NOTES ON QUASI-CATEGORIES

ANDRÉ JOYAL

Contents

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1. Introduction

The notion of quasi-category was introduced by M. Boardman and R.Vogt in their work on homotopy invariant algebraic structures [BV]. Our goal is to extend category theory to quasi-categories. The extended theory has applications to homotopy theory, higher category theory and topos theory.

A first draft of this paper was written in the Fall of 2004 in view its publication in the Proceedings of the Conference on higher categories which was held at the IMA in Minneapolis in June 2004. Its content is based on the talks I have given on quasi-categories over the last five years. It is a collection of assertions, many of which have not yet been proved formally (many have recently been proved by Jacob Lurie). I am preparing a book of two volumes on the theory of quasi-categories.

2. Elementary aspects

2.1. We fix three arbitrary Grothendieck universes $\mathbf{U}_1$, $\mathbf{U}_2$ and $\mathbf{U}_3$, with $\mathbf{U}_1 \in \mathbf{U}_2 \in \mathbf{U}_3$. Entities in $\mathbf{U}_1$ are *small*, entities in $\mathbf{U}_2$ are *large* and entities in $\mathbf{U}_3$ are *extra-large* (small entities are large and large entities are extra-large but the converse is not true). For example, a category is said to be *small* (resp. *large*, *extra-large*) if its set of arrows belong to $\mathbf{U}_1$ (resp. $\mathbf{U}_2$, $\mathbf{U}_3$). We denote by **Set** the category of small sets and by **SET** the category of large sets. A category is *locally small* if its hom sets are small. We denote by **Cat** the category of small categories and by **CAT** the category of locally small large categories. The category **Cat** is large and the category **CAT** extra-large. We shall denote small categories by ordinary capital letters and large categories by curly capital letters.

2.2. We shall denote by **Set** the category of (small) sets, by **S** the category of (small) simplicial sets and by **Cat** the category of small categories. By definition, we have

$$\mathbf{S} = [\Delta^o, \mathbf{Set}],$$

where Δ is the category of finite non-empty ordinals and order preserving maps. See the appendix 31 for terminology and notation on simplicial sets. The simplicial interval $\Delta[1]$ is denoted by I. The category Δ is a full subcategory of **Cat**. Recall that the *nerve* of a small category C is the simplicial set NC obtained by putting

$$(NC)_n = \mathbf{Cat}([n], C)$$

for every $n \geq 0$. The nerve functor $N : \mathbf{Cat} \to \mathbf{S}$ is full and faithful. We shall regard it as an inclusion $N : \mathbf{Cat} \subset \mathbf{S}$ by adopting the same notation for a category and its nerve. The functor N has a left adjoint

$$\tau_1 : \mathbf{S} \to \mathbf{Cat},$$

where $\tau_1 X$ is the *fundamental category* of a simplicial set X. The classical fundamental groupoid $\pi_1 X$ is obtained by formally inverting the arrows of the category $\tau_1 X$. The functor $\tau_1 : \mathbf{S} \to \mathbf{Cat}$ preserves finite products by a result in [GZ].

2.3. We shall say that an arrow $f : a \to b$ in a simplicial set X is *quasi-invertible*, or that it is a *quasi-isomorphism*, if its image by the canonical map $X \to \tau_1 X$ is invertibe in the category $\tau_1 X$.

2.4. Recall that a simplicial set X is called a *Kan complex* if every horn $\Lambda^k[n] \to X$ $(n > 0, k \in [n])$ has a filler $\Delta[n] \to X$, $\Lambda^k[n] \to X$

$$\begin{array}{ccc} \Lambda^k[n] & \xrightarrow{\forall} & X \\ \downarrow & \nearrow_{\exists} & \\ \Delta[n]. & & \end{array}$$

We say that a horn $\Lambda^k[n] \to X$ is *inner* if $0 < k < n$. We call a simplicial set X a *quasi-category* if every inner horn $\Lambda^k[n] \to X$ has a filler. This notion was introduced by M. Boardman and R.Vogt in their work on homotopy invariant algebraic structures [BV]. A quasi-categories is sometime called a *weak Kan complex* in the literature [KP]. A Kan complex and the nerve of a category are examples of quasi-categories. In fact, a simplicial set X is (isomorphic) to the nerve of a category iff every inner horn $\Lambda^k[n] \to X$has a *unique* filler $\Delta[n] \to X$. We shall often say that a vertex in a quasi-category is an *object* of this quasi-category and that an arrow is a *morphism*. A *map* of quasi-categories is defined to be a map of simplicial sets. We denote by **QCat** the category of quasi-categories. If X is a quasi-category then so is the simplicial set X^A for any simplicial set A. Hence the category **QCat** is cartesian closed.

2.5. A quasi-category can be large.

We fix three arbitrary Grothendieck universes $\mathbf{S} = \mathbf{U}_1$, $\mathbf{U}_2$ and $\mathbf{U}_3$, with $\mathbf{U}_1 \in \mathbf{U}_2 \in \mathbf{U}_3$. Entities in $\mathbf{U}_1$ are *small*, entities in $\mathbf{U}_2$ are *large* and entities in $\mathbf{U}_3$ are *extra-large* (small entities are large and large entities are extra-large but the converse is not true). For example, a category is said to be *small* (resp. *large*, *extra-large*) if its set of objects and it set of arrows belong to $\mathbf{U}_1$ (resp. $\mathbf{U}_2$, $\mathbf{U}_3$). We denote by **Set** the category of small sets and by **SET** the category of large sets. A category is *locally small* if its hom sets are small. We denote by **Cat** the category of small categories and by **CAT** the category of locally small large categories. The category **Cat** is large and the category **CAT** extra-large. We shall denote small categories by ordinary capital letters and large categories by curly capital letters. The *cardinality* of a small category is defined to be the cardinality of its set of arrows. A *diagram* in a category $\mathcal{E}$ is a functor $D : K \to \mathcal{E}$, where K is a small category; the *cardinality* of D is defined to be the cardinality of its domain K.

A large simplicial set is defined to be a functor $\Delta^o \to \mathbf{SET}$ where **SET** is the category of sets in a Grothendieck universe. A large simplicial set X is *locally small* if the vertex map $X_n \to X_0^{n+1}$ has small fibers for every $n \geq 0$. Most large quasi-categories considered in these notes are locally small.

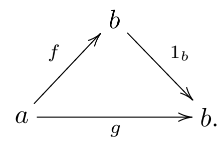
2.6. The fundamental category of a simplicial set X has a simpler description when X is a quasi-category. It is the *homotopy category* hoX described by Boardman and Vogt in [BV]. Here is a quick description of hoX. Consider the projection

$$p = (p_0, p_1) : X^I \to X^{\partial I} = X \times X$$

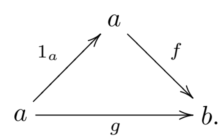
defined from the inclusion $\partial I = \{0, 1\} \subset I$. Its fiber $X(a, b)$ at $(a, b) \in X_0 \times X_0$ is the simplicial set of arrows $a \to b$ in X. It is a Kan complex when X is a quasi-category. We have

$$(hoX)(a, b) = \pi_0 X(a, b)$$

for every pair $a, b \in X_0 = Ob(hoX)$. We denote by $[f] : a \to b$ the homotopy class of an arrow $f : a \to b$. The homotopy relation $\sim$ between the arrows $a \to b$ has the following simple description. A *right homotopy* $u : f \Rightarrow_R g$ between two arrows $f, g : a \to b$ is defined to be a 2-simplex $u : \Delta[2] \to X$ with boundary $\partial u = (1_b, g, f)$,



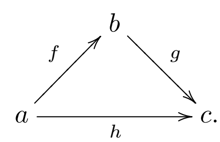
Dually a *left homotopy* $v : g \Rightarrow_L f$ is defined to be a 2-simplex $v : \Delta[2] \to X$ with boundary $\partial t = (f, g, 1_a)$,



It turns out that two arrows $f, g : a \to b$ are homotopic iff there exists a right homotopy $f \Rightarrow_R g$ iff there exists a right homotopy $g \Rightarrow_R f$ iff there exists a left homotopy $g \Rightarrow_L f$ iff there exists a left homotopy $f \Rightarrow_L g$. The composition law

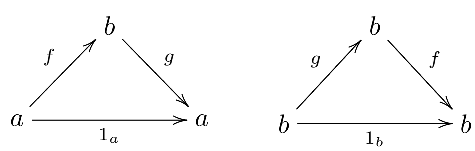
$$hoX(b,c) \times hoX(a,b) \;\; \to \;\; hoX(a,c)$$

of the category hoX can be described as follows. If $[f] : a \to b$ and $[g] : b \to c$, the horn $(g, \star, f) : \Lambda^1[2] \to X$ can be filled by a simplex $v : \Delta[2] \to X$,



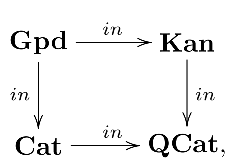
Then we have $[g][f] = [h]$, where $h = vd_1 : a \to c$.

2.7. If X is a quasi-category, then an arrow $f : a \to b$ in X is quasi-invertible iff there exists an arrow $g : b \to a$ together with two 2-simplices $u, v : \Delta[2] \to X$ with boundaries $\partial u = (g, 1_a, f)$ and $\partial v = (f, 1_b, g)$,



Let J be the groupoid generated by one isomorphism $0 \to 1$. It turns out that an arrow $f \in X$ is quasi-invertible, iff the map $I \to X$ which represents f can be extended along the inclusion $I \subset J$. See [J1].

2.8. There is an analogy between Kan complexes and groupoids. If **Gpd** denotes the category of groupoids and **Kan** denotes the category of Kan complexes, then we have **Kan** $\cap$ **Cat** $=$ **Gpd**, where the intersection is taken in **QCat** (or in **S**),



A quasi-category X is a Kan complex iff its homotopy category hoX is a groupoid [J1]. Hence we have a pullback square of categories,

$$\begin{array}{ccc} \mathbf{Gpd} & \xleftarrow{\pi_1} & \mathbf{Kan} \\ {\scriptstyle in}\downarrow & & \downarrow{\scriptstyle in} \\ \mathbf{Cat} & \xleftarrow{\tau_1} & \mathbf{QCat}. \end{array}$$

The inclusion functor $\mathbf{Gpd} \subset \mathbf{Cat}$ has a right adjoint $J : \mathbf{Cat} \to \mathbf{Gpd}$, where $J(C)$ is the groupoid of isomorphisms of a category C. Similarly, the inclusion functor $\mathbf{Kan} \subset \mathbf{QCat}$ has a right adjoint $J : \mathbf{QCat} \to \mathbf{Kan}$. The simplicial set $J(X)$ is the largest Kan subcomplex of a quasi-category X. The following naturality square is a pullback of simplicial sets,

$$\begin{array}{ccc} J(X) & \longrightarrow & J(hoX) \\ \downarrow & & \downarrow \\ X & \xrightarrow{h} & hoX, \end{array}$$

where h is the canonical map. Thus, a simplex $x : \Delta[n] \to X$ belongs to $J(X)$ iff the arrow $x(i-1, i)$ is a quasi-isomorphism for every $1 \leq i \leq n$. Moreover, the following two squares of functors commute up to a natural isomorphism,

$$\begin{array}{ccc} \mathbf{Gpd} & \xrightarrow{in} & \mathbf{Kan} \\ {\scriptstyle J}\uparrow & & \uparrow{\scriptstyle J} \\ \mathbf{Cat} & \xrightarrow{in} & \mathbf{QCat}, \end{array} \qquad \begin{array}{ccc} \mathbf{Gpd} & \xleftarrow{\pi_1} & \mathbf{Kan} \\ {\scriptstyle J}\uparrow & & \uparrow{\scriptstyle J} \\ \mathbf{Cat} & \xleftarrow{\tau_1} & \mathbf{QCat}. \end{array}$$

2.9. The category $\mathbf{S}$ has the structure of a 2-category. If A and B are simplicial sets, let us put

$$\tau_1(A, B) = \tau_1(B^A).$$

If we apply the functor τ_1 to the composition map $C^B \times B^A \to C^A$, we obtain the composition law

$$\tau_1(B, C) \times \tau_1(A, B) \to \tau_1(A, C)$$

of a 2-category $\mathbf{S}^{\tau_1}$ if we put $\mathbf{S}^{\tau_1}(A, B) = \tau_1(A, B)$. The 0-cells of this 2-category are simplicial sets and the 1-cells are the maps of simplicial sets. A 2-cell $\alpha : f \to g : A \to B$ is an arrow of the category $\tau_1(B^A)$. We shall say that α is a *natural transformation*. The 2-category $\mathbf{S}^{\tau_1}$ is cartesian closed.

2.10. There is a notion of equivalence in any 2-category. We shall say that a map of simplicial set is a *categorical equivalence* iff it is an equivalence in the 2-category $\mathbf{S}^{\tau_1}$. If X and Y are quasi-categories, a categorical equivalence $X \to Y$ is called an *equivalence of quasi-categories*. A map between quasi-categories $f : X \to Y$ is an equivalence iff there exists a map $g : Y \to X$ together with two quasi-isomorphisms $gf \to 1_X$ and $fg \to 1_Y$.

3. The model structure

3.1. We recall the construction of the homotopy category of simplicial sets $\mathbf{S}^{\pi_0}$ by Gabriel and Zisman in [GZ]. The category $\mathbf{S}$ is cartesian closed and the functor $\pi_0 : \mathbf{S} \to \mathbf{Set}$ preserves finite products. If $A, B \in \mathbf{S}$ let us put

$$\pi_0(A, B) = \pi_0(B^A).$$

If we apply the functor π_0 to the composition map $C^B \times B^A \to C^A$ we obtain a composition law $\pi_0(B, C) \times \pi_0(A, B) \to \pi_0(A, C)$ for a category $\mathbf{S}^{\pi_0}$, where we put $\mathbf{S}^{\pi_0}(A, B) = \pi_0(A, B)$. A map of simplicial sets is called a *homotopy equivalence* if it is invertible in the category $\mathbf{S}^{\pi_0}$.

3.2. A map of simplicial sets $u : A \to B$ is a *weak homotopy equivalence* if the map

$$\pi_0(u, X) : \pi_0(B, X) \to \pi_0(A, X)$$

is bijective for every Kan complex X. A map between Kan complexes is a weak homotopy equivalence iff it is a homotopy equivalence.

3.3. Recall that a map of simplicial sets $f : X \to Y$ is called a *Kan fibration* if it has the right lifting property with respect to the inclusion $\Lambda^k[n] \subset \Delta[n]$ for every $n > 0$ and $k \in [n]$,

$$\begin{array}{ccc} \Lambda^k[n] & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow f \\ \Delta[n] & \longrightarrow & Y \end{array} .$$

3.4. The category $\mathbf{S}$ admits a model structure in which the cofibrations are the monomorphisms and the weak equivalences are the weak homotopy equivalences [Q]. The fibrations are the Kan fibrations and the fibrant objects the Kan complexes. The acyclic fibrations are the *trivial fibrations* as defined in 31.10. The model structure is cartesian closed and proper. We shall denote it shortly by $(\mathbf{S}, \mathit{Who})$, where *Who* denotes the class of weak homotopy equivalences. We say that it is the *classical model structure* on $\mathbf{S}$.

3.5. We call a functor $p : E \to B$ (in $\mathbf{Cat}$) a *quasi-fibration* if for every object $x \in E$ and every isomorphism $g \in B$ with target $p(x)$, there exists an isomorphism $f \in E$ with target x such that $p(f) = g$. A functor $p : E \to B$ is a quasi-fibration iff the opposite functor $p^o : E^o \to B^o$ is a quasi-fibration. Hence a functor $p : E \to B$ is a quasi-fibration iff for every object $x \in E$ and every isomorphism $g \in B$ with source $p(x)$, there exists an isomorphism $f \in E$ with source x such that $p(f) = g$.

3.6. The category $\mathbf{Cat}$ admits a model structrure in which the weak equivalences are the equivalences of categories and the fibration are the quasi-fibrations [JT1]. A functor $u : A \to B$ is a cofibration iff the map $Ob(u) : ObA \to ObB$ is monic. Every object is fibrant and cofibrant. The model structure is cartesian and proper. We shall denote it shortly by $(\mathbf{Cat}, Eq)$, where Eq denotes the class of equivalences between categories.

3.7. If A is a simplicial set, we shall denote by $\tau_0 A$ the set of isomorphism classes of objects of the fundamental category $\tau_1 A$. The functor

$$\tau_0 : \mathbf{S} \to \mathbf{Set}$$

preserves finite products, since the functor $\tau_1 : \mathbf{S} \to \mathbf{Cat}$ preserves finite products. For any pair (A, B) of simplicial sets, let us put

$$\tau_0(A, B) = \tau_0(B^A).$$

If we apply the functor τ_0 to the the composition map $C^B \times B^A \to C^A$ we obtain the composition law $\tau_0(B, C) \times \tau_0(A, B) \to \tau_0(A, C)$ of a category $\mathbf{S}^{\tau_0}$, where we put $\mathbf{S}^{\tau_0}(A, B) = \tau_0(A, B)$. A map of simplicial sets $u : A \to B$ is a categorical equivalence iff u is invertible in the category $\mathbf{S}^{\tau_0}$.

3.8. We shall say that a map of simplicial sets $u : A \to B$ is a *weak categorical equivalence* if the map

$$\tau_0(u, X) : \tau_0(B, X) \to \tau_0(A, X)$$

is bijective for every quasi-category X. Equivalently, a map of simplicial sets $u : A \to B$ is a weak categorical equivalence iff the map

$$\tau_1(u, X) : \tau_1(B, X) \to \tau_1(A, X)$$

is an equivalence of categories for every quasi-category X iff the map

$$X^u : X^B \to X^A$$

is an equivalence of quasi-categories for every quasi-category X.

3.9. We shall say that a map of simplicial sets is a *quasi-fibration* if it has the right lifting property with respect to every monic weak categorical equivalences. The notion of quasi-fibration is self dual: a map $p : X \to Y$ is a quasi-fibration iff the opposite map $p^o : X^o \to Y^o$ is a quasi-fibration. The quasi-fibrations between quasi-categories have a simpler description. We shall say that a map of simplicial sets a *mid fibration* if it has the right lifting propery with respect to the inclusion $\Lambda^k[n] \subset \Delta[n]$ for every $0 < k < n$. Every quasi-fibration is a mid fibration. A mid fibration between quasi-categories $p : X \to Y$ is a quasi-fibration iff the functor $ho(p) : hoX \to hoY$ is a quasi-fibration in $\mathbf{Cat}$ iff for every object $x \in X$ and every quasi-isomorphism $g \in Y$ with target $p(x)$, there exists a quasi-isomorphism $f \in X$ with target x such that $p(f) = g$. A map between quasi-categories $p : X \to Y$ is a quasi-fibration iff the map

$$\langle j_0, p\rangle : X^J \to Y^J \times_Y X$$

obtained from the square

$$\begin{array}{ccc} X^J & \xrightarrow{X^{j_0}} & X \\ \downarrow & & \downarrow{\scriptstyle p} \\ Y^I & \xrightarrow{Y^{j_0}} & Y, \end{array}$$

is a trivial fibration, where j_0 denotes the inclusion $\{0\} \subset J$. See 31.10 for the notion of trivial fibration.

3.10. The category $\mathbf{S}$ admits a model structure in which the cofibrations are the monomorphisms and the weak equivalences are the weak categorical equivalences [J2]. The fibrations are the quasi-fibrations and the fibrant objects are the quasi-categories. The acyclic fibrations are the trivial fibrations. The model structure is cartesian closed and left proper. We shall say that it is the *model structure for quasi-categories*. We shall denote it shortly by $(\mathbf{S}, Wcat)$, where $Wcat$ denotes the class of weak categorical equivalences.

3.11. The pair of adjoint functors $\tau_1 : \mathbf{S} \leftrightarrow \mathbf{Cat} : N$ is a Quillen pair between the model categories $(\mathbf{S}, Wcat)$ and $(\mathbf{Cat}, Eq)$. A functor $u : A \to B$ in $\mathbf{Cat}$ is a quasi-fibration iff the map $Nu : NA \to NB$ is a quasi-fibration.

3.12. The classical model structure $(\mathbf{S}, Who)$ is a Bousfield localisation of the model structure $(\mathbf{S}, Wcat)$. Thus, a weak categorical equivalence is a weak homotopy equivalence and a Kan fibration is a quasi-fibration. The converse holds for a map between Kan complexes. A simplicial set A is weakly categorically equivalent to a Kan complex iff the category $\tau_1 A$ is a groupoid.

3.13. Consider the functor $k : \Delta \to \mathbf{S}$ defined by putting $k[n] = \Delta'[n]$ for every $n \geq 0$, where $\Delta'[n]$ denotes the (nerve of) the groupoid freely generated by the category $[n]$. If $X \in \mathbf{S}$, let us put

$$k^!(X)_n = \mathbf{S}(\Delta'[n], X)$$

for every $n \geq 0$. The functor $k^! : \mathbf{S} \to \mathbf{S}$ has a left adjoint $k_!$ which is the left Kan extension of the functor k along the Yoneda functor $y : \Delta \to \mathbf{S}$. The adjoint pair

$$k_! : \mathbf{S} \leftrightarrow \mathbf{S} : k^!$$

is a Quillen pair $k_! : (\mathbf{S}, Who) \leftrightarrow (\mathbf{S}, Wcat) : k^!$ and a homotopy coreflection [J2]. See 31.26 for the notion of homotopy coreflection. If X is a quasi-category, then the canonical map $k^!(X) \to X$ factors through the inclusion $J(X) \subseteq X$ and the induced map $k^!(X) \to J(X)$ is a trivial fibration.

3.14. A homotopy $\alpha : f \to g$ between two maps $f, g : A \to B$ is an arrow in the simplicial set X^A. The corresponding morphism $[\alpha] : f \to g$ in $\tau_1(A, B)$ is a natural transformation $f \to g$. When B is a quasi-category, the natural transformation $[\alpha]$ is invertible iff the arrow $\alpha(a) : f(a) \to g(a)$ is quasi-invertible in B for every vertex $a \in A$. We say that a map of simplicial sets $u : A \to B$ is *conservative* if the functor $\tau_1 u : \tau_1 A \to \tau_1 B$ is conservative. If a map of simplicial sets $u : A \to B$ is essential surjective then the map $X^u : X^B \to X^A$ is conservative for every quasi-category X.

3.15. Recall that a Kan complex X is said to be *minimal* if it contains no proper equivalent sub-complex $S \subset X$ (this means that the inclusion $S \subset X$ is not an homotopy equivalence for any proper subcomplex S of X. Every Kan complex contains a minimal subcomplex which is unique up to isomorphism. We say that a quasi-category X is *minimal* if it contains no proper equivalent sub-quasi-category $S \subset X$. Every quasi-category contains a minimal quasi-category which is unique up to isomorphism.

4. Equivalence with simplicial categories

Recall that a *simplicial category* is a category enriched over simplicial sets. We denote by **SCat** the category of simplicial categories and (strong) functors. The inclusion functor $\mathbf{Cat} \subset \mathbf{SCat}$ has a left adjoint $ho : \mathbf{SCat} \to \mathbf{Cat}$ which associates to a simplicial category X its *homotopy category* $ho(X)$. By construction, we have $ho(X)(a, b) = \pi_0 X(a, b)$ for every pair of objects $a, b \in X$.

