) \times \mathbb{Z}$

Definition: The hull of a discrete point set $\Lambda$ is defined as

$$\mathbb{X}(\Lambda) := \overline{\{t + \Lambda \mid t \in \mathbb{R}^d\}},$$

where the closure is taken in the local (rubber) topology.
Non-periodic point sets

Definition: A discrete point set $\Lambda \subset \mathbb{R}^d$ is called aperiodic when $\mathbb{X}(\Lambda)$ contains only non-periodic elements.

It is called strongly aperiodic when the remaining symmetry group of the hull is a finite group.
Aperiodic point sets

Silver mean substitution: \( a \mapsto aba, \ b \mapsto a \) \((\lambda_{PF} = 1 + \sqrt{2})\)

Silver mean point set: \( \Lambda = \{ x \in \mathbb{Z}[\sqrt{2}] \mid x' \in [-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}] \} \)
Model sets

\[ \mathbb{R}^d \xrightarrow{\pi} \mathbb{R}^d \times \mathbb{R}^m \xrightarrow{\pi_{\text{int}}} \mathbb{R}^m \]

\[ \cup \quad \cup \quad \cup \quad \text{dense} \]

CPS:
\[ \pi(\mathcal{L}) \xleftarrow{1-1} \mathcal{L} \xrightarrow{\pi_{\text{int}}} \pi_{\text{int}}(\mathcal{L}) \]

\[ \|
\]

Model set:
\[ \Lambda = \{ x \in L \mid x^* \in W \} \] (assumed regular)

\[ \text{with } W \subset \mathbb{R}^m \text{ compact, } \lambda(\partial W) = 0 \]

Diffraction:
\[ \hat{\gamma} = \sum_{k \in L^\circ} |A(k)|^2 \delta_k \]

\[ \text{with } L^\circ = \pi(\mathcal{L}^*) \quad \text{(Fourier module of } \Lambda) \]

and amplitude \( A(k) = \frac{\text{dens}(\Lambda)}{\text{vol}(W)} \mathbf{1}_W(-k^*) \)
Ammann-Beenker tiling

\[ L = \mathbb{Z}[\xi] \quad \mathcal{L} \sim \mathbb{Z}^4 \subset \mathbb{R}^2 \times \mathbb{R}^2 \quad O: \text{octagon} \]

\[ \xi = \exp(2\pi i / 8) \quad \phi(8) = 4 \quad \ast\text{-map: } \xi \mapsto \xi^3 \]

\[ \Lambda_{AB} = \{ x \in \mathbb{Z}1 + \mathbb{Z}\xi + \mathbb{Z}\xi^2 + \mathbb{Z}\xi^3 \mid x^* \in O \} \]
Ammann-Beenker tiling

physical space

internal space
Ammann-Beenker tiling
Aperiodic tilings
Aperiodic tilings

Many examples with hierarchical structure (see below).

**Exception:** The Kari-Culik prototile set
Question

Is there a single shape that tiles space without gaps or overlaps, but does not admit any periodic tiling?
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3D: Schmitt-Conway-Danzer ‘einstein’
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3D: Schmitt-Conway-Danzer ‘einstein’

2D: Penrose tiling (two tiles)
Question

Is there a single shape that tiles space without gaps or overlaps, but does not admit any periodic tiling?

3D: Schmitt-Conway-Danzer ‘einstein’

2D: Penrose tiling (two tiles)

No monotile known — but Penrose’s $1 + \varepsilon + \varepsilon^2$ tiling
The Taylor Tiling: Story

19 Feb 2010: Email from Joshua Socolar announcing

An aperiodic hexagonal tile

(joint preprint with Joan M. Taylor)
The Taylor Tiling: Story

19 Feb 2010: Email from Joshua Socolar announcing
An aperiodic hexagonal tile
(joint preprint with Joan M. Taylor)

28 Feb 2010: Visit Joan Taylor in Burnie, Tasmania
The Taylor Tiling: Story

19 Feb 2010: Email from Joshua Socolar announcing

*An aperiodic hexagonal tile*

(joint preprint with Joan M. Taylor)

based on Joan’s unpublished manuscript

*Aperiodicity of a functional monotile*

which is available (with hand-drawn diagrammes) from

http://www.math.uni-bielefeld.de/sfb701/preprints/view/420

(slight difference in definition of matching rules)
Robinson’s tiling
Robinson’s tiling
Half-hex tiling
Half-hex tiling
Half-hex tiling
Half-hex tiling
Half-hex tiling
Half-hex tiling

hexagonal tile

still admits periodic tilings of the plane
Half-hex tiling
Penrose’s $1 + \varepsilon + \varepsilon^2$ tiling

- 3 tiles: $1 + \varepsilon + \varepsilon^2$
- ‘key tiles’ encode matching rule information
- proof of aperiodicity (Penrose)
- the $\varepsilon$ tile transmits information along edge
The monotile

(figures from Socolar & Taylor *An aperiodic hexagonal monotile*)
The monotile

(figures from Socolar & Taylor *An aperiodic hexagonal monotile*)
Forced patterns

(figures from Socolar & Taylor *An aperiodic hexagonal monotile*)
Filling the gaps

(figures from Socolar & Taylor *An aperiodic hexagonal monotile*)
Filling the gaps

(figures from Socolar & Taylor *An aperiodic hexagonal monotile*)
Filling the gaps

(figures from Socolar & Taylor An aperiodic hexagonal monotile)
Composition-decomposition method

(Franz Gähler 1993)

- method to show that matching rules (local rules) enforce non-periodicity
- based on inflation (self-similarity)

requirements:
- Inflation rule has to respect matching rules:
  Tiles that match must have decompositions that match
- In any admitted tiling, each tile can be composed, together with part of its neighbours, to a unique supertile
- The supertiles inherit markings that enforce equivalent matching rules
Taylor’s substitution

(figures from Taylor’s manuscript *Aperiodicity of a functional monotile*)
Taylor’s substitution

(figures from Taylor’s manuscript *Aperiodicity of a functional monotile*)

Section 7.

Matching rules for the 8 prototile set can be simplified.

J. TAYLOR

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Fig. 7.1. The letter-labels can be replaced by color-tone markers to equivalent effect. The necessary black curved strips can be implemented as triangular lugs and sockets.

Fig. 7.2. Blue and red stripes can be used as a mnemonic for clockwise and anti-clockwise respectively. Recall that the wing is made up of 12 hexagonal cells - dashed line.
Inflation tiling
Inflation tiling

Relation to Penrose’s $1 + \varepsilon + \varepsilon^2$ tiling:
References