4.1. We shall say that a strong functor $f : X \to Y$ is *weakly fully faithful* if the map $X(a, b) \to Y(fa, fb)$ induced by f is a weak homotopy equivalence for every pair of objects $a, b \in X$. We shall say that f is *weakly essentially surjective* if the functor $ho(f) : ho(X) \to ho(Y)$ is essentially surjective. A functor $f : X \to Y$ is called a *Dwyer-Kan equivalence* if it is weakly fully faithful and weakly essentially surjective. Here is another description of the Dwyer-Kan equivalences. Let us denote by $Ho(\mathbf{S})$ the classical homotopy category, obtained by inverting the weak homotopy equivalences in $\mathbf{S}$. The canonical functor $\mathbf{S} \to Ho(\mathbf{S})$ induces a functor $Ho : \mathbf{SCat} \to Ho(\mathbf{S})\mathbf{Cat}$, where $Ho(\mathbf{S})\mathbf{Cat}$ denotes the category of categories enriched over $Ho(\mathbf{S})$. Then a functor $f : X \to Y$ in **SCat** is a Dwyer-Kan equivalence iff the functor $Ho(f) : Ho(X) \to Ho(Y)$ is an equivalence of categories enriched over $Ho(\mathbf{S})$.

4.2. A functor $f : X \to Y$ is called a *Dwyer-Kan fibration* if the map $X(a, b) \to Y(fa, fb)$ induced by f is a Kan fibration for every pair of objects $a, b \in X$ and the functor $ho(f) : ho(X) \to ho(Y)$ is a quasi-fibration in **Cat**. The category **SCat** admits a model structure in which a weak equivalence is a Dwyer-Kan equivalence and a fibration is a Dwyer-Kan fibration [B1]. The fibrant objects are the categories enriched over Kan complexes. A map $f : X \to Y$ is an acyclic fibration iff the map $Ob(f) : ObX \to ObY$ is surjective and the map $X(a, b) \to Y(fa, fb)$ is a trivial fibration for every pair of objects $a, b \in X$.

4.3. Recall that a *reflexive graph* is a 1-truncated simplicial set. Let **Grph** be the category of reflexive graphs. The obvious forgetful functor $U : \mathbf{Cat} \to \mathbf{Grph}$ has a left adjoint F. The composite $C = FU$ has the structure of a comonad on **Cat**. Hence the sequence $C_n A = C^{n+1}(A)$ for $n \geq 0$ has the structure of a simplicial object $C_*(A)$ in **Cat** for any small category A. The simplicial set $n \mapsto Ob(C_n A)$ is constant with value $Ob(A)$. It follows that $C_* A$ can be viewed as a simplicial category instead of a simplicial object in **Cat**. This defines a functor

$$C_* : \mathbf{Cat} \to \mathbf{SCat}.$$

If A is a category and X is a simplicial category, a *homotopy coherent diagram* $A \to X$ is defined to be a simplicial functor $C_*(A) \to X$, This notion was introduced by Vogt in [V]. The *coherent nerve* of a simplicial category X is the simplicial set $C^! X$ defined by putting

$$(C^! X)_n = \mathbf{SCat}(C_\star[n], X)$$

for every $n \geq 0$. This notion was introduced by Cordier in [C]. The simplicial set $C^!(X)$ is a quasi-category when X is enriched over Kan complexes [CP]. The functor $C^! : \mathbf{SCat} \to \mathbf{S}$ has a left adjoint $C_!$ which is the left Kan extension of the functor $[n] \mapsto C_\star[n]$ along the Yoneda functor $\Delta \to \mathbf{S}$. It turns out that we have $C_! A = C_\star A$ for every category A [J3].

4.4. The pair of adjoint functors

$$C_! : \mathbf{S} \leftrightarrow \mathbf{SCat} : C^!$$

is a Quillen equivalence between the model category for quasi-categories and the model category for simplicial categories [J3].

4.5. A simplicial category can be large. For example, the category **Kan** of Kan complexes is a large simplicial category. The coherent nerve of the category **Kan** is a large quasi-category that we shall denote by **Hot** or by $\mathbf{Hot}_1$. It plays an important role in the theory of quasi-categories. It is the analog of the category of sets.

4.6. The category **QCat** becomes enriched over Kan complexes if we put

$$Hom^J(X, Y) = J(Y^X)$$

for $X, Y \in \mathbf{QCat}$. The coherent nerve of **QCat** is a large quasi-category that we shall denote by $\mathbf{HOT_2}$. It is the analog of the category of small categories.

5. Left and right coverings

5.1. We recall that a pair $(\mathcal{A}, \mathcal{B})$ of classes of maps in a category $\mathcal{E}$ is said to be a *factorisation system* if the following conditions are satisfied:

- the classes $\mathcal{A}$ and $\mathcal{B}$ are closed under composition and contain the isomorphisms;
- every map $f : A \to B$ admits a factorisation $f = pu : A \to E \to B$ with $u \in \mathcal{A}$ and $p \in \mathcal{B}$, and the factorisation is unique up to unique isomorphism.

In this definition, the uniqueness of the factorisation of a map $f : A \to B$ means that for any other factorisation $f = qv : A \to F \to B$ with $v \in \mathcal{A}$ and $q \in \mathcal{B}$, there exists a unique isomorphism $i : E \to F$ such that $iu = v$ and $qi = p$,

$$\begin{array}{ccc} A & \xrightarrow{v} & F \\ {\scriptstyle u}\downarrow & \overset{i}{\nearrow} & \downarrow{\scriptstyle q} \\ E & \xrightarrow[p]{} & B. \end{array}$$

5.2. A functor $p : E \to B$ is said to be a *discrete fibration* if for every object $e \in E$ and every arrow $g \in B$ with target $p(e)$, there exists a unique arrow $f \in E$ with target e such that $p(f) = e$. There is a dual notion of *discrete opfibration* . If X is a presheaf on B, then the canonical functor $el(X) \to B$ is a discrete fibration, where $el(X) = B/X$ is the category of elements of X. Let us denote by $\mathbf{Disc}(B)$ the full subcategory of $\mathbf{Cat}/B$ whose objects are the discrete fibrations $E \to B$. Then the functor $X \mapsto el(X)$ induces an equivalence of categories

$$el : [B^o, \mathbf{Set}] \simeq \mathbf{Disc}(B).$$

5.3. We recall that a functor between small categories $u : A \to B$ is said to be *final*, but we shall say *0-final*, if the canonical map

$$\varinjlim_{C} Xu \to \varinjlim_{B} X$$

is an isomorphism for every diagram $X : B \to \mathcal{E}$ with values in a cocomplete category $\mathcal{E}$. A functor $u : A \to B$ is 0-final iff the canonical map

$$\varprojlim_{B} X \to \varprojlim_{C} Xu$$

is an isomorphism for every presheaf $X : B^o \to \mathbf{Set}$. Recall that a functor $u : A \to B$ induces a pair of adjoint functors between the categories of presheaves

$$u_! : [A^o, \mathbf{Set}] \to [B^o, \mathbf{Set}] : u^*.$$

A functor $u : A \to B$ is 0-final iff we have $u_!(1) = 1$, where 1 denotes terminal objects. If $X \in [A^o, \mathbf{Set}]$, then we have

$$u_!(X)(b) = \varinjlim_{b\backslash A} Xq_b$$

for every object $b \in B$, where the category $b\backslash A = (b\backslash B) \times_B A$ is defined by the pullback square

$$\begin{array}{ccc} b\backslash A & \xrightarrow{q_b} & A \\ \downarrow & & \downarrow u \\ b\backslash B & \longrightarrow & B. \end{array}$$

A functor $u : A \to B$ is 0-final iff the category $b\backslash A$ is connected for every object $b \in B$.

5.4. The category **Cat** admits a factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ is the class of 0-final functors and $\mathcal{B}$ is the class of discrete fibrations.

5.5. Dually, we shall say that a functor $u : A \to B$ is *0-initial*) if the opposite functor $u^o : A^o \to B^o$ is 0-final. The category **Cat** admits a factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ is the class of 0-initial functors and $\mathcal{B}$ is the class of discrete opfibrations.

5.6. We shall say that a functor $p : E \to B$ in **Cat** is a *covering* it is both a discrete fibration and a discrete opfibration. These notions coincide when B is a groupoid. The functor $\pi_1 : \mathbf{Cat} \to \mathbf{Grp}$ induces an equivalence between the category of coverings of B and the category of coverings of $\pi_1 B$. The inverse equivalence associates to a covering of $\pi_1 B$ its base change along the canonical functor $B \to \pi_1 B$. We shall say that a functor $u : A \to B$ in **Cat** is 0-*connected* if the functor $\pi_1 u : \pi_1 A \to \pi_1 B$ is 0-initial (or equivalently 0-final).

5.7. The category **Cat** admits a factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ is the class of 0-connected functors and $\mathcal{B}$ is the class of coverings.

5.8. Recall that a map $u : A \to B$ in a category $\mathcal{E}$ is said to be *left orthogonal* to a map $f : X \to Y$, and f *right orthogonal* to u, if every commutative square

$$\begin{array}{ccc} A & \xrightarrow{x} & X \\ {\scriptstyle u}\downarrow & \nearrow & \downarrow{\scriptstyle f} \\ B & \xrightarrow[y]{} & Y \end{array}$$

has a *unique* diagonal filler $d : B \to X$ (that is, $du = x$ and $fd = y$). We denote this relation by $u \perp f$.

5.9. We shall say that a map of simplicial sets $f : X \to Y$ is a *left covering* if it is right orthogonal to the inclusion $\{0\} \subseteq \Delta[n]$ for every $n \geq 0$. Dually, we shall say that f is a *right covering* if it is right orthogonal to the inclusion $\{n\} \subseteq \Delta[n]$ for every $n \geq 0$. We shall say that f is a *covering* if it is both a left and a right covering.

5.10. We shall say that a map of simplicial set $u : A \to B$ is *0-final* if the functor $\tau_1(u) : \tau_1 A \to \tau_1 B$ is 0-final. Dually shall say that u is *0-initial* if the functor $\tau_1(u) : \tau_1 A \to \tau_1 B$ is 0-initial. We shall say that u is *0-connected* if the functor $\pi_1(u) : \pi_1 A \to \pi_1 B$ is 0-connected.

5.11. Each of the following pair $(\mathcal{A}, \mathcal{B})$ of classes of maps in $\mathbf{S}$ is a factorisation system:

- $\mathcal{A}$ is the class of 0-connected maps and $\mathcal{B}$ the class of coverings;
- $\mathcal{A}$ is the class of 0-final maps and $\mathcal{B}$ the class of right coverings;
- $\mathcal{A}$ is the class of 0-initial maps and $\mathcal{B}$ the class of left coverings.

6. Join and slice

For any category C and any object $b \in C$ there is a category C/b of morphisms $a \to b$. We shall see that for any simplicial set X, there is a simplicial set X/a for any vertex $a \in X$. More generally, we shall construct a simplicial set X/a for any map of simplicial sets $a : A \to X$. For this, we introduce the join of simplicial sets. We use augmented simplicial sets for defining the join of simplicial sets.

6.1. The *join* of two categories A and B is the category $C = A \star B$ obtained as follows: $Ob(C) = Ob(A) \sqcup Ob(B)$ and for any pair of objects $x, y \in Ob(A) \sqcup Ob(B)$ we have

$$C(x,y) = \begin{cases} A(x,y) & \text{if } x \in A \text{ and } y \in A \\ B(x,y) & \text{if } x \in B \text{ and } y \in B \\ 1 & \text{if } x \in A \text{ and } y \in B \\ \emptyset & \text{if } x \in B \text{ and } y \in A. \end{cases}$$

Composition or arrows is obvious. Notice that the category $A \star B$ is a poset if A and B are posets: it is the *ordinal sum* of the posets A and B. The operation $(A, B) \mapsto A \star B$ is functorial and coherently associative. It defines a monoidal structure on $\mathbf{Cat}$, with the empty category as the unit object. The monoidal category $(\mathbf{Cat}\star)$ is not symmetric but there is a natural isomorphism

$$(A \star B)^o = B^o \star A^o.$$

The category $1 \star A$ is called the *projective cone with base* A and the category $A \star 1$ the *inductive cone with cobase* A. The object 1 is terminal in $A \star 1$ and initial in $1 \star A$. The category $A \star B$ can be equipped with the functor $A \star B \to I = 1 \star 1$ obtained by joining the functors $A \to 1$ and $B \to 1$. The resulting functor $\star : \mathbf{Cat} \times \mathbf{Cat} \to \mathbf{Cat}/I$ is right adjoint to the functor

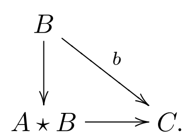
$$i^* : \mathbf{Cat}/I \to \mathbf{Cat} \times \mathbf{Cat},$$

where i denotes the inclusion $\{0,1\} = \partial I \subset I,$. This gives another description of the join operation.

6.2. The monoidal category $(\mathbf{Cat}, \star)$ is not closed. But the functor

$$(-) \star C : \mathbf{Cat} \to C \backslash \mathbf{Cat}$$

which associates to $X \in \mathbf{Cat}$ the inclusion $C \subseteq X \star C$ has a right adjoint for any category C. The right adjoint takes a functor $b : B \to C$ to a category that we shall denote by C/b, or more simply by C/B if the functor b is clear from the context. For any category A, there is a bijection between the functors $A \to C/b$ and the functors $A \star B \to C$ which extends b along the inclusion $B \subseteq A \star B$,



In particular, an object $1 \to C/b$ is a functor $c : 1 \star B \to C$ which extends b; it is a *projective cone with base* b in C.

6.3. Dually, the functor $A \star (-) : \mathbf{Cat} \to A \backslash \mathbf{Cat}$ has a right adjoint which takes a functor $a : A \to C$ to a category that we shall denote $a \backslash C$, or more simply by $A \backslash C$ if the functor a is clear from the context. An object $1 \to a \backslash C$ is a functor $c : A \star 1 \to C$ which extends a; it is an *inductive cone with cobase* a.

6.4. We shall denote by Δ_+ the category of all finite ordinals and order preserving maps, *including* the empty ordinal 0. We shall denote the ordinal n by n, so that we have $n = [n-1]$ for $n \geq 1$. We may occasionally denote the ordinal 0 by $[-1]$. Notice the isomorphism of categories $1 \star \Delta = \Delta_+$. The ordinal sum $(m, n) \mapsto m + n$ is functorial with respect to order preserving maps. This defines a monoidal structure on Δ_+,

$$+ : \Delta_+ \times \Delta_+ \to \Delta_+,$$

with 0 as the unit object.

6.5. Recall that an *augmented simplicial set* is defined to be a contravariant functor $\Delta_+ \to \mathbf{Set}$. We shall denote by $\mathbf{S}_+$ the category of augmented simplicial sets. By a general procedure due to Brian Day [Da], the monoidal structure of Δ_+ can be extended to $\mathbf{S}_+$ as a closed monoidal structure

$$\star : \mathbf{S}_+ \times \mathbf{S}_+ \to \mathbf{S}_+$$

with $0 = y(0)$ as the unit object. We call $X \star Y$ the *join* of the augmented simplicial sets X and Y. We have

$$(X \star Y)(n) = \bigsqcup_{i+j=n} X(i) \times Y(j)$$

for every $n \geq 0$.

6.6. From the inclusion $t : \Delta \subset \Delta_+$ we obtain a pair of adjoint functors

$$t^* : \mathbf{S}_+ \leftrightarrow \mathbf{S} : t_*.$$

The functor t^* removes the augmentation of an augmented simplicial set. The functor t_* gives a simplicial set A the trivial augmentation $A_0 \to 1$. Notice that $t_*(\emptyset) = 0 = y(0)$, where y is the Yoneda map $\Delta_+ \to \mathbf{S}_+$. The functor t_* is fully faithful and we shall regard it as an inclusion $t_* : \mathbf{S} \subset \mathbf{S}_+$. The operation $\star$ on $\mathbf{S}_+$ induces a monoidal structure on $\mathbf{S}$,

$$\star : \mathbf{S} \times \mathbf{S} \to \mathbf{S}.$$

By definition, $t_*(A \star B) = t_*(A) \star t_*(B)$ for any pair $A, B \in \mathbf{S}$. We call $A \star B$ the *join* of the simplicial sets A and B. It follows from the formula above, that we have

$$(A \star B)_n = A_n \sqcup B_n \sqcup \bigsqcup_{i+1+j=n} A_i \times A_j.$$

for every $n \geq 0$. Notice that we have

$$A \star \emptyset = A = \emptyset \star A$$

for any simplicial set A, since $t_*(\emptyset) = 0$ is the unit object for the operation $\star$ on $\mathbf{S}_+$. Hence the empty simplicial set is the unit object for the join operation on $\mathbf{S}$. The monoidal category $(\mathbf{S}, \star)$ is not symmetric but there is a natural isomorphism

$$(A \star B)^o = B^o \star A^o.$$

For every pair $m, n \geq 0$ we have

$$\Delta[m] \star \Delta[n] = \Delta[m + 1 + n]$$

since we have $[m] + [n] = [m + n + 1]$. In particular,

$$1 \star 1 = \Delta[0] \star \Delta[0] = \Delta[1] = I.$$

The simplicial set $1 \star A$ is called the *projective cone with base* A and the simplicial set $A \star 1$ the *inductive cone with cobase* A.

6.7. If A and B are simplicial sets, then the join of the maps $A \to 1$ and $B \to 1$ is a canonical map $A \star B \to I$. This defines a functor

$$\star : \mathbf{S} \times \mathbf{S} \to \mathbf{S}/I$$

which is right adjoint to the functor $i^* : \mathbf{S}/I \to \mathbf{S} \times \mathbf{S} = \mathbf{S}/\partial I$, where i denotes the inclusion $\{0, 1\} = \partial I \subset I$. This gives another description of the join operation for simplicial sets.

6.8. The monoidal category $(\mathbf{S}, \star)$ is not closed. But the functor

$$(-) \star B : \mathbf{S} \to B \backslash \mathbf{S}$$

which associates to X the inclusion $B \subseteq X \star B$ has a right adjoint for any simplicial set B. The right adjoint takes a map $b : B \to X$ to a simplicial set that we shall denote by X/b, or more simply by X/B if the map b is clear from the context. We shall say that X/b is obtained by *slicing* X *over* b. For any simplicial set A, there is a bijection between the maps $A \to X/b$ and the maps $A \star B \to X$ which extends b along the inclusion $B \subseteq A \star B$,

$$\begin{array}{ccc} B & & \\ \downarrow & \overset{b}{\searrow} & \\ A \star B & \longrightarrow & X. \end{array}$$

In particular, a vertex $1 \to X/b$ is a map $c : 1 \star B \to X$ which extends the map b; it a *projective cone with base* b in X. The simplicial set X/b is a quasi-category when X is a quasi-category.

6.9. Dually, the functor $A \star (-) : \mathbf{S} \to A\backslash\mathbf{S}$ has a right adjoint. The right adjoint takes a map $a : A \to X$ to a simplicial set that we shall denote $a\backslash X$, or more simply by $A\backslash X$ if the map a is clear from the context. We shall say that $a\backslash X$ is obtained by *slicing* X *under* a. A vertex $1 \to a\backslash X$ is a map $c : A \star 1 \to X$ which extends the map a; it is an *inductive cone with cobase* a in X.

6.10. If A, B and X are simplicial sets, we obtain a natural inclusion $A \star B \subseteq A \star X \star B$ by joining the maps $1_A : A \to A$, $\emptyset \to X$ and $1_B : B \to B$. The functor

$$A \star (-) \star B : \mathbf{S} \to (A \star B)\backslash\mathbf{S}$$

which associates to X the inclusion $A \star B \subseteq A \star X \star B$ has a right adjoint for any pair A and B. The right adjoint takes a map of simplicial sets $f : A \star B \to X$ to a simplicial set that we shall denote $\mathit{Fact}(f, X)$. A vertex $1 \to \mathit{Fact}(f, X)$ is a map $g : A \star 1 \star B \to X$ which extends f. When $A = B = 1$, it is a factorisattion of the arrow $f : I \to X$. If f is an arrow $a \to b$ then $\mathit{Fact}(f, X) = f\backslash(X/b) = (a\backslash X)/f$.

6.11. Recall that a model structure on a category $\mathcal{E}$ induces a model structure on the slice category $\mathcal{E}/B$ for each object $B \in \mathcal{E}$. In particular, we have a model category $(B\backslash\mathbf{S}, \mathit{Wcat})$ for each simplicial set B. The pair of adjoint functors $X \mapsto X \star B$ and $(X, b) \mapsto X/b$ is a Quillen pair between the model categories $(\mathbf{S}, \mathit{Wcat})$ and $(B\backslash\mathbf{S}, \mathit{Wcat})$.

6.12. There are other constructions of the join of two simplicial sets. For example, the *fat join* of A and B is the simplicial set $A \diamond B$ defined by the pushout square

$$\begin{array}{ccc} (A \times 0 \times B) \sqcup (A \times 1 \times B) & \longrightarrow & A \sqcup B \\ \downarrow & & \downarrow \\ A \times I \times B & \longrightarrow & A \diamond B. \end{array}$$

We have $A \sqcup B \subseteq A \diamond B$ and there is a canonical map $A \diamond B \to I$. The functor $(A, B) \mapsto A \diamond B$, from $\mathbf{S} \times \mathbf{S}$ to $\mathbf{S}/I$, is continuous. In the category $\mathbf{S} \times \mathbf{S}$ we have $(X, Y) = (X, 1) \times (1, Y)$. It follows that we have

$$X \diamond Y = (X \diamond 1) \times_I (1 \diamond Y).$$

For a fixed $B \in \mathbf{S}$, the functor $(-) \diamond B : \mathbf{S} \to B\backslash\mathbf{S}$ which takes a simplicial set X to the inclusion $B \subseteq X \diamond B$ has a right adjoint. The right adjoint takes a map $b : B \to X$ to a simplicial set that we shall denote by $X/\!\!/b$; we shall say that $X/\!\!/b$ is a *fat slice*. If $b \in X_0$, the simplicial set $X/\!\!/b$ is the fiber of the target map $X^I \to X$ at b.

6.13. The pair of adjoint functors $X \mapsto X \diamond B$ and $(X, b) \mapsto X /\!\!/ b$ is a Quillen adjoint pair between the model categories $(\mathbf{S}, Wcat)$ and $(B\backslash\mathbf{S}, Wcat)$.

6.14. The square

$$\begin{array}{ccc} A \sqcup B & \longrightarrow & A \star B \\ \downarrow & & \downarrow \\ A \diamond B & \longrightarrow & I \end{array}$$

has a unique diagonal filler θ_{AB}. This defines a natural transformation $\theta_{AB} : A \diamond B \to A \star B$. By adjointness, we obtain a natural map $\theta'_b : X/b \to X/\!\!/b$ for any map of simplicial sets $b : B \to X$. The map $\theta_{AB} : A \diamond B \to A \star B$ is a weak categorical equivalence for any pair of simploicial sets A and B and the map $\theta'_b : X/b \to X/\!\!/b$ is an equivalence of quasi-categories when X is a quasi-category.

7. Left and right fibrations

7.1. We call a map of simplicial sets $p : X \to B$ a *right fibration* if it has the right lifting property with respect to the inclusion $h_n^k : \Lambda^k[n] \subset \Delta[n]$ for every $0 < k \leq n$. The fibers of a right fibration are Kan complexes. A map $p : X \to B$ is a right fibration iff the map

$$\langle i_1, p\rangle : X^I \to B^I \times_B X$$

obtained from the inclusion $i_1 : \{0\} \subset I$ is a trivial fibration. If $f : X \to Y$ is a right fibration, then so is the map

$$\langle u, f\rangle : X^B \to Y^B \times_{Y^A} X^A.$$

for any monomorphism $u : A \to B$.

7.2. If A and B are categories, then a functor $p : A \to B$ is a right fibration iff p is a Grothendieck fibration whose fibers are groupoids.

7.3. We say that a map is *right anodyne* if it belongs to the saturated class generated by the inclusions $\Lambda^k[n] \subset \Delta[n]$ with $0 \leq k < n$. If $\mathcal{A}$ is the class of right anodyne maps and $\mathcal{B}$ is the class of right fibrations, then the pair $(\mathcal{A}, \mathcal{B})$ is a weak factorisation system on $\mathbf{S}$.

7.4. Dually, we call a map of simplicial sets $p : X \to Y$ a *left fibration* if it has the right lifting property with respect to the inclusion $h_n^k : \Lambda^k[n] \subset \Delta[n]$ for every $0 \leq k < n$. We say that a map is *left anodyne* if it belongs to the saturated class generated by the inclusions $\Lambda^k[n] \subset \Delta[n]$ with $0 \leq k < n$. If $\mathcal{A}$ is the class of left anodyne maps and $\mathcal{B}$ is the class of left fibrations, then the pair $(\mathcal{A}, \mathcal{B})$ is a weak factorisation system on $\mathbf{S}$.

7.5. A map $X \to B$ is a left fibration iff the opposite map $X^o \to B^o$ is a right fibration. A map $u : A \to B$ is left anodyne iff the opposite map $u^o : A^o \to B^o$ is right anodyne

7.6. If B is a simplicial set, we say that an object $X = (X, p)$ of the category $\mathbf{S}/B$ is a *simplicial set over* B and that $p : X \to B$ is its *structure map*. We say that a map $u : X \to Y$ in $\mathbf{S}/B$ is a *map over* B. The category $\mathbf{S}/B$ is enriched over $\mathbf{S}$. We shall denote by $[X, Y]_B$, or more simply by $[X, Y]$, the simplicial set of maps $X \to Y$ between two objects $X, Y \in \mathbf{S}/B$. We have a composition map

$$[Y, Z] \times [X, Y] \to [X, Z]$$

for every triple $X, Y, Z \in \mathbf{S}/B$.

7.7. The *homotopy category* $(\mathbf{S}/B)^{\pi_0}$ is defined by putting

$$(\mathbf{S}/B)^{\pi_0}(X, Y) = \pi_0[X, Y]$$

for every pair $X, Y \in \mathbf{S}/B$, The composition law

$$\pi_0[Y, Z] \times \pi_0[X, Y] \to \pi_0[X, Z]$$

is obtained by applying the functor π_0 to the composition map $[Y, Z] \times [X, Y] \to [X, Z]$. There is an obvious canonical functor $\mathbf{S} \to \mathbf{S}/B$. We say that a map in $\mathbf{S}/B$. is a *fibrewise homotopy equivalence* if the map is invertible in the category $(\mathbf{S}/B)^{\pi_0}$. Let $\mathbf{R}(B)$ the full subcategory of $\mathbf{S}/B$ spanned by the right fibrations $X \to B$. If $X, Y \in \mathcal{R}(B)$, then a map $u : X \to Y$ is a fiberwise homotopy equivalence iff the induced map between the fibers $u_b : X_b \to Y_b$ is a homotopy equivalence for every vertex $b \in B$.

7.8. We shall say that a map $u : M \to N$ in $\mathbf{S}/B$ is a *contravariant equivalence* if the map

$$\pi_0[u, X] : \pi_0[M, X] \to \pi_0[N, X]$$

is bijective for every $X \in \mathbf{R}(B)$. Every right anodyne map between two objects of $\mathbf{S}/B$ is a contravariant equivalence.

7.9. We shall say that a map in $\mathbf{S}/B$ is a *contravariant fibration* if it has the right lifting property with respect to the monic contravariant equivalences in $\mathbf{S}/B$. Every contravariant fibration in is a right fibration and the converse is true for a map in $\mathbf{R}(B)$.

7.10. The category $\mathbf{S}/B$ admits a left proper simplicial model structure in which a cofibration is a monomorphism, a weak equivalence is a contravariant equivalence and a fibration is a contravariant fibration. An object in $\mathbf{S}/B$ is fibrant iff it belongs to $\mathbf{R}(B)$. A contravariant fibration is acyclic iff it is a trivial fibration. This defines the *contravariant model structure* on $\mathbf{S}/B$. We shall denote it shortly by $(\mathbf{S}/B, \mathcal{W}^c(B))$, or more simply by $(\mathbf{S}/B, \mathcal{W}^c)$.

7.11. Every right fibration $X \to B$ has a minimal model which is unique up to isomorphism.

7.12. The model structure $(\mathbf{S}, Wcat)$ induces a model structure $(\mathbf{S}/B, Wcat)$ on the category $\mathbf{S}/B$ for each simplicial set B. The contravariant model structure $(\mathbf{S}/B, \mathcal{W}^c)$ is a Bousfield localisation of the model structure $(\mathbf{S}/B, Wcat)$.

7.13. Dually, let $\mathbf{L}(B)$ be the full subcategory of $\mathbf{S}/B$ whose objects are left fibrations $X \to B$. We say that a map $u : M \to N$ in $\mathbf{S}/B$ a *covariant equivalence* if the map

$$\pi_0[u, X] : \pi_0[M, X] \to \pi_0[N, X]$$

is bijective for every $X \in \mathbf{L}(B)$. We shall say that a map in $\mathbf{S}/B$ is a *covariant fibration* if it has the right lifting property with respect to the monic covariant equivalences in $\mathbf{S}/B$. Every covariant fibration in is a left fibration and the converse is true for a map in $\mathbf{L}(B)$.

7.14. The category $\mathbf{S}/B$ admits a simplical model structure in which a cofibration is a monomorphism, a weak equivalence is a covariant equivalence and a fibration is a covariant fibration. An object in $\mathbf{S}/B$ is fibrant iff it belongs to $\mathbf{L}(B)$. This defines the *covariant model structure* on $\mathbf{S}/B$. We shall denote it shortly by $(\mathbf{S}/B, \mathcal{W}_c(B))$, or more simply by $(\mathbf{S}/B, \mathcal{W}_c)$.

7.15. The functor $X \mapsto X^o$ induces an isomorphism of model categories

$$(\mathbf{S}/B, \mathcal{W}_c) \simeq (\mathbf{S}/B^o, \mathcal{W}^c).$$

Thus, a map $u : M \to N$ in $\mathbf{S}/B$ is a covariant equivalence iff the opposite map $u^o : M^o \to N^o$ is a contravariant equivalence in $\mathbf{S}/B^o$. A map $f : X \to Y$ in $\mathbf{S}/B$ is a covariant fibration iff the opposite map $f^o : X^o \to Y^o$ is a cntravariant fibration in $\mathbf{S}/B^o$.

7.16. Recall that a functor $u : C \to D$ in **Cat** induces a pair of adjoint functors between the categories of presheaves

$$u_! : [D^o, \mathbf{Set}] \to [C^o, \mathbf{Set}] : u^*.$$

Similarly, a map of simplicial sets $u : A \to B$ induces a pair of adjoint functors

$$u_! : \mathbf{S}/A \to \mathbf{S}/B : u^*.$$

The pair $(u_!, u^*)$ is a Quillen pair between the contravariant model structures

$$u_! : (\mathbf{S}/A, W^c) \to (\mathbf{S}/B, W^c) : u^*$$

and also between the covariant model structures

$$u_! : (\mathbf{S}/A, W_c) \to (\mathbf{S}/B, W_c) : u^*.$$

7.17. The pullback of a contravariant equivalence along a right fibration is a contravariant equivalence. The pullback of a weak categorical equivalence along a right fibration is a weak categorical equivalence.

8. Initial and final functors

8.1. A map of simplicial sets $u : A \to B$ induces a Quillen pair of adjoint functors

$$u_! : (\mathbf{S}/A, \mathcal{W}^c) \leftrightarrow (\mathbf{S}/B, \mathcal{W}^c) : u^*.$$

Hence also an adjoint pair of derived functors between the homotopy categories,

$$u_!^L : Ho(\mathbf{S}/A, \mathcal{W}^c) \leftrightarrow Ho(\mathbf{S}/B, \mathcal{W}^c) : u^{\star R}.$$

We say that u is *final* iff the functor $u_!^L$ preserves terminal objects. For each vertex $b \in B$, let us choose a factorisation $1 \to Lb \to B$ of the map $b : 1 \to B$ as a left anodyne map $1 \to Lb$ followed by a left fibration $Lb \to B$. Then a map $u : A \to B$ is final iff the simplicial set $Lb \times_B A$ is weakly contractible for every vertex $b \in B$. When B is a quasi-category, a map $u : A \to B$ is final iff the simplicial set $b\backslash A = (b\backslash B) \times_B A$ defined by the pullback square

$$\begin{array}{ccc} b\backslash A & \longrightarrow & A \\ \downarrow & & \downarrow u \\ b\backslash B & \longrightarrow & B \end{array}$$

is weakly contractible for every object $b \in B$.

8.2. If $u : A \to B$ is a final map, then a map $v : B \to C$ is final iff the composite $vu : A \to C$ is final. The notion of final map is invariant under weak categorical equivalences. More precisely, if the horizontal maps of a commutative square

$$\begin{array}{ccc} A & \longrightarrow & A' \\ \downarrow & & \downarrow \\ B & \longrightarrow & B' \end{array}$$

are weak categorical equivalences, then the map $A \to B$ is final iff the map $A' \to B'$ is final. A monomorphism is final iff it is right anodyne. A map $u : A \to B$ is final if it admits a factorisation $u = wi : A \to B' \to B$, with i a right anodyne map and w a weak categorical equivalence.

8.3. Let A be a simplicial set. We shall say that a vertex $a \in A$ is *terminal* if the map $a : 1 \to A$ is final or equivalently if it is right anodyne. A vertex $a \in A$ which is terminal in A is also a terminal object of the category $\tau_1 A$. The converse is true when A admits at least one terminal vertex. The notion of terminal vertex is invariant under weak categorical equivalence. More precisely, if $u : A \to B$ is a weak categorical equivalence, then a vertex $a \in A$ is terminal in A iff the vertex $u(a)$ is terminal in B. The vertex $1_a \in A/a$ is terminal in A/a for any simplical set A. Similarly for the vertex $1_a \in A/\!\!/a$.

8.4. If A is a quasi-category, then a vertex $a \in A$ is terminal iff the following equivalent conditions are satisfied:

- the simplicial set $A(x, a)$ is contractible for every $x \in A_0$;
- every simplical sphere $x : \partial\Delta[n] \to A$ with target $x(n) = a$ can be filled;
- the projection $A/a \to A$ is a weak categorical equivalence;
- the projection $A/a \to A$ is a trivial fibration;
- the projection $A/\!\!/a \to A$ is a weak categorical equivalence;
- the projection $A/\!\!/a \to A$ is a trivial fibration.

The full simplicial subset of terminal vertices of a quasi-category is a contractible Kan complex when non-empty.

8.5. In a simplicial set B, a vertex $b \in B$ is terminal iff the fiber inclusion $E(b) \subseteq E$ is a weak homotopy equivalence for every left fibration $p : E \to B$, where $E(b) = p^{-1}(b)$. Dually, a vertex $b\ inB$ is terminal iff the projection $\Gamma_B(E) \to E(b)$ is a homotopy equivalence for every right fibration $E \to B$, where $\Gamma_B(E)$ is the simplicial set of global sections of E.

8.6. We shall say that a map $u : A \to B$ is *initial* if the opposite map $u^o : A^o \to B^o$ is final. Similarly, we shall say that a vertex $b \in B$ is *initial* if the opposite vertex $b^o \in B^o$ is terminal.

8.7. The Yoneda lemma plays an important role in category theory. Let us describe its extensions to quasi-categories. Recall that in a simplicial set B, a node $b \in B$ can be viewed as a map $b : 1 \to B$; it is thus an object of the category $\mathbf{S}/B$. We say that an object (E, p) of the category $\mathbf{S}/B$ is *represented* by a node $u \in E(b)$ if the map $u : b \to E$ is a contravariant equivalence in $\mathbf{S}/B$. A node $u \in E$ which is terminal in the simplicial set E represents the object (E, p). The converse is true if the map $p : E \to B$ is a right fibration. In this case, the full simplicial subset of E which is spanned by the vertices which represents E is a contractible Kan complex when non-empty. The projection $B/b \to B$ is represented by the vertex $1_b \in B/b$. Similarly, the projection $B/\!\!/b \to B$ is represented by the vertex $1_b \in B/\!\!/b$. In general, a node $u \in E(b)$ represents an object (E, p) iff the canonical map

$$[u, X]_B : [E, X]_B \to [b, X]_B = X(b)$$

is a trivial fibration (resp. a weak homotopy equivalence) for every $X \in \mathcal{R}(B)$. If $b \in B$, let us choose a factorisation of the map $b : 1 \to B$ as a right cofibration $b' : 1 \to Rb$ followed by a right fibration $Rb \to B$. The object $Rb \to B$ is represented by the node b'. It follows that the canonical map

$$[b', X]_B : [R_b, X]_B \to X(b)$$

is a trivial fibration for every $X \in \mathcal{R}(B)$. This is a Yoneda lemma. If $p : E \to B$ is a right fibration, then every map $u : b \to E$ can be extended to a map $u' : Rb \to E$; the node u represents E iff the map $u' : Rb \to E$ is a fibrewise homotopy equivalence. If B is a quasi-category, then the projection $B/\!\!/a \to B$ is a right fibration for any vertex $a \in B$. Its fiber at $b \in B$ is equal to $B(b, a)$. Hence the canonical map

$$[B/\!\!/b, B/\!\!/a]_B \to B(b, a)$$

is a trivial fibration.

8.8. The Yoneda lemma be used for constructing a simplicial category $\tilde{B}$ whose coherent nerve is equivalent to the quasi-category B. By construction, $Ob\tilde{B} = B_0$ and

$$\tilde{B}(a, b) = [B/a, B/b]$$

for every $a, b \in B_0$. The category $\tilde{B}$ is actually enriched over Kan complexes.

8.9. Dually, we say that an object (E, p) of the category $\mathbf{S}/B$ is *corepresented* by a node $u \in E(b)$ if the map $u : b \to E$ is a covariant equivalence in the category $\mathbf{S}/B$.

9. Morita equivalence

9.1. Recall that a functor $u : A \to B$ induces a pair of adjoint functors between the categories of presheaves

$$u_! : [A^o, \mathbf{Set}] \to [B^o, \mathbf{Set}] : u^*.$$

A functor $u : A \to B$ is fully faithful iff the functor $u_!$ is fully faithful. Classically, a functor u is said to be final, but we shall say 0-*final*, iff the functor $u_!$ preserves terminal objects. A functor $u : A \to B$ is said to be *dominant*, , but we shall say

0-*dominant*, if the functor u^* is fully faithful. A functor $u : A \to B$ is dominant iff the category $Fact(u, f) = Fact(f, B) \times_B A$ defined by the pullback square

$$\begin{array}{ccc} Fact(f,B) \times_B A & \longrightarrow & A \\ \downarrow & & \downarrow u \\ Fact(f,B) & \longrightarrow & B \end{array}$$

is connected for every arrow $f \in B$, where $Fact(f, B) = f\backslash(A/b) = (a\backslash A)/f$ is the category of factorisations of the arrow $f : a \to b$.

9.2. A functor $u : A \to B$ is said to be a *Morita equivalence* if the adjoint pair $(u_!, u^*)$ is an equivalence of categories. A functor $u : A \to B$ is a Morita equivalence iff it is fully faithful and every object $b \in B$ is a retract of an object $u(a)$ for some object $a \in A$.

9.3. A map of simplicial sets $u : A \to B$ induces a Quillen pair of adjoint functors

$$u_! : (\mathbf{S}/A, \mathcal{W}^c) \leftrightarrow (\mathbf{S}/B, \mathcal{W}^c) : u^*.$$

Hence also an adjoint pair of derived functors between the homotopy categories,

$$u_!^L : Ho(\mathbf{S}/A, \mathcal{W}^c) \leftrightarrow Ho(\mathbf{S}/B, \mathcal{W}^c) : u^{\star R}.$$

We shall say that u is *fully faithful* if the functor $u_!^L$ is fully faithful. We shall say that u is *dominant* if the functor $u^{\star R}$ is fully faithful. We say that u is a *Morita equivalence* if the adjoint pair $(u_!^L, u^{\star R})$ is an equivalence.

9.4. A map of simplicial sets $u : A \to B$ is fully faithful iff the opposite map $u^o : A^o \to B^o$ is fully faithful. If $u : A \to B$ is fully faithful, then so is the functor $\tau_1(u) : \tau_1 A \to \tau_1 B$. A map between quasi-categories $f : X \to Y$ is fully faithful iff the map $X(a, b) \to Y(fa, fb)$ induced by f is a weak homotopy equivalenced for every pair $a, b \in X_0$.

9.5. A map of simplicial sets $u : A \to B$ is a Morita equivalence iff the opposite map $u^o : A^o \to B^o$ is a Morita equivalence. A map $u : A \to B$ is a Morita equivalence iff it is fully faithful and the functor $\tau_1(u) : \tau_1 A \to \tau_1 B$ is a Morita equivalence. Every weak categorical equivalence is a Morita equivalence.

9.6. A map of simplicial sets $u : A \to B$ is dominant iff the opposite map $u^o : A^o \to B^o$ is dominant. If $u : A \to B$ is dominant, then the functor $\tau_1(u) : \tau_1 A \to \tau_1 B$ is 0-dominant. When B is a quasi-category, a map $u : A \to B$ is dominant iff the category $Fact(u, f) = Fact(f, B) \times_B A$ defined by the pullback square

$$\begin{array}{ccc} Fact(f,B) \times_B A & \longrightarrow & A \\ \downarrow & & \downarrow u \\ Fact(f,B) & \longrightarrow & B \end{array}$$

is weakly contractible for every arrow $f \in B$, where $Fact(f, B) = f\backslash(A/b) = (a\backslash A)/f$ is the simplicial set of factorisations of the arrow $f : a \to b$.

10. Homotopy factorisation systems

Many of the class of maps of simplicial sets are part of a homotopy factorisation system. The idea of a homotopy factorisation system was first introduced by Bousfield in his work on localisation theory. We introduce a more general notion and give examples.

10.1. We first recall the notion of factorisation system. A pair $(\mathcal{A}, \mathcal{B})$ of classes of maps in a category $\mathcal{E}$ is called a *factorisation system* if the following two conditions are satisfied:

- the classes $\mathcal{A}$ and $\mathcal{B}$ are closed under composition and contain the isomorphisms;
- every map $f : A \to B$ admits a factorisation $f = pu : A \to E \to B$ with $u \in \mathcal{A}$ and $p \in \mathcal{B}$, and this factorisation is unique up to unique isomorphism.

In this definition, the uniqueness of the factorisation of the map means that for any other factorisation $f = qv : A \to F \to B$ with $v \in \mathcal{A}$ and $q \in \mathcal{B}$, there exists a unique isomorphism $i : E \to F$ such that $iu = v$ and $qi = p$. Every factorisation system is a weak factorisation system (see the appendix for this notion).

10.2. We say that a class of maps $\mathcal{M}$ in a category $\mathcal{E}$ has the *right cancellation property* if the implication

$$vu \in \mathcal{M} \text{ and } u \in \mathcal{M} \ \Rightarrow\ v \in \mathcal{M}$$

is true for any pair of maps $u : A \to B$ and $v : B \to C$. Dually, we say that $\mathcal{M}$ has the *left cancellation property* if the implication

$$vu \in \mathcal{M} \text{ and } v \in \mathcal{M} \ \Rightarrow\ u \in \mathcal{M}$$

is true. The left class of a factorisation system has the right cancellation property and the right class has the left cancellation property.

10.3. We say that a class of maps $\mathcal{M}$ in model category is *invariant under weak equivalences* if for every commutative square

$$\begin{array}{ccc} A & \longrightarrow & A' \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle u'} \\ B & \longrightarrow & B' \end{array}$$

in which the horizontal maps are weak equivalences, we have $u \in \mathcal{M} \Leftrightarrow u' \in \mathcal{M}$.

10.4. Let $\mathcal{E}$ is a Quillen model category with model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$. If $\mathcal{M} \subseteq \mathcal{E}$ is a class of maps, let us put $\mathcal{M}_{cf} = \mathcal{M} \cap \mathcal{E}_{fc}$, where $\mathcal{E}_{fc}$ is the full subcategory of fibrant-cofibrant objects. We say that a pair $(\mathcal{A}, \mathcal{B})$ of classes of maps in $\mathcal{E}$ is a *homotopy factorisation system* if the following conditions are satisfied:

- the classes $\mathcal{A}$ and $\mathcal{B}$ are invariant under weak equivalences;
- the pair $(\mathcal{A} \cap \mathcal{C}_{cf}, \mathcal{B} \cap \mathcal{F}_{cf})$ is a weak factorisation system on $\mathcal{E}_{cf}$;
- the class $\mathcal{A}$ has the right cancellation property;
- the class $\mathcal{B}$ has the left cancellation property.

The conditions (iii) and (iv) turn out to be equivalent in the presence of the others. We call $\mathcal{A}$ the *left class* and $\mathcal{B}$ the *right class* of the homotopy factorisation system.

10.5. A homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ induces a weak factorisation system $(Ho(\mathcal{A}), Ho(\mathcal{B}))$ on the homotopy category $Ho(\mathcal{E})$. The weak factorisation system $(Ho(\mathcal{A}), Ho(\mathcal{B}))$ is not a factorisation system in general. However, the class $Ho(\mathcal{A})$ has the right cancellation property and the class $Ho(\mathcal{B})$ has the left cancellation property. The system $(\mathcal{A}, \mathcal{B})$ is determined by the system $(Ho(\mathcal{A}), Ho(\mathcal{B}))$ since a map $u \in \mathcal{E}$ belongs to $\mathcal{A}$ (resp. $\mathcal{B}$) iff its image in $Ho(\mathcal{E})$ belongs to $Ho(\mathcal{A})$ (resp. $Ho(\mathcal{B})$). Not every factorisation system on the category $Ho(\mathcal{E})$ induces a homotopy factorisation on $\mathcal{E}$ (even if the classes have the cancellation properties).

10.6. Each class of a homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ is closed under composition and retracts. The intersection $\mathcal{A} \cap \mathcal{B}$ is the class of weak equivalences. The left class is closed under homotopy cobase change and the right class is closed under homotopy base change.

10.7. Each class of a homotopy factorisation system determines the other. We shall often use this property in specifying a homotopy factorisation system. A homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ is also determined by each of the classes of the weak factorisation system $(\mathcal{A} \cap \mathcal{C}_{cf}, \mathcal{B} \cap \mathcal{F}_{cf})$.

10.8. The model category $(\mathbf{Cat}, Eq)$ admits a homotopy factorisation system in which the left class is the class of essentially surjective functors and the right class is the class of full and faithful functors. Similarly, the model category $(\mathbf{S}, Wcat)$ admits a homotopy factorisation system in which the left class is the class of essentially surjective maps and the right class is the class of fully faithful maps. We notice here that the notion of essentially surjective map is easy to define for a general simplicial sets (a map $u : A \to B$ is essentially surjective iff the map $\tau_0(u) : \tau_0(A) \to \tau_0(B)$ is surjective). The fully faithful maps are harder to describe. But they are entirely determined by the essentially surjective maps.

10.9. The model category (Cat, Eq) admits a homotopy factorisation system in which the left class is the class of 0-final functors; a functor $f : A \to B$ belongs to the right class iff it admits a factorisation $f = pi : A \to A' \to B$ with i an equivalence and p a discrete fibration. Similarly, the model category $(\mathbf{S}, Wcat)$ admits a homotopy factorisation system in which the left class is the class of of final maps; a map of simplicial sets $f : X \to Y$ belongs to the right class iff it admits a factorisation $f = pi : X \to X' \to Y$ with i a weak categorical equivalence and p a right fibration. Dually, the model category $(\mathbf{S}, Wcat)$ admits a homotopy factorisation system in which the left class is the class of of initial maps; a map of simplicial sets $f : X \to Y$ belongs to the right class iff it admits a factorisation $f = pi : X \to X' \to Y$ with i a weak categorical equivalence and p a left fibration.

10.10. We say that a map of simplicial sets $u : A \to B$ is *homotopy monic* if the square

$$\begin{array}{ccc} A & \xrightarrow{1_A} & A \\ {\scriptstyle 1_A}\downarrow & & \downarrow{\scriptstyle u} \\ A & \xrightarrow{u} & B \end{array}$$

is homotopy cartesian in the model category $(\mathbf{S}, Who)$. A map $u : A \to B$ is homotopy monic iff it admits a factorisation $u = u'i : A \to A' \to B$ with i a weak homotopy equivalence and u' an inclusion of components $A' \subseteq A' \sqcup B' = B$.

Equivalently, u is homotopy monic iff its homotopy fibers are empty or contractible. We shall say that a map $u : A \to B$ is a *weak epimorphism* if the map $\pi_0 u : \pi_0 A \to \pi_0 B$ is surjective. Equivalently, u is weakly epic iff its homotopy fibers are non-empty. The model category $(\mathbf{S}, Who)$ admits a homotopy factorisation system in which the left class is the class of homotopy monomorphisms and the right class is the class of homotopy epimorphisms.

10.11. A map of simplicial sets $u : A \to B$ is *0-connected* if its homotopy fibers are connected. The model category $(\mathbf{S}, Who)$ admits a homotopy factorisation system in which the left class is the class of 0-connected maps; a map $f : X \to Y$ belongs to the right classs iff it admits a factorisation $f = f'i$, with i a weak homotopy equivalence and f' a covering space map. We shall say that a map in the right class is a *0-cover*.

10.12. For every $n \geq 0$, the model category $(\mathbf{S}, Who)$ admits a homotopy factorisation system in which the left class is the class of n-connected maps and in which the right class is the class on n-covers. A fibration $f : X \to Y$ is a n-cover iff the diagonal map $X \to X \times_Y X$ is a $n-1$-cover. A (-1)-cover is a homotopy monomorphism.

10.13. We say that a functor $u : A \to B$ is a *0-cover* if it is both a discrete fibration and a discrete opfibration. We say that a functor $u : A \to B$ is *0-connected* is the functor $\pi_1(u) : \pi_1(A) \to \pi_1(B)$ is 0-final (or equivalently 0-initial). The model category $(\mathbf{Cat}, Eq)$ admits a homotopy factorisation system in which the left class $\mathcal{A}$ is the class of 0-connected functors. A functor $f : X \to Y$ belongs to the right class iff it admits a factorisation $f = pi : A \to A' \to B$ with i an equivalence and p a 0-covers.

10.14. We say that a Grothendieck fibration $p : E \to B$ is a *1-fibration* if its fibers are groupoids. We say that a category is *simply connected* if the canonical functor $\pi_1 C \to 1$ is an equivalence, where $\pi_1 C$ is the groupoid freely generated by C. We say that a functor $u : A \to B$ is *1-final* if the category $b\backslash A = (b\backslash B) \times_B A$ is simply connected for every object $b \in B$. The model category $(\mathbf{Cat}, Eq)$ admits a homotopy factorisation system in which the left class $\mathcal{A}$ is the class of 1-final functors. A functor $f : X \to Y$ belongs to the right class iff it admits a factorisation $f = pi : A \to A' \to B$ with i an equivalence and p a 1-fibration.

10.15. The model category $(\mathbf{S}, Wcat)$ admits a homotopy factorisation system in which the left class is the class of weak homotopy equivalence. A map $f : X \to Y$ belongs to the right classs iff it admits a factorisation $f = pi : X \to X' \to Y$, with i a weak categorical equivalence and p a Kan fibration. A map in the left class is *infinitely connected*. We may call a map in the right class an ∞-*cover*.

10.16. Let $F : \mathcal{E} \leftrightarrow \mathcal{E}' : G$ be a Quillen pair between two model categories. If $(\mathcal{A}, \mathcal{B})$ is a homotopy factorisation system in $\mathcal{E}$ and $(\mathcal{A}', \mathcal{B}')$ a homotopy factorisation system in $\mathcal{E}'$, then the conditions $F(\mathcal{A}) \subseteq \mathcal{A}'$ and $G(\mathcal{B}') \subseteq \mathcal{B}$ are equivalent. We shall say that the system $(\mathcal{A}, \mathcal{B})$ is the *inverse* image of $(\mathcal{A}', \mathcal{B}')$ by F if we have $\mathcal{A} = F^{-1}(\mathcal{A}')$. We shall say that $(\mathcal{A}', \mathcal{B}')$ is the *inverse image* of $(\mathcal{A}, \mathcal{B})$ by G if we have $\mathcal{B}' = G^{-1}(\mathcal{B})$. These two conditions are equivalent when the pair (F, G) is a Quillen equivalence. When satisfied, we say that $(\mathcal{A}', \mathcal{B}')$ is obtained by *transporting*

$(\mathcal{A}, \mathcal{B})$ *across* the Quillen equivalence. Any homotopy factorisation system can be transported across a Quillen equivalence.

10.17. Let $F : \mathcal{E} \leftrightarrow \mathcal{E}'$ be a Quillen pair of adjoint functors between two model categories. If $(\mathcal{A}, \mathcal{B})$ is a homotopy factorisation system in $\mathcal{E}$ and $(\mathcal{A}', \mathcal{B}')$ a homotopy factorisation system in $\mathcal{E}'$, then the conditions $F(\mathcal{A}) \subseteq \mathcal{A}'$ and $G(\mathcal{B}') \subseteq \mathcal{B}$ are equivalent. We shall say that the system $(\mathcal{A}, \mathcal{B})$ is the *inverse* image of $(\mathcal{A}', \mathcal{B}')$ by F if we have $\mathcal{A} = F^{-1}(\mathcal{A}')$. We shall say that $(\mathcal{A}', \mathcal{B}')$ is the *inverse image* of $(\mathcal{A}, \mathcal{B})$ by G if we have $\mathcal{B}' = G^{-1}(\mathcal{B})$. These two conditions are equivalent when the pair (F, G) is a Quillen equivalence. When satisfied, we say that $(\mathcal{A}', \mathcal{B}')$ is obtained by *transporting* $(\mathcal{A}, \mathcal{B})$ *across* the Quillen equivalence. Any homotopy factorisation system can be transported across a Quillen equivalence.

10.18. We saw in **??** that the adjoint pair of functors

$$C_! : \mathbf{S} \leftrightarrow \mathbf{SCat} : C^!$$

is a Quillen equivalence between the model category for quasi-categories and the model category for simplicial categories. This defines a bijection between the homotopy factorisation systems in $(\mathbf{S}, \mathit{Wcat})$ and the homotopy factorisation systems in the model category $\mathbf{SCat}$. We shall call the coresponding classes of maps by the same name. For example, the final maps between simplicial categories introduced by Dwyer and Kan correspond to the final maps between simplicial set introduced here. We shall say that a map of simplicial categories $f : X \to Y$ is a *right fibration* if it is a Dwyer-Kan fibration and the map $C^!(f) : C^!(X) \to C^!(Y)$ is a right fibration.

11. Grothendieck fibrations

11.1. There is notion of Grothendieck fibration for maps between quasi-categories. Let us first recall the notion of cartesian arrow with respect to a functor $p : E \to B$. An arrow $f : a \to b$ in E is said to be *cartesian* if for every arrow $g : c \to b$ in E and every factorisation $p(g) = p(f)u : p(c) \to p(a) \to p(b)$ in B there is a unique arrow $v : c \to a$ such that $g = fv$ and $p(v) = u$. It is easy to verify that and arrow $f : a \to b$ is cartesian iff the square of categories

$$\begin{array}{ccc} E/a & \longrightarrow & E/b \\ \downarrow & & \downarrow \\ B/pa & \longrightarrow & B/pb \end{array}$$

is cartesian where the functor $E/a \to E/b$ (resp. $B/pa \to B/pb$) is defined by composing with f (resp. pf). A functor $p : E \to B$ is a *Grothendieck fibration* if for any vertex $a \in E_0$ and any arrow $g \in B_1$ with target $p(a)$ there exists a cartesian arrow $f \in E_1$ with target a such that $p(f) = g$. There a dual notions of cocartesian arrow and of Grothendieck *opfibration.* We can now define the notion of cartesian arrow with respect to a map between quasi-categories $p : E \to B$. We say that an arrow $f \in E$ with target $b \in E$ is *cartesian* if the map $E/f \to B/pf \times_{B/pb} E/b$

obtained from the commutative square

$$\begin{array}{ccc} E/f & \longrightarrow & E/b \\ \downarrow & & \downarrow \\ B/pf & \longrightarrow & B/pb \end{array}$$

is a trivial fibration. Here the map $E/f \to E/b$ is defined from the inclusion $\{1\} \subset I$. We shall say that a map between quasi-categories $p : E \to B$ is a *Grothendieck fibration* if it is a mid fibration and for every vertex $b \in E$ and every arrow $g \in B$ with target $p(b)$, there exists a cartesian arrow $f \in E$ with target b such that $p(f) = g$. A right fibration in $QCat$ is a Grothendieck fibration. The source map $X^I \to X$ is a Grothendieck fibration for any quasi-category X. Every Grothendieck fibration is a quasi-fibration. The class of Grothendieck fibrations is closed under composition and base change in $QCat$. Every Grothendieck fibration is a quasi-fibration. There is a dual notion of *cocartesian* arrow and of *Grothendieck opfibration*.

11.2. A map between quasi-categories $u : A \to B$ admits a factorisation $u = qi : A \to P \to B$ with i a fully faithful right adjoint and q a Grothendieck fibration. To see this, consider the path object $P(u)$ defined by the pullback square

$$\begin{array}{ccc} P(u) & \xrightarrow{h} & B^I \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle t} \\ A & \xrightarrow{u} & B. \end{array}$$

The projection $q = sh : P(u) \to B$ is a Grothendieck fibration. There is a unique map $i : A \to P(u)$ such that $pi = 1_A$ and $hi = \delta u$, where $\delta : B \to B^I$ is the diagonal. We have $p \vdash i$ and the counit of the adjunction is the identity of the maps $pi = 1_A$. Thus, i is fully and faithful.

12. Proper and smooth maps

12.1. Recall that a map of simplicial sets $u : A \to B$ induces an adjoint pair of derived functors

$$u_!^L : HoR(B) \leftrightarrow HoR(A) : u^{\star R}.$$

We obtain a pseudo-2-functor

$$HoR : \mathbf{S}^{\tau_1} \to ADCAT.$$

where $ADCAT$ is the category of big categories and adjoint maps. The functor associates to an arrow $\alpha : u \to v : A \to B$ of the category $\tau_1(A, B)$, a pair $(\alpha_!, \alpha^*)$ of adjoint natural transformations

$$\alpha_! : u_!^L \to v_!^L \quad \text{and} \quad \alpha^* : v^{*R} \to u^{*R}$$

If $(\alpha, \beta) : u \vdash v$ is an adjunction between two maps $u : A \to B$ and $v : B \to A$, then the pair $(\alpha_!, \beta_!)$ is an adjunction $u_!^L \vdash v_!^L$ and the pair (β^*, α^*) is an adjunction $u^{*R} \vdash v^{*R}$. It follows that we have a canonical isomorphism $u^{*R} = v_!^L$.

12.2. We introduce the notions of proper map and of smooth map (following a terminology of Grothendieck). We shall say that a map of simplicial sets $p : E \to B$ is *proper* if the pullback functor $p^* : \mathbf{S}/B \to \mathbf{S}/E$ takes a right cofibration to a right cofibration. A left fibration is an example of a proper map. A Grothendieck opfibration (resp. fibration) in $QCat$ is another example. A map $p : E \to B$ is proper (resp. smooth) iff the inclusion $p^{-1}(b(n)) \subseteq b^*(E)$ is a right (resp. left) cofibration for every simplex $b : \Delta[n] \to B$. The proper maps are closed under composition and base change. When p is proper, the pair of adjoint functors

$$p^* : \mathbf{S}/B \leftrightarrow \mathbf{S}/E : p_*$$

is a Quillen pair between the model categories $(\mathbf{S}/B, W_r)$ and $(\mathbf{S}/E, W_r)$. Moreover, the functor p^* takes a right equivalence over B to a right equivalence over E. We thus obtain a pair of adjoint functors

$$p^* : HoR(B) \leftrightarrow HoR(E) : p_*^R$$

where $p^* = Ho(p^*)$. Suppose that we have a cartesian square

$$\begin{array}{ccc} F & \xrightarrow{v} & E \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ A & \xrightarrow{u} & B \end{array}$$

in which the map p is proper. Then the following square commutes up to a natural isomorphism:

$$\begin{array}{ccc} HoR(F) & \xrightarrow{v_!^L} & HoR(E) \\ {\scriptstyle q^*}\uparrow & & \uparrow{\scriptstyle p^*} \\ HoR(A) & \xrightarrow{u_!^L} & HoR(B), , \end{array}$$

hence also the following square of right adjoints

$$\begin{array}{ccc} HoR(F) & \xleftarrow{v^{*R}} & HoR(E) \\ {\scriptstyle q_*^R}\downarrow & & \downarrow{\scriptstyle p_*^R} \\ HoR(A) & \xleftarrow{u^{*R}} & HoR(B) \end{array}$$

12.3. Dually, we say that a map of simplicial sets $p : E \to B$ is *smooth* if the pullback functor $p^* : \mathbf{S}/B \to \mathbf{S}/E$ takes a left cofibration to a left cofibration. A right fibration is an example of a smooth map. A map $p : E \to B$ is smooth iff the opposite map $p^o : E^o \to B^o$ is proper. When p is smooth, the pair of adjoint functors

$$p^* : \mathbf{S}/B \leftrightarrow \mathbf{S}/E : p_*$$

is a Quillen pair between the model categories $(\mathbf{S}/B, W_l)$ and $(\mathbf{S}/E, W_l)$.

12.4. The functor $u^{*R} : HoR(B) \to HoR(A)$ admits a right adjoint

$$R(u_*) : HoR(A) \to HoR(B)$$

for *any map* $u : A \to B$. To see this, it suffices by Morita equivalence, to consider the case where A and B are quasi-categories. In this case, we have a factorisation $u = pi : A \to C \to B$, with i a left adjoint and p a left Grothendieck fibration. The map p is proper since a left Grothendieck fibration is a proper map. If $q : C \to A$ and $i \vdash q$, then $i^{*R} \vdash q^{*R}$. Thus,

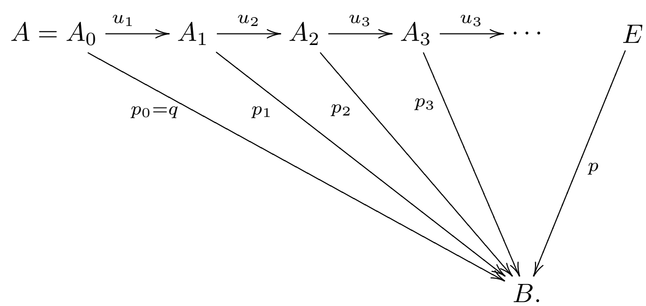
$$u^{*R} = i^{*R} p^{*R} \vdash p^R_* q^{*R} = R(u_*).$$

13. Localisation

13.1. Recall that a functor $p : C \to D$ is said to be *conservative* if the implication

$$p(f) \text{ invertible } \Rightarrow f \text{ invertible}$$

is true for every arrow $f \in C$. The model category (Cat, Eq) admits a homotopy factorisation system in which the right class is the class of conservative functors; a map in the left class is called a *0-localisation*. Let us describe the 0-localisations explicitly. If S is a set of arrows in a small category A, then there is a functor $u_S : A \to S^{-1}A$ which inverts universally every arrow in S. The universality means that if a functor $v : A \to B$ inverts every arrow in S, then there exists a unique functor $v' : S^{-1}A \to B$ such that $v' u_S = v$. The functor u_S is an example of a 0-localisation but it is not the most general one. Every functor $q : A \to B$ admits a factorisation $q = p_1 u_1 : A \to S_0^{-1}A \to B$, where S_0 is the set of arrows inverted by q. Let us put $A_1 = S_0^{-1}A$. The functor $p_1 : A_1 \to B$ need not be conservative, in which case it admits a factorisation $p_1 = p_2 u_2 : S_1^{-1}A_1 \to A_2 \to B$, where S_1 is the set of arrows inverted by p_1. Let us put $A_2 = S_1^{-1}A_1$. The construction may never stop. If we iterate we obtain a commutative diagram of categories and functors,



The category E is defined to be the colimit of the sequence of functors (u_n). The functor p is the unique functor $E \to B$ which extends the functor p_n for each $n \geq 0$. If $u : A_0 \to E$ is the canonical functor, we have a factorisation $q = pu$. The functor p is conservative. The functor $q : A \to B$ is a 0-localisation iff the functor $p : E \to B$ is an equivalence.

13.2. We say that a functor $f : X \to Y$ in **SCat** is *conservative* if the functor $Ho(f) : Ho(X) \to Ho(Y)$ is conservative. There is then a homotopy factorisation system in the model category **SCat** in which the left class is the class of conservative functors. We shall say that a map in the right class is a *Dwyer-Kan localisation*. We saw in **??** that the adjoint pair of functors

$$C_! : \mathbf{S} \leftrightarrow \mathbf{SCat} : C^!$$

is a Quillen equivalence between the model category for quasi-categories and the model category for simplicial categories. We saw in **??** that the adjoint pair of functors

$$C_! : \mathbf{S} \leftrightarrow \mathbf{SCat} : C^!$$

is a Quillen equivalence between the model category for quasi-categories and the model category for simplicial categorie. There is thus a unique homotopy factorisation system in $(\mathbf{S}, Wcat)$ which is obtained by transporting the Dwyer-Kan system. We shall describe this system explicitly.

13.3. Recall that a map of simplicial sets $u : A \to B$ is said to be *conservative* if the functor $\tau_1(u) : \tau_1(A) \to \tau_1(B)$ is conservative. The model category $(\mathbf{S}, Wcat)$ admits a homotopy factorisation system in which the right class is the class of conservative maps; a map in the left class is called a *localisation*. A monomorphism of simplicial sets is a localisation iff it has the left lifting property with respect to the conservative quasi-fibrations. The functor $\tau_1 : \mathbf{S} \to: \mathbf{Cat}$ takes a localisation to a 0-localisation.

13.4. Every localisation is a dominant map. A weak left adjoint is a localisation iff it is a weak reflection. The base change of a localisation along a left or a right fibration is a localisation.

13.5. Let us say that a monic localisation $A \to A'$ is *dominates* another monic localisation $A \to A'$" if the map $A'' \to A'_A A$" is a weak categorical equivalences. This defines a preoder relation among the monic localisations of a simplicial set A. Let us denote the resulting poset by $Loc(A)$. There is a similar poset $Loc_0(C)$ for the 0-localisations (monic on objects) of a category C. The map $Loc(A) \to Loc_0(\tau_1 A)$ induced by the functor τ_1 is an isomorphismes of posets. There is thus a bijection between the (equivalence classes of) localisations of A and the (equivalence classes of) 0-localisations of its fundamental category $\tau_1 A$. If S is a set of arrows in a simplicial set A, consider the simplicial set $L(A, S)$ defined by the pushout square

$$\begin{array}{ccc} S \times I & \longrightarrow & A \\ \downarrow & & \downarrow \\ S \times J & \longrightarrow & L(A,S), \end{array}$$

where J is the groupoid generated by one arrow $0 \to 1$. The inclusion $A \subseteq \langle L(A, S)$ is a monic localisation and we have $\tau_1 L(A, S) = S^{-1}\tau_1(A)$. If X is a quasi-category, then the map

$$X^{L(A,S)} \to X^A$$

is fully faithful and its (essential) image is spanned by the maps $f : A \to X$ such that $f(S) \subseteq J(X)$.

13.6. Every quasi-category X is the localisation of its category of elements $el(X) = \Delta/X$. More generally, every simplicial set A is the localisation of its category of elements $el(A) = A/\Delta$. Let us describe the canonical map $\theta_A : \Delta/A \to A$. Observe first that for any category C, there is a canonical fuctor $\theta_C : \Delta/C \to C$ which associates to a chain $x : [n] \to C$ the top object $x(n) \in C$. The map θ_C is natural in $C \in \mathbf{Cat}$. It can be extended uniquely as a natural transformation $\theta_A : \Delta/A \to A$ in $A \in \mathbf{S}$. This is because the functor $A \mapsto \Delta/A$, from $\mathbf{S}$ to itself, is cocontinuous.

The map θ_A is characterised by the property that the square

$$\begin{array}{ccc} \Delta/\Delta[n] & \xrightarrow{\Delta/a} & \Delta/A \\ {\scriptstyle \theta_{\Delta[n]}}\downarrow & & \downarrow{\scriptstyle \theta_A} \\ \Delta[n] & \xrightarrow{a} & A \end{array}$$

commutes for every simplex $a : \Delta[n] \to A$. Let $\Sigma \subset \Delta$ be the subcategory of Δ consisting of the maps $f : [m] \to [n]$ such that $f(m) = n$. Let $\Sigma_A \subseteq \Delta/A$ be the inverse image of Σ by the canonical functor $\Delta/A \to \Delta$. The map $\theta_A : \Delta/A \to A$ takes every arrow in Σ_A to units arrows in A. There is a canonical map

$$L(A/\Delta, \Sigma_A) \to A$$

and it is a weak categorical equivalence.

13.7. To every model category $\mathcal{E}$ we can associate the quasi-category

$$L(\mathcal{E}) = L(\mathcal{E}, \mathcal{W})$$

where $\mathcal{W}$ is the class of weak equivalence in $\mathcal{E}$. We shall say that $L(\mathcal{E})$ is the *quasi-localisation* of $\mathcal{E}$. It follows from [] that the quasi-category $L(\mathcal{E})$ is equivalent to the coherent nerve of the simplicial category of fibrant-cofibrant objects of $\mathcal{E}$, when the model category is simplicial. It follows that $L(\mathcal{E})$ is finitely bicomplete in this case and that it is bicomplete if $\mathcal{E}$ is bicomplete. The quasi-category $L(\mathcal{E})$ is cartesian closed if the model category $\mathcal{E}$ is cartesian closed.

14. Adjoint maps

Recall from 2.9 that the category $\mathbf{S}$ has the structure of a 2-category $\mathbf{S}^{\tau_1}$.

14.1. Recall the notion of adjoint map in a general 2-category. An *adjunction* between a pair of 1-cells $u : A \to B$ and $v : B \to A$ is a pair of 2-cells $\alpha : 1_A \to vu$ and $\beta : uv \to 1_B$ for which the *adjunction identities* hold:

$$(\beta \circ u)(u \circ \alpha) = 1_u \quad \text{and} \quad (v \circ \beta)(\alpha \circ v) = 1_v.$$

We write $(\alpha, \beta) : u \vdash v$ to indicate that the pair (α, β) is an adjunction between u and v. The 1-cell u is called the *left adjoint* and the 1-cell v the *right adjoint*. The cell α is called the *unit* of the adjunction and the cell β the *counit*. Each of the 2-cells α and β determines the other.

14.2. If $u : A \to B$ and $v : B \to A$ are maps of simplicial sets, an *adjunction* $(\alpha, \beta) : u \vdash v$ is defined to be an adjunction in the 2-category $\mathbf{S}^{\tau_1}$. We say that a homotopy $\alpha : 1_A \to vu$ is an *adjunction unit* if the correponding 2-cell $[\alpha] : 1_A \to vu$ is the unit of an adjunction in the 2-category $\mathbf{S}^{\tau_1}$. Dually, we shall say that a homotopy $\beta : uv \to 1_B$ is an *adjunction counit* if the 2-cell $[\beta] : uv \to 1_B$ is the counit of an adjunction. The functor $\tau_1 : \mathbf{S} \to \mathbf{Cat}$ is actually a 2-functor. Hence it takes an adjunction to an adjunction.

14.3. A map between quasi-categories $v : B \to A$ admits a left adjoint iff the quasi-category $a\backslash B = (a\backslash A) \times_A B$ defined by the pullback square

$$\begin{array}{ccc} a\backslash B & \longrightarrow & B \\ \downarrow & & \downarrow v \\ a\backslash A & \longrightarrow & A \end{array}$$

admits an initial vertex for every vertex $a \in A$. This condition can also be expressed by saying that the projection $a\backslash B \to B$ is representable for every vertex $a \in A$. A vertex of the quasi-category $a\backslash B$ is a pair (f, b) consisting of a vertex $b \in B_0$ together with an arrow $f : a \to v(b)$. An arrow $f : a \to v(b)$ is said to be *universal* if the pair (f, b) is an initial vertex of $a\backslash B$. If u is a map $A \to B$, then a homotopy $\alpha : 1_A \to vu$ is an adjunction unit iff the arrow $\alpha_a : a \to vu(a)$ is universal for every vertex $a \in A$. Equivalently, a homotopy $\alpha : 1_A \to vu$ is an adjunction unit iff the pair (α, u) is universal with respect to the map $v^A : B^A \to A^A$. The notion of couniversal arrow is defined dually and there is a dual characterisation of adjunction counit.

14.4. Recall that a full subcategory $A \subseteq B$ is said to be *reflective* (resp. *coreflective*) if the inclusion functor $A \subseteq B$ has a left (resp. right) adjoint called a *reflection* (resp. *coreflection*). If $u : A \leftrightarrow B : v$ is a pair of adjoint functors, then the right adjoint v is fully faithful iff the adjunction counit $uv \to 1_B$ is an isomorphism. Dually, the left adjoint u is fully faithful iff the adjunction unit $1_A \to vu$ is an isomorphism. In general, we shall say that a functor $u : A \to B$ is *reflective* if it is fully faithful and has a right adjoint $v : B \to A$, in which case v is called a *reflection*. Dually, we shall say that a functor $v : B \to A$ is *reflective* if it is fully faithful and has a left adjoint $u : A \to B$, in which case u is called a *coreflection*. These notions can be defined in any 2-category and, in particular in the 2-category $\mathbf{S}^{\tau_1}$.

14.5. The notion of adjoint maps in $\mathbf{S}^{\tau_1}$ can be weakened. Observe that a 1-cell $u : A \to B$ in a 2-category $\mathcal{E}$ is a left adjoint iff the functor

$$\mathcal{E}(u, C) : \mathcal{E}(B, C) \to \mathcal{E}(A, C)$$

is a right adjoint for every object $C \in \mathcal{E}$. This motivates the following definition. We say that a map of simplicial sets $u : A \to B$ is a *weak left adjoint* if the functor

$$\tau_1(u, X) : \tau_1(B, X) \to \tau_1(A, X)$$

is a right adjoint for every quasi-category X. Dually, we shall say that u is a *weak right adjoint* if the functor $\tau_1(u, X)$ is a left adjoint for every quasi-category X. A map of simplicial sets $u : A \to B$ is a weak left adjoint iff the opposite map $u^o : A^o \to B^o$ is a weak right adjoint. A map between quasi-categories is a left (resp. right) adjoint iff it is a weak left (resp. right) adjoint. The notion of weak left (resp. right) adjoint is invariant under weak categorical equivalence. The composite of two weak left (resp. right) adjoints is a weak left (resp. right) adjoint. If a map $u : A \to B$ is a weak left (resp. right) adjoint, then the map $X^u : X^B \to X^A$ is a right (resp. left) adjoint for every quasi-category X. The functor $\tau_1 : \mathbf{S} \to \mathbf{Cat}$ takes a weak left (resp. right) adjoint to a left (resp. right) adjoint.

14.6. We say that a map of simplicial sets $u : A \to B$ is a *weak left adjoint* if the functor

$$\tau_1(u, X) : \tau_1(B, X) \to \tau_1(A, X)$$

is a right adjoint for any quasi-category X. This notion is invariant under weak categorical equivalence. Every left adjoint is a weak left adjoint and the converse is true for maps between quasi-categories. Every weak left adjoint is an initial map. If $u : A \to B$ is a weak left adjoint, then the map $X^u : X^B \to X^A$ is a right adjoint for any quasi-category X. A map $u : A \to B$ is a weak left adjoint iff the pullback functor $u^* : \mathcal{R}(B) \to \mathcal{R}(A)$ takes a a representable object to a representable object. The functor $\tau_1 : \mathbf{S} \to Cat$ takes a weak left adjoint to a left adjoint. The map $A \to 1$ is a weak left adjoint iff A admits a terminal vertex. There is a dual notion of weak right adjoint and dual results.

14.7. We say that a simplicial subset $A \subseteq B$ is *weakly reflective* if the restriction functor $\tau_1(B, X) \to \tau_1(A, X)$ is a coreflection for every quasi-category X. More generally, we say that a map of simplicial sets $u : A \to B$ is *weakly reflective* if the functor

$$\tau_1(u, X) : \tau_1(B, X) \to \tau_1(A, X)$$

is a coreflection for every quasi-category X. Dually, we say that a map $u : A \to B$ is a *weak reflection* if the functor

$$\tau_1(u, X) : \tau_1(B, X) \to \tau_1(A, X)$$

is coreflective for every quasi-category X. There are also dual notions of weakly coreflective maps and of weak coreflection.

14.8. A weak left adjoint is a reflection iff it is a localisation. Dually, a weak right adjoint is a coreflection iff it is a localisation.

15. Cylinders, distributors and spans

15.1. Let A be a category. Recall that a *sieve* in A is a full subcategory $S \subseteq A$ for which the implication $\mathrm{target}(f) \in S \Rightarrow \mathrm{source}(f) \in S$ is true for every arrow $f \in A$. Dually, a *cosieve* in A is a full subcategory $S \subseteq A$ for which the implication $\mathrm{source}(f) \in S \Rightarrow \mathrm{target}(f) \in S$ is true for every arrow $f \in A$. If $S \subseteq A$ is a sieve (resp. cosieve) there exists a unique functor $p : A \to I$ such that $S = p^{-1}(0)$ (resp. $S = p^{-1}(1)$); we shall say that the sieve $p^{-1}(0)$ and the cosieve $p^{-1}(1)$ are *complementary.* There is a bijection between the sieves and the cosieves of A.

15.2. We shall say that an object of the category $\mathbf{Cat}/I$ a *0-cylinder* (the notion of *cylinder* is defined below). The *base* of a 0-cylinder $p : C \to I$ is the category $C(1) = p^{-1}(1)$ and its *cobase* is the category $C(0) = p^{-1}(0)$. The base of (C, p) is a cosieve in C and its cobase is a sieve. If $i : \partial I \subset I$ is the inclusion, then the functor

$$i^* : \mathbf{Cat}/I \to \mathbf{Cat} \times \mathbf{Cat}$$

is Grothendieck fibration. Its fiber at (A, B) is the category $\mathcal{C}^0(A, B)$ of 0-cylinders with cobase A and base B. The join $A \star B$ is naturally equipped with a map $A \star B \to I$; the resulting cylinder is the terminal object of the category $\mathcal{C}^0(A, B)$. Similarly the coproduct $A \sqcup B$ is naturally equipped with a map $A \sqcup B \to I$; the resulting cylinder is the initial object of the category $\mathcal{C}^0(A, B)$.

15.3. Recall that a *distributor* $D : A \Rightarrow B$, but we shall say a 0-*distributor*, is defined to be a functor $D : A^o \times B \to \mathbf{Set}$. We shall denote by $\mathcal{D}^0(A, B)$ the category of 0-distributors from A to B. To each 0-distributor $D \in \mathcal{D}^0(A, B)$ we can associate a category $A \star_D B$ obtained by *collage* of A with B along D. The category $C = A \star_D B$ is constructed as follows: $Ob(C) = Ob(A) \sqcup Ob(B)$ and for any pair $x, y \in Ob(C)$, we have

$$C(x,y) = \begin{cases} A(x,y) & \text{if } x \in A \text{ and } y \in A \\ B(x,y) & \text{if } x \in B \text{ and } y \in B \\ D(x,y) & \text{if } x \in A \text{ and } y \in B \\ \emptyset & \text{if } x \in B \text{ and } y \in A. \end{cases}$$

Composition of arrows is obvious. The category $A \star_D B$ is equipped with a canonical functor $A \star_D B \to I$; it is thus a 0-cylinder from A to B. The resulting functor $\mathcal{D}^0(A, B) \to \mathcal{C}^0(A, B)$ is an equivalence of categories. We have $A \star_1 B = A \star B$ and $A \star_\emptyset B = A \sqcup B$, where $1 \in \mathcal{D}^0(A, B)$ is the terminal 0-distributor and $\emptyset$ the initial 0-distributor. Notice that we have $A \star_H A = A \times I$, where H denotes the 0-distributor $Hom : A^o \times A \to \mathbf{Set}$.

15.4. If X is a simplicial set, we shall say that a full simplicial subset $S \subseteq X$ is a *sieve* if the implication target$(f) \in S \Rightarrow$ source$(f) \in S$ is true for every arrow $f \in X$. The notion of cosieve is defined similarly. If $h : X \to \tau_1 X$ is the canonical map, then the map $S \mapsto h^{-1}(S)$ induces a bijection between the sieves (resp. cosieves) of $\tau_1 X$ and the sieves of X. If $S \subseteq X$ is a sieve (resp. cosieve) there exists a unique map $p : X \to I$ such that $S = p^{-1}(0)$ (resp. $S = p^{-1}(1)$); we shall say that the sieve $p^{-1}(0)$ and the cosieve $p^{-1}(1)$ are *complementary*. There is a bijection between the sieves and the cosieves of X.

15.5. We call an object of the category $\mathbf{S}/I$ a *cylinder*. The *base* of a cylinder $p : C \to I$ is the simplicial set $C(1) = p^{-1}(1)$ and its *cobase* is the simplicial set $C(0) = p^{-1}(0)$. The cobase of a cylinder (C, p) is a sieve in C and its cobase is a cosieve. If $C(1) = 1$ (resp. $C(0) = 1$) we say that C is an *inductive cone* (resp *projective cone*). If $C(0) = C(1) = 1$ we say that C is a *spindle*. If $i : \partial I \subset I$ is the inclusion, then the functor $i^* : \mathbf{S}/I \to \mathbf{S} \times \mathbf{S}$ is Grothendieck fibration. Its fiber at $(A, B) \in \mathbf{S} \times \mathbf{S}$. is the category $\mathcal{C}(A, B)$ of cylinders with cobase A and base B. The join $A \star B$ is naturally equipped with a map $A \star B \to I$; the resulting cylinder is the terminal object of the category $\mathcal{C}(A, B)$ Similarly the coproduct $A \sqcup B$ is naturally equipped with a map $A \sqcup B \to I$; the resulting cylinder is the initial object of the category $\mathcal{C}(A, B)$.

15.6. Let $\mathbf{S}^{(2)} = [\Delta^o \times \Delta^o, \mathbf{Set}]$ be the category of double simplicial sets. If $A, B \in \mathbf{S}$, let us put

$$(A \Box B)_{mn} = A_m \times B_n$$

for $m, n \geq 0$. We shall say that an object of the category $\mathbf{S}^{(2)}/(A \Box B)$ is a *distributor* from A to B. We shall denote by $\mathcal{D}(A, B)$ the category of distributors from A to B. Let $\sigma : \Delta \times \Delta \to \Delta$ be the ordinal sum functor. The functor $\sigma^* : \mathbf{S} \to \mathbf{S}^{(2)}$ has a left adjoint $\sigma_!$ and a right adjoint σ_*. We have

$$\sigma_!(A \Box B) = A \star B$$

for any pair of simplicial sets A and B.

WRONG!

If $D \in \mathcal{D}(A,B)$, then $\sigma_!(D) \in \mathcal{C}(A,B)$. We shall say that that the cylinder $\sigma_!(D)$ is obtained by *collage* of A and B along D and we shall denote it $A \star_D B$. The collage functor $D \mapsto \sigma_!(D)$ induces an equivalence of categories

$$\mathcal{D}(A,B) \simeq \mathcal{C}(A,B).$$

The inverse equivalence associates to a cylinder $X \in \mathcal{C}(A,B)$ a bisimplicial set $D(X)$ equipped with a map $D(X) \to A\Box B$. By construction, we have

$$D(X)_{mn} = Hom_I(\Delta[m] \star \Delta[n], X),$$

for every $m, n \geq 0$, where the hom set is taken in the category $\mathcal{S}/I$. From the inclusion $[m] \subseteq [m+n]$ we obtain a natural transformation $p_1 \to \sigma$, where p_1 is the first projection $\Delta \times \Delta \to \Delta$. Similarly, from the inclusion $[n] \subseteq [m+n]$ we obtain a natural transformation $p_2 \to \sigma$. We have $p_1^*(A) \times p_2^*(A) = A\Box A$. We thus obtain a natural map $\sigma^*(A) \to A\Box A$. This defines the *unit distributor* $I_A \in \mathcal{D}(A,A)$. Its collage cylinder $A \star_{I_A} A$ is isomorphic to the cylinder $A \times I$.

15.7. Let $\mathcal{D}$ be the category whose objects are the distributors $p : D \to A\Box B$ and whose maps $p \to p'$ are the triple of maps $f : D \to D'$, $a : A \to A'$ and $b : B \to B'$ fitting in a commutative square

$$\begin{array}{ccc} D & \xrightarrow{f} & D' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ A\Box B & \xrightarrow{a\Box b} & A'\Box B'. \end{array}$$

Then the collage functor $D \mapsto A \star_D B$ induces an equivalence of categories

$$\mathcal{D} \simeq \mathbf{S}/I$$

15.8. A *span* $S = (s, S, t)$ between two simplicial sets A and B is defined to be a pair of maps

$$A \xleftarrow{s} S \xrightarrow{t} B$$

Equivalently, a span is a contravariant functot $P \to \mathbf{S}$, where P is the poset of non-empty subsets of $\{0,1\}$. We shall denote by $Span(\mathbf{S})$ the category of spans $[P^o, \mathbf{S}]$. Let ∂P be the subcategory of P whose objects are $\{0\}$ and $\{1\}$. If i is the inclusion $\partial P \subset P$, then the functor

$$i^* : Span(\mathbf{S}) \to \mathbf{S} \times \mathbf{S}$$

is a Grothendieck fibration. If $A, B \in \mathbf{S}$, we shall denote by $Span(A,B)$ the fiber of i^* at (A,B). An object $S = (s,S,t)$ of this category is a map $(s,t) : S \to A \times B$. There is thus an isomorphism of categories

$$Span(A,B) = \mathbf{S}/(A \times B).$$

15.9. Let δ be the diagonal functor $\Delta \to \Delta \times \Delta$. The functor $\delta^* : \mathbf{S}^{(2)} \to \mathbf{S}$ has a left adjoint $\delta_!$ and a right adjoint δ_*. For any pair of simplicial sets A and B, we have

$$\delta^*(A\Box B) = A \times B.$$

Hence the functor δ^* induces a functor

$$\delta^* : \mathcal{D}(A,B) \to Span(A,B).$$

Let $\mathcal{D}$ be the category whose objects are the distributors $p : D \to A\Box B$ and whose maps $p \to p'$ are the triple of maps $f : D \to D'$, $a : A \to A'$ and $b : B \to B'$ fitting in a commutative square

$$\begin{array}{ccc} D & \xrightarrow{f} & D' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ A\Box B & \xrightarrow{a\Box b} & A'\Box B'. \end{array}$$

Then the collage functor $D \mapsto A \star_D B$ induces an equivalence of categories

$$\mathcal{D} \simeq \mathbf{S}/I$$

15.10. The category $\mathbf{Cat}/I$ is cartesian closed. The model category $(\mathbf{Cat}, Eq)$ induces a model structure on the category $\mathbf{Cat}/I$ and the induced model structure is cartesian closed. We shall denote it by $(\mathbf{Cat}/I, Eq)$.

15.11. The category $\mathbf{S}/I$ is cartesian closed. The model category $(\mathbf{S}, Wcat)$ induces a model structure on the category $\mathbf{S}/I$ and the induced model structure is cartesian closed. We shall denote it by $(\mathbf{S}/I, Wcat)$. If i denotes the inclusion $\{0,1\} = \partial I \subset I$, then the functor

$$i^* : \mathbf{S}/I \to \mathbf{S} \times \mathbf{S},$$

has a right adjoint i_* given by $i_*(A, B) = A \star B$. The pair (i^*, i_*) is a Quillen adjoint pair between the model categories $(\mathbf{S}/I, Wcat)$ and $(\mathbf{S}, Wcat) \times (\mathbf{S}, Wcat)$. It follows in particular that the join of two quasi-categories is a quasi-category.

15.12. The functor $\tau_1 : \mathbf{S} \to \mathbf{Cat}$ induced by the functor $\tau_1 : \mathbf{S}/I \to \mathbf{Cat}/I$ preserves finite products. The resulting pair of adjoint functors

$$\tau_1 : \mathbf{S}/I \to \mathbf{Cat}/I : N$$

is a Quillen adjoint pair between the model categories $(\mathbf{S}/I, Wcat)$ and $(\mathbf{Cat}/I, Eq)$.

15.13. The model category $(\mathbf{S}, Wcat)$ induces a model structure on the category $Cyl(A, B)$ for each pair of simplicial sets A and B. More precisely, a map in $Cyl(A, B)$ is a weak equivalence (resp. a cofibration, a fibration) iff its underlying map in $\mathbf{S}$ is a weak categorical equivalence (resp. a monomorphism, a quasi-fibration) in the model category $(\mathbf{S}, Wcat)$. The induced model structure is simplicial and cartesian closed. A cylinder $C \in Cyl(A, B)$ is fibrant iff the structure map $C \to A \star B$ is a mid-fibration. We shall denote this model structure by $(Cyl(A, B), Wcat)$. We shall say that a map $S \to T$ in $D(A, B)$ is a *mid equivalence* if the corresponding map $A \star_S B \to A \star_T B$ is a weak categorical equivalence. There is then a model structure on $D(A, B)$ in which the weak equivalences are the mid equivalences and the cofibrations are the monomorphisms. We shall denote this model structure by $(Cyl(A, B), W_m(A, B))$ or more simply by $(Cyl(A, B), W_m)$.

If we transport the model structure $(Cyl(A, B), Wcat)$ along the equivalence $D(A, B) \simeq Cyl(A, B)$, we obtain a model structure on $D(A, B)$ The model structure can be transported along the equivalence $D(A, B) \simeq Cyl(A, B)$.

15.14. for each pair of simplicial sets A and B.

We shall denote this model structure by $(Cyl(A,B), Wcat)$.

Let $\mathcal{S}^{(2)} = [\Delta^o \times \Delta^o, Set]$ be the category of double simplicial sets. If $A, B \in \mathcal{S}$, let us put

$$(A\Box B)_{mn} = A_m \times B_n$$

for $m, n \geq 0$. We shall say that an object of the category $\mathcal{S}^{(2)}/(A\Box B)$ is a *distributor* from A to B. We shall denote by $D(A,B)$ the category of distributors from A to B. If $Tot : \mathcal{S}^{(2)} \to \mathcal{S}$ denotes the left Kan extension of ordinal sum functor $\Delta \times \Delta \to \Delta \to \mathcal{S}$, then we have

$$Tot(A\Box B) = A \star B$$

for any pair of simplicial sets A and B. If $D \in D(A,B)$, then $Tot(D) \in Cyl(A,B)$. We shall say that the cylinder $Tot(D)$ is obtained by *collage* of A and B along D and we shall denote it $A \star_D B$. The functor $D \mapsto A \star_D B$ induces an equivalence of categories

$$D(A,B) \simeq Cyl(A,B).$$

15.15.

16. Limits and colimits

16.1. There is a notion of limit and of colimit for a diagram with values in a quasi-category. A *diagram* in a quasi-category X is defined to be a map of simplicial sets $T \to X$. For example, a map $I \times I \to X$ is called a *commutative square* in X. A map $c : 1 \star T \to X$ is called a *projective cone* in X; the *base* of the cone is obtained by composing c with the inclusion $T \subset 1 \star T$. The cones $1 \star T \to X$ with a fixed base $d : T \to X$ are the vertices of a simplicial set X/d. The simplicial set X/d is a quasi-category when X is a quasi-category. We shall say that a cone $1 \star T \to X$ with base $d : T \to X$ is *exact* if it is a terminal vertex of the quasi-category X/d. When c is exact, the vertex $l = c(1) \in X$ is called the (*homotopy*) *limit* of d:

$$l = \varprojlim_{t \in T} d(t).$$

The full simplicial subset of X/d spanned by the exact cones is a contractible Kan complex when non-empty. It follows that the limit of a diagram is homotopy unique when it exists. The notion of limit can also be defined by using fat cones $1 \diamond T \to X$ instead of cones $1 \star T \to X$. The two notions are equivalent. The notions of coexact inductive cone $T \star 1 \to X$ and of colimit are defined dually.

16.2. We shall say that a diagram $d : T \to X$ is *discrete* if T is a discrete simplicial set. The limit l (resp. colimit l) of a discrete diagram $d : T \to X$ is called a *product* (resp. *coproduct*) and it is denoted

$$l = \prod_{t \in T} d(t) \qquad \Big(\text{resp.} \quad l = \coprod_{t \in T} d(t) \quad \Big).$$

The limit (resp. colimit) of the empty diagram $\emptyset \to X$ is a terminal (resp. initial) vertex in X. We shall say that a quasi-category X *admits finite products* (resp. *finite coproducts*) if every finite discrete diagram $T \to X$ has a limit (resp. colimit).

16.3. The square $I \times I$ is both a projective cone $1 \star \Lambda^2[2]$ and an inductive cone $\Lambda^0[2] \star 1$. A commutative square $I \times I \to X$ is said to be *cartesian* (resp. *cocartesian*) if it is exact as a projective cone (resp. inductive cone). A cartesian (resp. cocartesian) square is also called a *pullback square* (resp. *pushout square*). We shall say that a quasi-category X *admits pullbacks* (resp. *pushouts*) if every diagram $\Lambda^2[2] \to X$ (resp. $\Lambda^0[2] \to X$) has a limit (resp. a colimit).

16.4. A diagram $d : T \to X$ is *finite* if T has only a finite number of non-degenerated cells. We shall say that a quasi-category X is *finitely complete* (resp. *finitely cocomplete*) if every finite diagram $T \to X$ has a limit (resp. colimit). We shall say that X is *finitely bicomplete* if it is finitely complete and cocomplete. The localisation of a model category is an example of a finitely bicomplete quasi-category.

16.5. A (big) quasi-category X is said to be *complete* (resp. *cocomplete*) if every small diagram with values in X has a limit (resp. colimit). It is said to be *bicomplete* if it is complete and cocomplete. The localisation of a bicomplete model category is a bicomplete quasi-category. In particular, the coherent nerve of the simplicial category $\mathbf{K}$ of Kan complexes is a bicomplete quasi-category $\mathbf{HOT} = \mathbf{HOT_0}$. The coherent nerve of the simplicial category $\mathbf{QCat}$ is a bicomplete quasi-category $\mathbf{HOT_1}$.

16.6. From a diagram $d : T \to X$ and a map $u : S \to T$ we obtain a canonical arrow

$$\varinjlim_{s \in S} d(u(s)) \to \varinjlim_{t \in T} d(t)$$

in the homotopy category hoX, when the colimits exist. The arrow is invertible if the map u is final, and in particular when u is a weak categorical equivalence. If $T = \sqcup_{i \in I} T_i$ and the colimits exist, we have a canonical isomorphism

$$\coprod_{i \in I} \varinjlim_{t \in T_i} d(t) \to \varinjlim_{t \in T} d(t)$$

in the category hoX. Suppose that $i : A \to B$ is a monic map and that we have a pushout square of simplicial sets

$$\begin{array}{ccc} A & \longrightarrow & S \\ {\scriptstyle i}\downarrow & & \downarrow \\ B & \xrightarrow{v} & T. \end{array}$$

If X is a quasi-category and $d : T \to X$, we have a pushout square in X,

$$\begin{array}{ccc} \varinjlim_{a \in A} d(a) & \longrightarrow & \varinjlim_{s \in S} d(s) \\ \downarrow & & \downarrow \\ \varinjlim_{b \in B} dv(b) & \longrightarrow & \varinjlim_{t \in T} d(t) \end{array}$$

if the colimits exist. When X is finitely cocomplete, the colimit of any finite non-empty diagram $d : T \to X$ of dimension n can be computed iteratively by taking pushouts and the colimit of finite diagrams of dimension $< n$. A quasi-category with an initial vertex is finitely cocomplete iff it admits pushouts. Dually, a quasi-category with a terminal vertex is finitely complete iff it admits pullbacks.

16.7. If X is a quasi-category and T is a simplicial set, then the diagonal $X \to X^T$ has a right (resp. left) adjoint iff every diagram $T \to X$ has a limit (resp. colimit).

16.8. We say that a map between quasi-categories $f : X \to Y$ *preserve* the limit of a diagram $d : T \to X$ if this limit exists and f takes an exact projective cone with base d to an exact projective cone. A weak right adjoint preserves the limit of every diagram. We shall say that a map between quasi-categories $X \to Y$ is *continuous* if it takes every exact projective cone in X to an exact cone in Y. There is a dual notion of cocontinuous map.

16.9. Many properties of cartesian squares in categories remains valid for cartesian squares in quasi-categories. A useful property of cartesian squares is the following: let $t : \Delta[2] \to X^I$ is a commutative triangle of commutative squares in X,

$$\begin{array}{ccccc} a'' & \longrightarrow & a' & \longrightarrow & a \\ \downarrow & td_2 & \downarrow & td_0 & \downarrow \\ b'' & \longrightarrow & b' & \longrightarrow & b \end{array}$$

If the square td_0 is cartesian, then the square td_2 is cartesian iff the composite square td_1 is cartesian.

16.10. An arrow $f : a \to b$ in a simplicial set X is the same thing as a map $f : I \to X$. There is thus a slice simplicial set X/f. From the inclusion $\{i\} \subset I$ we obtain a projection p_i,

$$X/a \xrightarrow{p_0} X/f \xrightarrow{p_1} X/b.$$

The projection p_0 is a trivial fibration when X is a quasi-category. In this case we can define a map

$$f_! : X/a \to X/b,$$

by putting $f_! = p_1 s$, where s is a section of p_0. The map $f_!$ is unique up to a unique invertible 2-cell in the 2-category $QCat$. There is also a similar map map $f_! : X /\!\!/ a \to X /\!\!/ b$. When X admits pullbacks the map $f_! : X/a \to X/b$ has a right adjoint

$$f^* : X/b \to X/a$$

for every arrow $f : a \to b$; we shall say that f^* is the *base change map* along f.

16.11. Every finitely cocomplete quasi-category X admits a natural action $x \mapsto A \cdot x$ by an arbitrary finite simplicial set A. The action is defined as follows. The diagonal $\Delta_A : X \to X^A$ has a left adjoint

$$\varinjlim_A : X^A \to X$$

since X is finitely complete. If $x \in X$, then $A \cdot x = \varinjlim_A \Delta_A(x)$. The element $A \cdot x$ is the colimit of the constant diagram $c : A \to X$ with value x. There is a a canonical homotopy equivalence

$$X(A \cdot x, y) \simeq X(x, y)^A$$

for every $y \in X$. The action map $x \mapsto A \cdot x$ is inducing a map $\mathcal{S}_0 \times X \to X$, where $\mathcal{S}_0$ is the category of finite simplicial sets. For a fixed node $x \in X$, the map $A \mapsto A \cdot x$ takes homotopy pushout squares in $\mathcal{S}_0$ to pushout squares in X. For example, the 1-sphere S^1 can be defined by the pushout square

$$\begin{array}{ccc} \partial I & \xrightarrow{i} & I \\ {\scriptstyle i}\downarrow & & \downarrow \\ I & \longrightarrow & S^1 \end{array}$$

where i is the inclusion. The map $I \to 1$ is a weak homotopy equivalence. By combining this, we obtain a pushout square

$$\begin{array}{ccc} x \sqcup x & \longrightarrow & x \\ \downarrow & & \downarrow \\ x & \longrightarrow & S^1 \cdot x \end{array}$$

for each $x \in X$. Dually, a finitely complete quasi-category X admits a contravariant action $(x, A) \mapsto x^A$ by finite simplicial sets. The vertex $x^A \in X$ is the limit of the constant diagram $A \to X$ with value x. There is a a canonical homotopy equivalence

$$X(y, x^A) \simeq X(y, x)^A$$

for every $y \in X$. The covariant and contravariant actions are related by the formula $x^A = (A^o \cdot x^o)^o$. The map $A \mapsto x^A$ takes homotopy pushout squares in $\mathcal{S}_0$ to pullback squares in X. We thus have a pullback square

$$\begin{array}{ccc} x^{S^1} & \longrightarrow & x \\ \downarrow & & \downarrow \\ x & \longrightarrow & x \times x \end{array}$$

for each $x \in X$.

17. Kan extensions

17.1. If X is a cocomplete (big) quasi-category, then so is the (big) simplicial set X^A for any (small) simplicial set A. If $u : A \to B$, then the map $X^u : X^B \to X^A$ has a left adjoint

$$\Sigma_u : X^A \to X^B.$$

If $f : A \to X$, the map $\Sigma_u(f) : B \to X$ is called the *left Kan extension of f along u*. If $u \dashv v$ then $X^v \dashv X^u$, and hence hence $\Sigma_u = X^v$.

17.2. Consider a cartesian square

$$\begin{array}{ccc} F & \xrightarrow{v} & E \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ A & \xrightarrow{u} & B \end{array}$$

in which p is a proper map *or* u is a smooth map. Then the following square commutes (up to an isomorphism):

$$\begin{array}{ccc} X^F & \xleftarrow{X^v} & X^E \\ {\scriptstyle \Sigma_q}\downarrow & & \downarrow{\scriptstyle \Sigma_p} \\ X^A & \xleftarrow{X^u} & X^B. \end{array}$$

For example, if B is a quasi-category and $u : A \to B$ let us compute the composite

$$X^A \xrightarrow{\Sigma_u} X^B \xrightarrow{e_b} X$$

where $b : 1 \to B$ and $e_b = X^b$. The projection $p : B/b \to B$ is smooth since a right fibration is smooth. Hence the following square commutes:

$$\begin{array}{ccc} X^{A/b} & \xleftarrow{X^q} & X^A \\ {\scriptstyle \Sigma_v}\downarrow & & \downarrow{\scriptstyle \Sigma_u} \\ X^{B/b} & \xleftarrow{X^p} & X^B. \end{array}$$

The terminal vertex $t : 1 \to B/b$ is right adjoint to the map $r : B/b \to 1$. Thus, $X^t = \Sigma_r$. But we have $X^b = X^t X^p$ since $b = pt$. Thus

$$e_b\Sigma_u = X^b\Sigma_u = X^t X^p \Sigma_u = X^t \Sigma_v X^q = \Sigma_r \Sigma_v X^q = \Sigma_{rv} X^q.$$

But Σ_{rv} is the map

$$\varinjlim : X^{A/b} \to X.$$

Hence the square

$$\begin{array}{ccc} X^{A/b} & \longleftarrow & X^A \\ {\scriptstyle \varinjlim}\downarrow & & \downarrow{\scriptstyle \Sigma_u} \\ X & \xleftarrow{e_b} & X^B \end{array}$$

commutes. In particular, if $f : A \to X$, we obtain Kan's formula

$$\Sigma_u(f)(b) = \varinjlim_{u(a)\to b} f(a).$$

Dually, if X is a complete quasi-category then so is the (big) simplicial set X^A for any (small) simplicial set A. If $u : A \to B$, then the map $X^u : X^B \to X^A$ has a right adjoint

$$\Pi_u : X^A \to X^B.$$

If $f : A \to X$, the map $\Pi_u(f) : B \to X$ is called the *right Kan extension of f along u*. If B is a quasi-category, then the square

$$\begin{array}{ccc} X^{b\backslash A} & \longleftarrow & X^A \\ {\scriptstyle \varprojlim}\downarrow & & \downarrow{\scriptstyle \Pi_u} \\ X & \xleftarrow[e_b]{} & X^B \end{array}$$

commutes for any $b \in B_0$. In particular, if $f : A \to X$, we obtain the second Kan's formula:

$$\Pi_u(f)(b) = \varprojlim_{b\to u(a)} f(a).$$

17.3. If X is a quasi-category, then the contravariant functor $A \mapsto ho(A, X)$ is a kind of cohomology theory with values in Cat. When X is bicomplete, the map $ho(u, X) : ho(B, X) \to ho(A, X)$ has a left adjoint $ho(\Sigma_u)$ and a right adjoint $ho(\Pi_u)$ for any map $u : A \to B$. If we restrict the functor $A \mapsto ho(A, X)$ to the subcategory $Cat \subset \mathcal{S}$, we obtain a *homotopy theory* in the sense of Heller, also called a *derivateur* by Grothendieck. Most derivateurs occuring naturally in mathematics can be represented by a bicomplete quasi-categories.

17.4. If X is a bicomplete quasi-categories, then the map $X^u : X^B \to X^A$ is a Grothendieck bifibration for any fully faithful monomorphism $u : A \to B$.

18. Span

18.1. A *span* $S = (s, S, t)$ between two simplicial sets A and B is defined to be a pair of maps

$$A \xleftarrow{\ s\ } S \xrightarrow{\ t\ } B$$

Equivalently, a span is a contravariant functot $P \to \mathbf{S}$, where P is the poset of non-empty subsets of $\{0, 1\}$. We shall denote by $Span(\mathbf{S})$ the category of spans $[P^o, \mathbf{S}]$. Let ∂P be the subcategory of P whose objects are $\{0\}$ and $\{1\}$. If i is the inclusion $\partial P \subset P$, then the functor

$$i^* : Span(\mathbf{S}) \to \mathbf{S} \times \mathbf{S}$$

is a Grothendieck fibration. If $A, B \in \mathbf{S}$, we shall denote by $Span(A, B)$ the fiber of i^* at (A, B). An object $S = (s, S, t)$ of this category is a map $(s, t) : S \to A \times B$. There is thus an isomorphism of categories

$$Span(A, B) = \mathbf{S}/(A \times B).$$

The composite of $S \in Span(A, B)$ with $T \in Span(B, C)$ is defined to be the span $S \circ T = S \times_B T \in Span(A, C)$,

$$\begin{array}{ccccc} & & S \times_B T & & \\ & \swarrow & & \searrow & \\ & S & & T & \\ {}^{s}\swarrow & & {}^{t}\searrow \quad {}^{s}\swarrow & & \searrow^{t} \\ A & & B & & C \end{array}$$

The composition functor

$$- \circ - : Span(A, B) \times Span(B, C) \to Span(A, C)$$

is coherently associative. The unit span $\Delta_A \in Span(A, A)$ is the diagonal $(1_A, 1_A) : A \to A \times A$. The spans form a bicategory $SPAN$. By definition, a 0-cell of $SPAN$ is a simplicial set, a 1-cell $A \to B$ is a span $S \in Span(A, B)$ and a 2-cell is a map $S \to T$ in $Span(A, B)$. The bicategory $SPAN$ is symmetric monoidal. The *tensor product* $S \otimes T$ of a span $(s, t) : S \to A \times B$ with a span $(u, v) : T \to A \times B$ is defined to be the span

$$((s, u)(, t, v)) : S \times T \to (A \times C) \times (B \times C).$$

This defines a functor of two variables

$$\otimes : Span(A, B) \times Span(C, D) \to Span(A \times C, B \times D).$$

The operation $\otimes$ is compatible with the composition operation:

$$(S \otimes T) \circ (U \otimes V) \simeq (S \circ U) \otimes (T \circ U).$$

The unit for the tensor product is the terminal span. The *dual* S^* of a span $(s,t) : S \to A \times B$ is defined to be the span $(t^o, s^o) : {}^tS^o \to B^o \times A^o$. The duality operation is a global automorphism which reverses the direction of 1-cells but preserves the direction of 2-cells.

18.2. To every span $(s,t) : S \to A \times B$ we associate the cylinder $C(S) = C(s, S, t)$ defined by the pushout square

$$\begin{array}{ccc} S \sqcup S & \xrightarrow{s \sqcup t} & A \sqcup B \\ \downarrow & & \downarrow \\ S \times I & \longrightarrow & C(S) \end{array}$$

Notice that $C(S) = (s,t)_{!!}(S \times I)$. The functor $C(-)$ has a right adjoint which we now describe. If $X = (X, p)$ is a cylinder, let us denote by $Ar(X)$ the simplicial set $Hom_I(I, X)$ of sections of the structure map $p : X \to I$. A vertex of $Ar(X)$ is a *crossing arrow* of the cylinder X. From the inclusions $\{0\} \subset I$ and $\{1\} \subset I$, we obtain a two projections $Ar(X) \to X(0)$ and $Ar(X) \to X(1)$. The functor

$$Ar : \mathbf{S}/I \to Span(\mathbf{S})$$

is right adjoint to the functor $C(-)$. The adjoint pair $(C(-), Ar)$ induces a pair of adjoint functors

$$C(-) : Span(A, B) \leftrightarrow Cyl(A, B) : Ar$$

for each pair of simplicial sets A and B.

18.3. We shall say that a map of spans $f : S \to T$ in $Span(A, B)$ a *mixed equivalence* if the map of cylinders

$$C(f) : C(S) \to C(T)$$

is a weak categorical equivalence in $Cyl(A, B)$. The category $Span(A, B)$ admits a model structure in which the weak equivalences are the mixed equivalences and the cofibrations are the monomorphisms. We shall denote this model structure shortly by $(Span(A, B), Wm(A, B))$ or by $(Span(A, B), Wm)$ where $Wm(A, B)$ is the class of mixed equivalences in $Span(A, B)$. We shall say that a map $S \to A \times B$ is a *mixed fibration* if the span S is fibrant with respect to this model structure. The category $Span(A, B)$ is enriched over simplical sets and the model structure is simplicial. The pair of adjoint functors

$$C(-) : Span(A, B) \leftrightarrow Cyl(A, B) : Ar$$

is a Quillen equivalence between the model categories $(Span(A, B), Wm)$ and $(Cyl(A, B), Wcat)$.

18.4. The duality functor

$$(-)^* : Span(A, B) \to Span(B^o, A^o)$$

is an equivalence of model categories.

18.5. If A and B are two simplicial sets, we shall denote by $A \boxdot B$ the span

$$A \longleftarrow A \times B \longrightarrow B$$

This defines a functor of two variables

$$\boxdot : \mathbf{S} \times \mathbf{S} \to \mathit{Span}(\mathbf{S}).$$

If $u : C \to D$ and $v : U \to V$ are two maps of simplicial sets, we denote by $u \boxdot' v$ the map

$$(C \boxdot V) \sqcup_{C \boxdot U} (D \boxdot U) \to D \boxdot V.$$

obtained from the commutative square

$$\begin{array}{ccc} C \boxdot U & \longrightarrow & D \boxdot U \\ \downarrow & & \downarrow \\ C \boxdot V & \longrightarrow & D \boxdot V. \end{array}$$

The map $u \boxdot' v$ is actually a map in $\mathit{Span}(D, V)$,

$$\begin{array}{ccccc} D & \longleftarrow & (C \times V) \sqcup_{C \times U} (D \times U) & \longrightarrow & V \\ \downarrow & & \downarrow{\scriptstyle u \times' v} & & \downarrow \\ D & \longleftarrow & D \times V & \longrightarrow & V \end{array}$$

The span $A \boxdot B$ is the terminal object of the category $\mathit{Span}(A, B)$. Thus, any map $(s, t) : X \to A \times B$ can be viewed as a map of spans $(s, t) : X \to A \boxdot B$. The map $(s, t) : X \to A \boxdot B$ is a mixed fibration iff it has the right lifting property with respect to the following two kinds of maps:

- $u \boxdot' v$ for u a monomorphism and v a left cofibration;
- $u \boxdot' v$ for u a right cofibration and v a monomorphism.

Equivalently, $(s, t) : X \to A \boxdot B$ is a mixed fibration iff it has the right lifting property with respect to the the following two kinds of maps:

- $\delta_m \boxdot' h^j_n$ for $m \geq 0$ and $0 \leq j < n$;
- $h^i_m \boxdot' \delta_n$ for $0 < i \leq m$ and $n \geq 0$.

18.6. If A and B are quasi-categories and $(s, t) : S \to A \times B$ is a fibrant span, then the map $s : S \to A$ is a Grothendieck fibration and the map $t : S \to B$ is a Grothendieck opfibration.

18.7. The functor

$$i^* : \mathit{Span}(\mathbf{S}) \to \mathbf{S} \times \mathbf{S}$$

is a Grothendieck bifibration. To each pair of maps of simplicial sets $u : A \to A'$ and $v : B \to B'$ we can thus associate a pair of adjoint functors

$$(u \times v)_! : \mathit{Span}(A, B) \leftrightarrow \mathit{Span}(A', B') : (u \times v)^*.$$

It is a Quillen pair between the model categories $(\mathit{Span}(A, B), Wm)$ and $(\mathit{Span}(A', B'), Wm)$. And it is a Quillen equivalence if the maps u and v are weak categorical equivalences. It follows from this result that the model category $(\mathit{Span}(A, B), Wm)$ is equivalent to a model category $(\mathit{Span}(A', B'), Wm)$ in which A' and B' are quasi-categories.

18.8. The equivalence between the category $Span(A,1)$ and the category $\mathbf{S}/A$ induces an equivalence of model categories

$$(Span(A,1), Wm(A,1)) = (\mathbf{S}/A, Wr(A)).$$

Thus, a mixed fibration $X \to A \times 1$ is the same thing as a right fibration $X \to A$. The *inductive mapping cone* of a map $f : X \to A$ is defined to be the cylinder $Ci(f,X) = C(f,X,p_X)$, where $p_X : X \to 1$. By construction, we have a pushout square

$$\begin{CD} X @>{f}>> A \\ @VVV @VVV \\ X \diamond 1 @>>> Ci(f,X). \end{CD}$$

The functor $Ci : \mathbf{S}/A \to Cyl(A,1)$ has a right adjoint which associates to a direct cone $Y \in Cyl(A,1)$ its simplicial set of crossing arrows $Ar(Y)$. The pair of adjoint functors (Ci, Ar) is a Quillen equivalence

$$Ci : (\mathbf{S}/A, Wr(A)) \leftrightarrow (Cyl(A,1), Wcat) : Ar.$$

18.9. The equivalence between the category $Span(1,B)$ and the category $\mathbf{S}/B$ induces an equivalence of model categories

$$(Span(1,B), Wm(1,B)) = (\mathbf{S}/B, Wl(B)).$$

Thus, a mixed fibration $X \to 1 \times B$ is the same thing as a left fibration $X \to B$. The *projective mapping cone* of a map $f : X \to B$ is defined to be the cylinder $Cp(X,f) = C(p_X, X, f)$, where $p_X : X \to 1$. By construction, we have a pushout square

$$\begin{CD} X @>{f}>> A \\ @VVV @VVV \\ 1 \diamond X @>>> Cp(X,f). \end{CD}$$

The functor $Cp : \mathbf{S}/B \to Cyl(1,B)$ has a right adjoint which associates to an inverse cone $Y \in Cyl(1,B)$ its simplicial set of crossing arrows $Ar(Y) = Y(0,1)$. The pair of adjoint functors (Cp, Ar) is a Quillen equivalence

$$Cp : (\mathbf{S}/B, Wr(B)) \leftrightarrow (Cyl(1,B), Wcat) : Ar.$$

18.10. The equivalence between the category $Span(1,1)$ and the category $\mathbf{S}$ induces an equivalence of model categories

$$(Span(1,1), Wm(1,1)) = (\mathbf{S}, Who).$$

Hence a map $X \to 1 \times 1$ is a mid fibration iff X is a Kan complex. The cylinder $C[p_X, X, p_X]$ is a the (unreduced) *suspension* ΣX of the simplicial set X. By construction, we have a pushout square

$$\begin{CD} \partial I \times X @>>> \partial I \\ @VVV @VVV \\ I \times X @>>> \Sigma(X). \end{CD}$$

The functor $\Sigma : \mathbf{S} \to Cyl(1,1)$ has a right adjoint which associates to a spindle $Y \in Cyl(1,1)$ the simplicial set of crossing arrows $Ar(Y)$. The pair of adjoint functors (Σ, Ar) is a Quillen equivalence

$$\Sigma : (\mathbf{S}, Who) \leftrightarrow (Cyl(1,1), Wcat) : Ar.$$

18.11. We shall denote by $Span_f(A,B)$ the full subcategory of $Span(A,B)$ generated by the fibrant spans. The unit span $\Delta_A \in Span(A,A)$ is not fibrant unless the simplicial set A is discrete. A fibrant replacement of Δ_A is obtained by factoring the diagonal $(1_A, 1_A) : A \to A \times A$ as a weak equivalence $A \to U_A$ followed by a mixed fibration $U_A \to A \times A$. When A is a quasi-category, the map $(s,t) : A^I \to A \times A$ is a mixed fibration and we can take $U_A = A^I$. If $S \in Span_f(A,B)$, then the functor

$$S \circ (-) : Span_f(B,C) \to Span_f(A,C)$$

is a left Quillen functor. Similarly, if $T \in Span_f(B,C)$, then the functor

$$(-) \circ T : Span_f(A,B) \to Span_f(A,C)$$

is a left Quillen functor. The composition law

$$- \circ - : Span_f(A,B) \times Span_f(B,C) \to Span(A,C),$$

induces a composition law between the homotopy categories:

$$- \circ - : HoSpan(A,B) \times HoSpan(B,C) \to HoSpan(A,C).$$

If $S \in Span_f(A,B)$ and $U \in Span_f(C,D)$, then the functor $T \mapsto S \circ T \circ U$ from $Span(B,C)$ to $Span(A,D)$ is the composite of two left Quillen functors and in two ways:

$$T \mapsto (S \circ T) \circ U \quad \text{and} \quad T \mapsto S \circ (T \circ U).$$

It follows that we have

$$(S \circ T) \circ U = S \circ (T \circ U)$$

for the *derived* composition functor. We thus obtain a bicategory $HoSPAN$.

18.12. The tensor product functor

$$\otimes : Span(A,B) \times Span(C,D) \to Span(A \times C, B \times D)$$

is a left Quillen functor of two variables with respect to the model structures on the categories of spans. We thus obtain a symmetric monoidal structure on the bicategory $HoSPAN$.

19. Duality

19.1. We shall say that a map of simplicial sets $M \to A^o \times B$ is a *distributor from A to B* and we shall write $M : A \Rightarrow B$. The category of distributors $A \Rightarrow B$ is the category $\mathbf{S}/(A^o \times B)$ and we shall denote it by $D(A,B)$. We give the category $D(A,B)$ the model structure $(\mathbf{S}/(A^o \times B), Wl(A^o \times B))$. Hence a fibrant distributor $A \Rightarrow B$ is a left fibration $M \to A^o \times B$.

19.2. The *twisted category of arrows* $T(A)$ of a category A is defined to be the category of elements of the hom functor $A^o \times A \to Set$. A simplex $[n] \to T(A)$ is a map $[n]^o \star [n] \to A$. Let $a : \Delta \to \Delta$ the functor defined by putting $a([n]) = [n]^o \star [n]$ (ordinal sum) for every $n \geq 0$. If A is a general simplicial set, we shall put $T(A) = a^*(A)$. The functor $a^* : \mathbf{S} \to \mathbf{S}$ is a right Quillen functor for the model structure $(\mathbf{S}, Wcat)$. From the inclusions $[n]^o \subset [n]^o \star [n]$ and $[n] \subset [n]^o \star [n]$ we obtain a natural map $(s_A, t_A) : T(A) \to A^o \times A$. It is a left fibration when A is a quasi-category.

19.3. The *dual twisted category of arrows* $T^o(A)$ of a category A is defined to be the category of element of the hom functor $A^o \times A \to Set$ viewed as a contravariant functor $A \times A^o \to Set$. A simplex $[n] \to T^o(A)$ is a map $[n] \star [n]^o \to A$. Let $b : \Delta \to \Delta$ the functor defined by putting $b([n]) = [n] \star [n]^o$ (ordinal sum) for every $n \geq 0$. If A is a general simplicial set, we shall put $T^o(A) = b^*(A)$. From the inclusions $[n] \subset [n]^o \star [n]$ and $[n]^o \subset [n]^o \star [n]$ we obtain a map $(s^o_A, t^o_A) : T^o(A) \to A \times A^o$. We have a duality

$$T^o(A) = (T(A^o))^o, \quad s^o_A = (s_{A^o})^o \quad \text{and} \quad t^o_A = (t_{A^o})^o.$$

19.4. If A is a quasi-category, consider the spans $\eta_A \in Span(1, A^o \times A)$ and $\epsilon_A \in Span(A \times A^o, 1)$ respectively defined by the diagrams

$$\begin{array}{ccc} & T(A) & \\ \swarrow & & \searrow^{(s_A, t_A)} \\ 1 & & A^o \times A, \end{array} \qquad \begin{array}{ccc} & T^o(A) & \\ {}^{(s^o_A, t^o_A)}\swarrow & & \searrow \\ A \times A^o & & 1. \end{array}$$

If $\Delta_A \in Span(A, A)$ is the unit span, then there is a canonical isomorphism

$$(A \otimes \eta_A) \circ (\epsilon_A \otimes A) \simeq \Delta_A$$

in the homotopy category $HoSpan(A, A)$. There is also a dual canonical isomorphism

$$(\eta_A \otimes A^o) \circ (A^o \otimes \epsilon_A) \simeq \Delta_{A^o}$$

in the homotopy category $HoSpan(A^o, A^o)$. Hence the quasi-categories A and A^o are dual to each other in the symmetric monoidal bicategory $HoSPAN$. If A is a general simplicial set, then the spans η_A and ϵ_A are defined as above, but by using a fibrant replacement of the distributor TA. The monoidal category $HoSPAN$ is compact closed.

19.5. The span $S^* \in Span(B^o, A^o)$ is the transpose of the span $S \in Span(A, B)$ in the monoidal category $HoSPAN$. This means that we have a canonical isomorphism

$$S = (A \otimes \eta_B)(A \otimes S^* \otimes B)(\epsilon_A \otimes B)$$

in $HoSpan(A, B)$ and a canonical isomorphism

$$S^* = (\eta_A \otimes B^o)(A^o \otimes u \otimes B^o)(A^o \otimes \epsilon_B)$$

in $HoSpan(B^o, A^o)$.

19.6. It follows from the duality above that for any simplicial sets A, B and C, there is an equivalence of homotopy categories

$$HoSpan(A \times B, C) \to HoSpan(B, A^o \times C).$$

The equivalence is defined by the left Quillen functor which associates to a span $S \in Span(A \times B, C)$ the span

$$S' = (\eta_A \otimes B) \circ (A^o \otimes S)$$

calculated by the following diagram with a pullback square,

$$\begin{array}{ccccccc} & & S' & \longrightarrow & S & \longrightarrow & B \times C \\ & & \downarrow & & \downarrow & & \\ A^o & \xleftarrow{s_A} & T(A) & \xrightarrow{t_A} & A & & \end{array}$$

The inverse Quillen equivalence associates to a span $V \in Span(B, A^o \times C)$, the span

$$V' = (A \otimes V) \circ (\epsilon_A \otimes C)$$

calculated by the following diagram with a pullback square

$$\begin{array}{ccccccc} B \times C & \longleftarrow & V & \longleftarrow & V' & & \\ & & \downarrow & & \downarrow & & \\ & & A^o & \xleftarrow{t^o_A} & T^o(A) & \xrightarrow{s^o_A} & A. \end{array}$$

19.7. It follows from the duality above that for any simplicial sets A and B, there is an equivalence of homotopy categories

$$HoSpan(A, B) \to HoD(A, B).$$

The equivalence is induced by the functor which associates to a span $S \in Span(A, B)$ the distributor $S' \in D(A, B)$ calculated by following diagram with a pullback square,

$$\begin{array}{ccccccc} & & S' & \longrightarrow & S & \longrightarrow & B \\ & & \downarrow & & \downarrow & & \\ A^o & \xleftarrow{s_A} & T(A) & \xrightarrow{t_A} & A. & & \end{array}$$

The inverse equivalence is induced by the functor which associates to a distributor $V \in D(A, B)$, the span $V' \in Span(A, B)$ calculated by the following diagram with a pullback square,

$$\begin{array}{ccccccc} & & V' & \longrightarrow & V & \longrightarrow & B \\ & & \downarrow & & \downarrow & & \\ A & \xleftarrow{s^o_A} & T^o(A) & \xrightarrow{t^o_A} & A^o. & & \end{array}$$

19.8. To each map of simplicial sets $u : A \to B$ we can associate two spans $P(u) \in Span(A, B)$ and $P^*(u) \in Span(B, A)$ described by the following diagrams

$$\begin{array}{ccccc} P(u) & \longrightarrow & B^I & \xrightarrow{t} & B \\ \downarrow & & \downarrow{\scriptstyle s} & & \\ A & \xrightarrow{u} & B, & & \end{array} \qquad \begin{array}{ccccc} & & P^*(u) & \longrightarrow & A \\ & & \downarrow & & \downarrow{\scriptstyle u} \\ B & \xleftarrow{s} & B^I & \xrightarrow{t} & B, \end{array}$$

Where the squres are pullback. We have $P(u)^* = P^*(u^o)$. If B is a quasi-category, the span $P(u) \in Span(A, B)$ is left adjoint to the span $P^*(u) \in Span(B, A)$ in the bicategory $HoSPAN$. If A and B are quasi-categories, then for any span $(u, v) : S \to A \times B$ we have a canonical decomposition

$$S = P^*(u) \circ P(v)$$

in the bicategory $HoSPAN$.

20. The quasi-category **Hot**

By definition, the quasi-category **Hot** is the coherent nerve of the (simplicial) category of Kan complexes.

20.1. The quasi-category **Hot** is freely generated by its terminal node $1 \in \mathbf{Hot}$ as a cocomplete quasi-category. This means that for any cocomplete quasi-category X and any vertex $x \in X$ there exists a cocontinuous map $e_x : \mathbf{Hot} \to X$ such that $e_x(1) = x$, and moreover that e_x is unique up to a unique invertible 2-cell. We have $e_x(A) = A \cdot x$ for every $A \in \mathbf{Hot}$.

20.2. Every cocomplete quasi-category X supports a natural action $\mathbf{S} \times X \to X$ by the category $\mathbf{S}$. For a fixed $x \in X$, the map $A \mapsto A \cdot x$ takes a weak homotopy equivalence to a quasi-isomorphism. It induces an action of the quasi-category **Hot** on X. The action map $\mathbf{Hot} \times X \to X$ is cocontinuous in each variable. Dually, every complete quasi-category X supports a natural contravariant action $X \times \mathbf{S}^o \to X$ by the category $\mathbf{S}$ and a contravariant action $X \times \mathbf{Hot}^o \to X$ of the quasi-category **Hot**.

20.3. We say that a quasi-category X is *cartesian closed* if it admits finite products and the product map $a \times - : X \to X$ has a right adjoint $[a, -] : X \to X$ for every vertex $a \in X$. The quasi-category $\mathbf{Hot}_1$ is cartesian closed. The slice quasi-category $\mathbf{Hot}_1/I$ is also cartesian closed.

20.4. We say that a quasi-category X is *locally cartesian closed* if it is finitely complete and the slice quasi-category X/a is cartesian closed for every vertex $a \in X$. A finitely complete quasi-category is locally cartesian closed iff the base change map

$$f^* : X/b \to X/a$$

has a right adjoint for every arrow $f : a \to b$. The quasi-category **Hot** is locally cartesian closed.

20.5. If 1 denotes the terminal object of the quasi-category **Hot**, then the left fibration

$$1\backslash\mathbf{Hot} \to \mathbf{Hot}$$

is universal. Let us put $1\backslash\mathbf{Hot} = \mathbf{Hot}_\bullet$. The universality means that the following two properties are satisfied: (i) for every left fibration $p : E \to B$ there exists a commutative square

$$\begin{array}{ccc} E & \xrightarrow{f'} & \mathbf{Hot}_\bullet \\ \downarrow{\scriptstyle p} & & \downarrow \\ B & \xrightarrow{f} & \mathbf{Hot}, \end{array}$$

such that the induced map $E \to f^*\mathbf{Hot}_\bullet$ is a fiberwise homotopy equivalence in the category $\mathbf{S}/B$; (ii) the pair of maps (f, f') is unique up to a unique invertible 2-cell. Here a 2-cell $(f, f') \to (g, g')$ is defined to be a pair of 2-cells $\alpha : f \to g$ and $\alpha' : f' \to g'$ (in the 2-category of (big) simplicial sets) such that $\pi \circ \alpha' = \alpha \circ p$. We shall say that the pair (f, f') is *classifying* the left fibration $p : E \to B$.

20.6. Remark. The objet Ω of a topos is classifying the monomorphisms in this topos. There is an analogy between the objet Ω and the quasi-category **Hot** since the latter is classifying left fibrations in the category of simplicial sets. It may be possible to devellop some parts of the theory of quasi-categories by using an axiomatic approach. The axiomatic approach in category theory was advocated by Benabou, Lawvere, Street and Tierney. The category $QCat$ has many of the nice properties of Cat, for example it is cartesian closed. Many properties of **Hot** can be proved by using the universality of the left fibration $\mathbf{Hot}_\bullet \to \mathbf{Hot}$. For example, it can be proved that **Hot** is a complete quasi-category. This is because the right adjoint to the diagonal $\mathbf{Hot} \to \mathbf{Hot}^A$ is the map which classifies the left fibration $\mathbf{Hot}_\bullet^A \to \mathbf{Hot}^A$. We shall see that **Hot** is an ∞-topos. There might be an elementary notion of ∞-topos. We may also axiomatise the quasi-category $\mathbf{Hot}_1$.

20.7. We shall say that a map $f : A^o \to \mathbf{Hot}$ is a *prestack*. If A is a simplicial set, we shall put

$$\mathcal{P}(A) = \mathbf{Hot}^{A^o} = [A^o, \mathbf{Hot}].$$

If $u : A \to B$, then the map $u^* = [u^o, \mathbf{Hot}] : \mathcal{P}(B) \to \mathcal{P}(A)$ has a left adjoint $u_! = \Sigma_{u^o}$ and a right adjoint $u_* = \Pi_{u^o}$.

20.8. Let $T(A)$ be the twisted simplicial set of arrows of a simplicial sets A. See **??**. When A is a quasi-category, the canonical map $(s, t) : T(A) \to A^o \times A$ is a left fibration. In which case it is classified by a pair of maps (hom_A, hom'_A):

$$\begin{array}{ccc} T(A) & \xrightarrow{hom'_A} & \mathbf{Hot}_\bullet \\ {\scriptstyle (s,t)}\downarrow & & \downarrow \\ A^o \times A & \xrightarrow{hom_A} & \mathbf{Hot}. \end{array}$$

This defines the map $hom_A : A^o \times A \to \mathbf{Hot}$ from which the *Yoneda map*

$$y_A : A \to \mathcal{P}(A) = \mathbf{Hot}^{A^o}$$

is obtained. The quasi-category $\mathbf{Hot}^A$ is equivalent to the coherent nerve of $\mathcal{L}(A)$. Dually, the quasi-category $\mathbf{Hot}^{A^o}$ is equivalent to the coherent nerve of $\mathcal{R}(A)$.

21. The trace

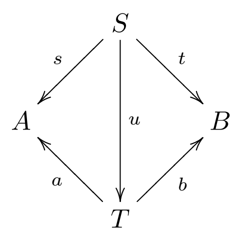
21.1. Let $\mathbf{C}$ be the category of cocomplete quasi-categories and cocontinuous maps. If $X \in \mathbf{C}$ then $X^A \in \mathbf{C}$ for any simplicial set A. The category $\mathbf{C}$ has the structure of a 2-category: it is a sub-2-category of the 2-category of big simplicial sets, where the 2-category of (small) simplicial sets is defined to be $\mathbf{S}^{\tau_1}$. If X is a cocomplete quasi-category, then so is the quasi-category $X(A) = X^{A^o}$ for any simplicial set A. If $S \in Span(A, B)$ the composite

$$X(A) \xrightarrow{s^*} X(S) \xrightarrow{t_!} X(B).$$

is a cocontinuous map

$$X\langle S\rangle : X(A) \to X(B).$$

If $u : S \to T$ is a map in $Span(A, B)$, then from the commutative diagram



and the counit $u_!u^* \to id$, we obtain a 2-cell

$$X\langle u\rangle : X\langle S\rangle = t_!s^* = b_!u_!u^*a^* \to b_!a^* = X\langle T\rangle.$$

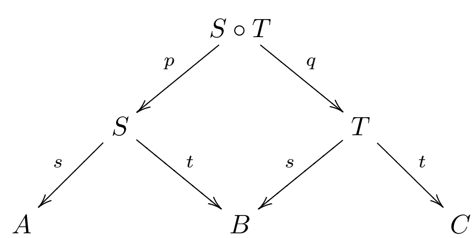
This defines a functor

$$X\langle -\rangle : Span(A, B) \to \mathbf{C}(X(A), X(B)).$$

The functor takes a mixed equivalence to an isomorphism. We thus obtain a functor

$$X\langle -\rangle : HoSpan(A, B) \to \mathbf{C}(X\langle A\rangle, X\langle B\rangle).$$

21.2. Recall that the composite of a span $S \in Span(A, B)$ with a span $T \in Span(B, C)$ is defined to be the span $S \circ T = S \times_B T$,



Let us suppose that A and B are quasi-categories and that S is a fibrant span. In this case the target map $t : S \to C$ is Grothendieck opfibration. It is thus a proper map. Hence the following diagram commutes up to a canonical isomorphism,

$$\begin{array}{ccccccccc} & & & & X(S\circ T) & & & & \\ & & & \overset{p^*}{\nearrow} & & \overset{q_!}{\searrow} & & & \\ & & X(S) & & & & X(T) & & \\ & \overset{s^*}{\nearrow} & & \overset{t_!}{\searrow} & & \overset{s^*}{\nearrow} & & \overset{t_!}{\searrow} & \\ X(A) & & & & X(B) & & & & X(C) \end{array}$$

It follows that

$$X\langle S\circ T\rangle \simeq X\langle T\rangle \circ X\langle S\rangle.$$

This defines a pseudo 2-functor

$$X : HoSPAN \to \mathbf{C}.$$

The functor is contravariant with respect to 1-cells but covariant with respect to 2-cells.

21.3. If the quasi-category X is bicomplete, then the map $X\langle S\rangle : X(A) \to X(B)$ has a right adjoint

$$X(B) \xrightarrow{t^*} X(S) \xrightarrow{s_*} X(A).$$

The opposite of this right adjoint is the map

$$X^o\langle S^o\rangle : X^o(B^o) \xrightarrow{(t^o)^*} (X^o)(S^o) \xrightarrow{(s^o)_!} X^o(A^o)$$

where S^o denotes the span $(t^o, s^o) : S^o \to B^o \times A^o$.

21.4. If A is a quasi-category, then we have a span $\epsilon_A \in Span(A\times A^o, 1)$,

$$\begin{array}{ccc} & T^o(A) & \\ \overset{(s^o_A, t^o_A)}{\swarrow} & & \searrow \\ A\times A^o & & 1. \end{array}$$

The *trace map*

$$tr_A : X^{A^o\times A} \to X$$

is defined to be the map $X\langle \epsilon_A\rangle : X(A\times A^o) \to X(1)$. By definition, it is the composite

$$X^{A^o\times A} \xrightarrow{X^{(s_{A^o}, t_{A^o})}} X^{T(A^o)} \xrightarrow{\text{colim}} X,$$

In category theory, the trace of a map $f : A^o\times A \to X$ is called a *coend* and denoted

$$coend_A(f) = tr_A(f) = \int^{a\in A} f(a,a).$$

We shall use the same notation. We have $coend_A(f) = coend_{A^o}({}^t f)$, where ${}^t f : A\times A^o \to X$ is the transpose of f.

21.5. If X is a complete quasi-category, the *cotrace map*

$$cotr_A : X^{A^o \times A} \to X$$

is defined to be the composite

$$X^{A^o \times A} \xrightarrow{X^{(s_A, t_A)}} X^{T(A)} \xrightarrow{\lim} X,$$

where (s_A, t_A) is the canonical map $T(A) \to A^o \times A$. In category theory, the cotrace of a map $f : A^o \times A \to X$ is called an *end* and denoted

$$end_A(f) = \int_{a \in A} f(a, a).$$

We shall use the same notation. We have $end_A(f) = end_{A^o}({}^t f)$, where ${}^t f : A \times A^o \to X$ is the transpose of f. The trace and cotrace are dual to each other. We have

$$end_A(f)^o = coend_{A^o}(f^o),$$

where $f^o : A \times A^o \to X^o$ is the opposite of $f : A^o \times A \to X$.

21.6. If $X \in \mathbf{C}$, then we have a canonical isomorphism $X(A)(B) = X(A \times B)$ for any pair of simplicial sets A and B. The functor $(X, A) \mapsto X(A)$ is contravariant in $A \in SPAN$. Hence the category $\mathbf{C}$ is cotensored over the monoidal category $SPAN$. It is also cotensored over the homotopy category $HoSPAN$ by the result above. But $HoSPAN$ is a compact closed 2-category by **??**. It follows by duality that $\mathbf{C}$ is tensored over $HoSPAN$. In other words the functors $X \mapsto X(A)$ and $X \mapsto X(A^o)$ are mutually adjoints. This means that for $X, Y \in \mathbf{C}$ there is a natural equivalence of categories

$$\mathbf{C}(X(A^o), Y) \simeq \mathbf{C}(X, Y(A)).$$

The counit of the adjunction is the trace map $tr_A : Y^{A \times A^o} \to Y$. The unit $\eta_A : X \to X^{A \times A^o}$ associates to $x \in X$ the map $(a, b) \mapsto hom_A(b, a) \cdot x$. The map

$$y : A \times X \to X^{A^o}$$

given by $(a, x) \mapsto (a \mapsto hom_A(b, a) \cdot x)$ is cocontinuous in the second variable, and it is universal with respect to that property. This means that if Y is a cocomplete quasi-category and $f : A \times X \to Y$ is a map cocontinuous in the second variable, then there exists a cocontinuous map $\bar{f} : X^{A^o} \to Y$ together with an invertible 2-cell $\alpha : f \simeq \bar{f} y$, and that the pair $(\bar{f}, \alpha)$ is unique up to a unique invertible 2-cell.

21.7. To each map of simplicial sets $u : A \to B$ we can associate two spans $P(u) \in Span(A, B)$ and $P^*(u) \in Span(B, A)$. If B is a quasi-category, the span $P(u) \in Span(A, B)$ is right adjoint to the span $P^*(u) \in Span(B, A)$ in the bicategory $HoSPAN$.

Thus, if $X \in \mathbf{C}$, the map $X\langle P(u) \rangle : X(A) \to X(B)$ is thus left adjoint to the map $X\langle P^*(u) \rangle : X(B) \to X(A)$, since the functor $X\langle - \rangle$ is contravariant on 1-cells. But the map $X\langle P^*(u) \rangle$ is isomorphic to the map $u^* : X(B) \to X(A)$. Hence the map $X\langle P(u) \rangle : X(A) \to X(B)$ is isomorphic to the map $u_! : X(A) \to X(B)$. The spans $P(u)$ and $P^*(u^o)$ are also mutually dual in the monoidal category $HoSPAN$. Hence we have

$$P(u) = (A \otimes \eta_B)(A \otimes P^*(u^o) \otimes B)(\epsilon_A \otimes B)$$

in the monoidal bicategory $HoSPAN$. This formula implies that for every $X \in \mathbf{C}$, we have a decomposition

$$u_! = X\langle \epsilon_A \otimes B\rangle \circ X\langle P^*(u^o) \otimes B\rangle \circ X\langle A \otimes \eta_B\rangle.$$

But we have $X\langle P^*(u^o) \otimes B\rangle = (u^o \times B)^*$. In other words, if $f : A^o \to X$, then we have the formula

$$u_!(f)(b) = \int^{a\in A} B(b, u(a)) \cdot f(a).$$

Dually, if X is a complete quasi-category and $f : A^o \to X$, then we have the formula

$$u_*(f)(b) = \int_{a\in A} f(a)^{B(u(a),b)}.$$

21.8. The quasi-category of prestacks $\mathcal{P}(A) = \mathbf{Hot}^{A^o}$ is cocomplete and freely generated by the Yoneda map $y_A : A \to \mathcal{P}(A)$. This means that if X is a cocomplete quasi-category, then for every map $f : A \to X$, there exists a cocontinuous map $\bar{f} : \mathcal{P}(A) \to X$ together with an invertible 2-cell $\alpha : f \simeq \bar{f} y_A$, and that the pair $(\bar{f}, \alpha)$ is unique up to an unique invertible 2-cell. The map $\bar{f}$ is the left Kan extension of f along y_A. It follows that we have

$$\bar{f}(e) = \varinjlim_{A/e} f p_e$$

for every $e \in \mathcal{P}(A)$, where A/e and p_e are defined by the pullback square

$$\begin{array}{ccc} A/e & \xrightarrow{p_e} & A \\ \downarrow & & \downarrow{\scriptstyle y_A} \\ \mathcal{P}(A)/e & \longrightarrow & \mathcal{P}(A). \end{array}$$

When $f = y_A$, the map $\bar{f}$ is the identity of $\mathcal{P}(A)$. This means that we have

$$e = \varinjlim_{A/e} p_e$$

for every $e \in \mathcal{P}(A)$.

21.9. Dually, the quasi-category $\mathcal{P}^o(A) = \mathcal{P}(A^o)^o$ is complete and freely generated by the dual Yoneda map $y^o_A = (y_{A^o})^o : A \to \mathcal{P}^o(A)$. This means that if X is a complete quasi-category, then for every map $f : A \to X$ there exists a continuous map $\bar{f} : \mathcal{P}^o(A) \to X$ together with an invertible 2-cell $\alpha : f \simeq \bar{f} y^o_A$, and moreover that the pair $(\bar{f}, \alpha)$ is unique up to an unique invertible 2-cell.

22. Factorisation systems in quasi-categories

22.1. There is a notion of factorisation system in a quasi-category. Let us define the orthogonality relation $i \perp f$ between the arrows of a quasi-category X. A commutative square

$$\begin{array}{ccc} a & \longrightarrow & x \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f} \\ b & \longrightarrow & y \end{array}$$

is a map $s : I \times I \to X$ which extends the map $(i, f) : I \sqcup I \to X$ along the inclusion $I \sqcup I = \partial I \times I \subset I \times I$. A *diagonal filler* for s is defined to be a map $I \star I \to X$ which extends s along the inclusion $I \times I \subset I \star I$. The fiber at s of the projection $X^{I \star I} \to X^{I \times I}$ is a Kan complex $Fill(s)$. If $Fill(s)$ is contractible for every square s which extends the given pair (i, f), we shall say that i is *strictly left orthogonal* to f, or that f is *strictly right orthogonal* to i, and write $i \perp f$. If $h : X \to hoX$ is the canonical map, then the relation $i \perp f$ in X implies the relation $h(i) \perp h(f)$ in hoX, but *the converse is not necessarly true*. However, if $h(i) = h(i')$ and $h(f) = h(f')$, then the relations $i \perp f$ and $i' \perp f'$ are equivalent. Hence the relation $i \perp f$ depends only on the homotopy classes of i and f. If Σ is a set of arrows in X we shall put

$$\Sigma^{\perp} = \{f \in X_1 : \forall i \in \Sigma,\ \ i \perp f\}, \qquad {}^{\perp}\Sigma = \{f \in X_1 : \forall i \in \Sigma,\ \ f \perp i\}.$$

The set $\Sigma^{\perp}$ contains the quasi-isomorphisms and it is closed under composition and retract. It is also closed under base change when they exist. If $v \in \Sigma^{\perp}$ then a composite $f = vu$ (in hoX) belongs to $\Sigma^{\perp}$ iff $u \in \Sigma^{\perp}$.

22.2. We shall say that a pair (A, B) of class of arrows in a (big) quasi-category X is a *factorisation system* if the following two conditions are satisfied:

- $A^{\perp} = B$ and $A = {}^{\perp}B$;
- every arrow $f \in X$ admits a factorisation $f = pu$ (in hoX), with $u \in A$ and $p \in B$.

We say that A is the *left class* and that B is the *right class* of the factorisation system.

22.3. If (A, B) is a factorisation system in the quasi-category X, then the pair $(h(A), h(B))$ is a weak factorisation system in the category hoX. Moreover, we have $A = h^{-1}h(A)$ and $B = h^{-1}h(B)$. In particular, the intersection $A \cap B$ is the class of quasi-isomorphism of X. If (C, D) is a weak factorisation system in $ho(X)$, then the pair $(h^{-1}(C), h^{-1}(D))$ is a factorisation system in X iff we have $i \perp u$ for every $i \in h^{-1}(C)$ and $u \in h^{-1}(D)$. A factorisation system (A, B) in a quasi-category X, induces a factorisation system on the slice quasi-categories X/b and $b\backslash X$ for every node $b \in X$, and a factorisation system on the quasi-category X^S for every simplicial set S. When X has finite products we shall say that a factorisation system (A, B) in X is *stable under product* if we have $a \times A \subseteq A$ for every $a \in X_0$. When X is finitely complete we shall say that (A, B) is *stable under base change* if A is closed under base change.

22.4. Let $p : E \to B$ be a right Grothendieck fibration between quasi-categories. If Qi is the set of quasi-isomorphisms in B, and C is the set of cartesian arrows in E, then the pair $(p^{-1}(Qi), C)$ is a factorisation system in E. If X is a quasi-category with pullbacks, then the target map $t : X^I \to X$ is a Grothendieck fibration. Hence the category X^I admits a factorisation system $(t^{-1}(Qi), C)$, where C is the set of cartesian square in X.

22.5. We shall say that an arrow $u : a \to b$ in a quasi-category X is a *monomorphism* or that it is *monic* if the commutative square

$$\begin{array}{ccc} a & \xrightarrow{1_a} & a \\ {\scriptstyle 1_a}\downarrow & & \downarrow{\scriptstyle u} \\ a & \xrightarrow{u} & b \end{array}$$

is cartesian. A monomorphism in X is monic in hoX. A map between Kan complexes $u : A \to B$ is monic in the quasi-category **Hot** iff it is homotopy monic (see **??**). We say that an arrow in a quasi-category X is *surjective*, or that is a *surjection*, if it is left orthogonal to every monomorphism of X. A map between Kan complexes $u : A \to B$ is a surjective arrow in **Hot** iff the map $\pi_0 u : \pi_0 A \to \pi_0 B$ is surjective. If A is the class of surjections and let B be the class of monomorphisms, then the pair (A, B) is a factorisation system iff every arrow in X admits a factorisation $f = up$, with u a monomorphism and p a surjection. In this case, we shall say that the quasi-category admits *surjection-mono* factorisations. The quasi-category HOT admits this kind of factorisation. If a quasi-category X admits surjection-mono-factorisations then so are the slice quasi-categories $b\backslash X$ and X/b for every vertex $b \in X$, and the quasi-category X^S for every simplicial set S.

22.6. We say that a simplicial set is *0-object* if it is a disjoint union of weakly contractible spaces. A simplicial set A is a 0-object iff the diagonal $A \to A \times A$ is homotopy monic. A Kan complex K is 0-object iff the projection $K^{S^1} \to K$ is a homotopy equivalence. If X is a quasi-category, we shall say that a vertex $a \in X$ is *discrete* or that it is a 0-*object* if the simplicial set $X(x, a)$ is a 0-object for every node $x \in X$; we shall say that an arrow $u : a \to b$ is a *0-cover* if it is a 0-object of the slice quasi-category X/b. When the product $a \times a$ exists, a vertex $a \in X$ is 0-object iff the diagonal $a \to a \times a$ is monic. When the exponential a^{S^1} exists, a vertex $a \in X$ is a 0-object iff the projection $a^{S^1} \to a$ is quasi-invertible. A map of simplicial sets $u : A \to B$ is an arrow in the quasi-category **Hot**; the arrow is a 0-cover in **Hot** iff the map is a 0-cover in $\mathcal{S}$ (see **??**). An arrow $u : a \to b$ in a quasi-category X is a 0-cover iff the map $X(x, u) : X(x, a) \to X(x, b)$ is a 0-cover for every node $x \in X$. We shall say that an arrow $u : a \to b$ in X is *0-connected* if it is left orthogonal to every 0-cover in X. A map of simplicial sets $u : A \to B$ is 0-connected iff its homotopy fibers are connected. In a quasi-category, we shall say that the factorisation of an arrow as a 0-connected arrow followed by 0-cover is a *0-factorisation* of this arrow. We shall say that a quasi-category X *admits 0-factorisations* if every arrow in X admits a 0-factorisation. In this case the quasi-category admits a factorisation system in which the left class is the class of 0-connected arrows and the right class the class of 0-covers. The quasi-category **Hot** admits 0-factorisations and these factorisations are stable under base change. If a quasi-category X admits 0-factorisations, then so are the slice quasi-categories $b\backslash X$ and X/b for every vertex $b \in X$, and the quasi-category X^S for every simplicial set S.

22.7. There is a notion of n-cover and of n-connected arrow in every quasi-category for every $n \geq 0$. It can be defined by induction on n. If $n > 0$, we shall say that a simplicial set A is a *n-object* if its diagonal $A \to A \times A$ is a $(n-1)$-cover. (more concretely, a simplicial set A is a *n-object* if we have $\pi_i(A, a) = 0$ for every $i > n$ and every $a \in A_0$). If X is a quasi-category, we shall say that a vertex $a \in X$ is a *n-object* if the simplicial set $X(x, a)$ is a n-object for every vertex $x \in X$. When the product $a \times a$ exists, the vertex a is a n-object iff the diagonal $a \to a \times a$ is a $(n-1)$-cover. When the exponential $a^{S^{n+1}}$ exists, the vertex a is a n-object iff the projection $a^{S^{n+1}} \to a$ is quasi-invertible. We shall say that an arrow $u : a \to b$ is a *n-cover* if it is a n-object of the slice quasi-category X/b. A map between

Kan complexes $u : A \to B$ is a n-cover in HOT iff its homotopy fibers are n-objects. We shall say that an arrow in a quasi-category X is *n-connected* if it is left orthogonal to every n-cover. A map between Kan complexes $u : A \to B$ is n-connected in **Hot** iff its homotopy fibers are n-connected. In a quasi-category, we shall say that the factorisation of an arrow as a n-connected arrow followed by n-cover is a *n-factorisation* of this arrow. We shall say that a quasi-category X *admits n-factorisations* if every arrow in X admits a n-factorisation. In this case the quasi-category admits a factorisation system in which the left class is the class of n-connected arrows and the right class the class of n-covers. The quasi-category HOT admits n-factorisations for each $n \geq 0$, and these factorisations are stable under base change. If a quasi-category X admits n-factorisations, then so are the slice quasi-categories $b\backslash X$ and X/b for every vertex $b \in X$, and the quasi-category X^S for every simplicial set S.

22.8. If X admits n-factorisations for every $0 \leq k \leq n$, we obtain a factorisation systems (A_k, B_k) for each $0 \leq k \leq n$. Notice the inclusions

$$A_0 \supseteq A_1 \supseteq A_2 \supseteq A_3 \cdots \supseteq A_n$$

$$B_0 \subseteq B_1 \subseteq B_2 \subseteq B_3 \cdots \subseteq B_n.$$

An *Eilenberg-MacLane n-cover* in a quasi-category X is a n-cover which is $(n-1)$-connected. A *Postnikov tower* (of height n) for an arrow $a \to b$ is defined to be a factorisation of length $n+1$ (in hoX)

$$a \xleftarrow{p_0} x_0 \xleftarrow{p_1} x_1 \xleftarrow{p_2} \qquad \cdots x_{n-1} \xleftarrow{p_n} x_n \xleftarrow{q_n} b$$

with p_0 a 0-cover, p_k is an EM k-cover for each $0 < k \leq n$ and q_n a n-connected arrow. The Postnikov tower of an arrow is unique (up to a unique isomorphism in the quasi-category of towers) when it exists. If X admits k-factorisations for each $0 \leq k \leq n$, then every arrow in X admits a Postnikov n-tower.

22.9. To each model category $\mathcal{E}$ we can associate its quasi-localisation $L(\mathcal{E})$. Let $p : \mathcal{E} \to L(\mathcal{E})$ be the canonical map. We conjecture that if (A, B) is a factorisation system in $L(\mathcal{E})$, then the pair $(p^{-1}(A), p^{-1}(B)$ is a homotopy factorisation system in $L(\mathcal{E})$ and that this defines a bijection between the factorisation systems in $L(\mathcal{E})$ and the homotopy factorisation systems in $\mathcal{E}$. This conjecture can be proved in the case of a simplicial category if we use **??**.

22.10. We say that a factorisation system (A, B) in a quasi-category X is *generated* by a set Σ of of arrows in X if we have $B = \Sigma^{\perp}$. Let X be a cartesian closed quasi-category. We shall say that a factorisation system (A, B) in X is *stably* generated by a set of arrows Σ if it is generated by the set

$$\Sigma' = \bigcup_{a \in X_0} a \times \Sigma.$$

For example, the surjection-mono-factorisation system in **Hot** is stably generated by the map $S^0 \to 1$. More generally, the the n-factorisation system is stably generated by the map $S^{n+1} \to 1$. The system of essentially surjective maps and of fully faithful maps in $\mathbf{Hot}_1$, is stably generated by the inclusion $\partial I \subset I$; the system of initial maps and of left fibrations is stably generated by the inclusion $\{0\} \subset I$; the dual system of final maps and of right fibrations is stably generated by the inclusion $\{1\} \subset I$; the system of weak homotopy equivalences and of Kan fibrations is stably generated by the two inclusions $\{0\} \subset I$ and $\{1\} \subset I$; the system of localisations and of conservative maps is stably generated by the map $I \to 1$.

23. Quasi-algebra

23.1. All universal algebra can be extended to this quasi-categories. A *Lawvere theory* or a *product theory* is defined to be a small quasi-category with finite products T. A *model* of T (with values in **Hot**) is defined to be a product preserving map $T \to \mathbf{Hot}$. The quasi-category of models $Mod(T)$ is the full simplicial subset of $[T, \mathbf{Hot}] = \mathbf{Hot}^T$ which is spanned by the models. The quasi-category $Mod(T)$ is complete and cocomplete. If t is an object of T, then the map $hom(t,-) : T \to \mathbf{Hot}$ is a *free model* of T. It follows from Yoneda lemma that T is equivalent to the opposite of the quasi-category of finitely generated free models of T. For example, the product theory of monoids Mo is the opposite of the category of finitely generated free monoids. A *model* of a product theory T with values in a quasi-category with finite products X is a defined to be a product preserving map $T \to X$. The quasi-category of models $Mod(T, X)$ is the full simplicial subset of $[T, X] = X^T$ which is spanned by the models. The quasi-category $Mod(T, X)$ is complete if X is complete. An *interpretation* of a product theory S in a product theory T is a model $u : S \to T$. The induced map $Mod(u) : Mod(T) \to Mod(S)$ has a left adjoint $u_! : Mod(S) \to Mod(T)$. The *tensor product* $S \odot T$ of two theories is the target of a map $S \times T \to S \odot T$ preserving finite products in each variables and which is universal with respect to that property. The (2-)category of product theories is symmetric (pseudo-)monoidal closed. We have $Mod(S \odot T, U) = Mod(S, Mod(T, U)$ for every quasi-category with product U. In particular, we have

$$Mod(S \odot T) = Mod(S, Mod(T)) = Mod(T, Mod(S).$$

For example, the tensor square $Mo^{\odot 2} = Mo \odot Mo$ is the theory of braided monoids. More generally, $Mo^{\odot n}$ is the theory of E_n-monoids for each $n \geq 2$. If Gr denotes the theory of groups, then $Gr^{\odot n}$ is the theory of n-fold loop spaces for each $n \geq 1$.

23.2. The notion of algebraic structure can be extended to include partially defined finitary operations. This is called an *essentially algebraic structure.* For example, the notion of category is essentially algebraic, since the composition of arrows in a category is only partially defined. A *finitary limit sketch* is a pair (A, P), where A is a simplicial set and P is a family of projective cones with finite base $u_i : 1 \star C_i \to A$, $(i \in I)$. A map $f : A \to X$ with codomain a finitely complete quasi-category X is said to be a *model* of the sketch if each cone $fu_i : 1 \star C_i \to X$ is exact in X. The quasi-category of models $Mod(A, P; X)$ is defined to be the full simplicial subset of $[A, X]$ whose objects are the models. Let us say for short that a quasi-category is *left exact* if it admits finite limits and also that a map between left exact categories is *left exact* if it preserves finite limits. If X and Y are left exact categories and X is small, then there is a left exact category $Lex(X, Y)$ of left exact maps $X \to Y$. By definition, it is the full simplicial subset of $[X, Y] = Y^X$ which is spanned by the left exact maps $X \to Y$. Every finitary limit sketch (A, P) has a universal model $u : A \to T(A, P)$ with values in a left exact quasi-category $T(A, P)$; the universality means that u induces an equivalence of quasi-categories $Lex(T(A, P), X) \to Mod(A, P; X)$ for any left exact quasi-category X. We now give a few examples of essentially algebraic notions. A *category object* (in a left exact quasi-category X) is defined to be a simplicial object $C : \Delta^o \to X$ satisfying the Segal conditions. The conditions say that the natural projection

$$C_n \to C_1 \times_{\partial_0,\partial_1} C_1 \times \cdots \times_{\partial_0,\partial_1} C_1,$$

is quasi-invertible for every $n \geq 2$. Another example of structure that can be defined by a limit sketch is the notion of monomorphism: an arrow $a \to b$ is monic iff the diagonal $a \to a \times_b a$ is invertible. A third example is the notion of *discrete object*: an object a is discrete iff the diagonal $a \to a \times a$ is monic. It follows that the notion of a category object $C : \Delta^o \to X$ with discrete object of objects C_0 is also (quasi-)algebraic. The (2-)category of small left exact quasi-categories is symmetric monoidal closed. The tensor product $S \odot T$ is the target of a map $S \times T \to S \odot T$ left exact in each variables and universal with respect to that property. There is an equivalence of quasi-categories

$$Mod(S \odot T) = Mod(S, Mod(T)) = Mod(T, Mod(S)).$$

For example, if Ca denotes the theory of categories, then $Ca^{\odot 2} = Ca \odot Ca$ is the theory of double categories.

23.3. The notion of algebraic structure can be extended to include partially defined infinitary operations. A sheaf on a space is an example since gluing the sections of a sheaf on the open sets of a cover is an infinitary operation if the cover is inifinite. A *limit sketch* is a pair (A, P), where A is a simplicial set and P is a family of projective cones $u_i : 1 \star C_i \to A$, $(i \in I)$. We say that a map $f : A \to X$ with codomain a complete quasi-category X is a *model* of the sketch if each cone $fu_i : 1 \star C_i \to X$ is exact in X. The quasi-category of models $Mod(A, P; X)$ is defined to be the full simplicial subset of $[A, X]$ spanned by the models. The quasi-category $Mod(A, P; X)$ is an example of what we call a *locally presentable quasi-category*. Up to equivalence, it is the most general example. We shall not give an abstract definition of the notion of locally presentable quasi-category here. Every locally presentable quasi-category is complete and cocomplete. If X is locally presentable then so are the slice quasi-categories $a\backslash X$ and X/a for any object $a \in X$ and the quasi-category X^A for any simplicial set A. If X is locally presentable and Y is cocomplete, then every cocontinuous map $f : X \to Y$ has a right adjoint $Y \to X$. We shall denote by **LP** the (2-)category of locally presentable quasi-categories and cocontinuous maps.

23.4. Let Σ be a set of arrows in a locally presentable quasi-category X. Then the pair $({}^\perp(\Sigma^\perp), \Sigma^\perp)$ is a factorisation system. We say that an object $a \in X$ is Σ-*local* if the terminal map $a \to 1$ belongs to $\Sigma^\perp$. Let us denote by X^Σ the simplicial subset of X spanned by the Σ-local objects. The quasi-category X^Σ is locally presentable, an the inclusion $X_\Sigma \subseteq X$ has a left adjoint $r : X \to X_\Sigma$. The map r is cocontinuous and it inverts universally the arrows in Σ. We shall say that it is a *localisation*. More generally, we say that a map $f : X \to Y$ in **LP** is a *localisation* iff its right adjoint $Y \to X$ is fully faithful. Every map $X \to Y$ in **LP** can be factored as a localisation $X \to L$ followed by conservative map $L \to Y$, and this factorisation is unique up to equivalence (with an equivalence which is unique up to a unique 2-cell). The factorisation can be constructed as follows. Let W be the class of arrows in X which are quasi-inverted by f. Then the pair $(W, W^\perp)$ is a factorisation system. The quasi-category X^W of W-local object is locally presentable and the inclusion $X^W \subseteq X$ has a left adjoint $r : X \to X^W$. The restriction $f \mid X^W$ is a conservative map $g : X^W \to Y$ and we have $f \simeq gr$. A true factorisation $f = g'r' : X \to L \to Y$ can be obtained by factoring g as an equivalence $i : X^W \to L$ followed by a quasi-fibration $g' : L \to X$. There then is a map $r' : X \to L$ such that $g'r' = f$. The factorisation system $(W, W^\perp)$ is always generated by a set $\Sigma \subseteq W$.

23.5. If X and Y are locally presentables then so is the quasi-category $Map(X, Y)$ of cocontinuous maps $X \to Y$. The 2-category **LP** is symmetric (pseudo-)monoidal closed. The *tensor product* of two locally presentable quasi-categories X and Y is the target of a map $X \times Y \to X \otimes Y$ cocontinuous in each variable and universal with respect to that property. The unit object for this tensor product is the quasi-category **Hot**. If $X \in \mathbf{LP}$, the equivalence $\mathbf{Hot} \otimes X \simeq X$ is induced by the action map $(A, x) \mapsto A \cdot x$, where for a simplicial set A, the object $A \cdot x$ is the colimit of the constant diagram $A \to X$ with value x. The quasi-category $\mathcal{P}(A)$ is locally presentable and freely generated by the simplicial set A. For any pair of simplicial sets A and B, there is an external product for prestacks $\mathcal{P}(A) \times \mathcal{P}(B) \to \mathcal{P}(A \times B)$ and it induces an equivalence of quasi-categories

$$\mathcal{P}(A) \otimes \mathcal{P}(B) \simeq \mathcal{P}(A \times B).$$

The quasi-categories $\mathcal{P}(A)$ and $\mathcal{P}(A^o)$ are dual objects in **LP**, with the duality pairing

$$\langle -, - \rangle : \mathcal{P}(A^o) \times \mathcal{P}(A) \to \mathbf{Hot}$$

given by the cocontinuous extension of the map $hom : A^o \times A \to \mathbf{Hot}$. It follows from this duality that we have an equivalence

$$\mathcal{P}(A) \otimes X = Map(\mathcal{P}(A^o), X) = X^{A^o}.$$

The equivalence

$$Map(X^{A^o}, Y) = Map(X, Y^A)$$

also follows from this duality.

23.6. The opposite of a locally presentable quasi-category is cocomplete but it is not locally presentable in general. For example, the quasi-category $\mathbf{Hot}^o$ is not locally presentable. We shall say that a complete quasi-category Z is a (generalised) *algebraic theory* if the quasi-category Z^o is locally presentable. Every limit sketch (A, P) has a universal model $u : A \to U(A, P)$ with values in the algebraic theory $U(A, P) = Mod(A, P)^o$. The map $u^o : A^o \to Mod(A, P)$ is obtained by composing the Yoneda map $A^o \to [A, \mathbf{Hot}]$ with the reflection map $[A, \mathbf{Hot}] \to Mod(A, P)$. Let us denote by **AT** the category whose objects are the generalised algebraic theories and whose maps are the continuous maps. The 2-categories **AT** and **LP** are isomorphic (beware that the isomorphism reverses the direction of 2-cells). Hence the 2-category **AT** has the structure of a symmetric (pseudo)-monoidal closed category. We shall denote by $T \odot U$ the tensor product of two algebraic theories T and U. By definition, we have $(T \odot U)^o = T^o \otimes U^o$. We shall denote by $Mod(T, U)$ the quasi-category of continuous maps $T \to U$. We have $Mod(T, U)^o = Map(T^o, U^o)$. The category **LP** is cotensored over the category **AT**. More precisely, if X is locally presentable quasi-categorie and T is an algebraic theory, then the quasi-category of continuous maps $T \to X$ is a locally presentable quasi-category $Mod(T, X)$. In fact, we have

$$Mod(T, X) = T^o \otimes X.$$

In particular, $Mod(T, \mathbf{Hot}) = T^o$. It follows that we have $X \otimes Y = Mod(X^o, Y)$ for any pair of locally presentable quasi-categories X and Y. We also have the formula

$$Mod(S \odot T) = Mod(S) \otimes Mod(T)$$

for any pair of algebraic theories S and T. Dually, the category **AT** is cotensored over the category **LP**. More precisely, if T is an algebraic theory and X is a locally presentable quasi-category, then the quasi-category of cocontinuous maps $X \to T$ is an algebraic theory $Map(X, T)$. In fact, we have

$$Map(X, T) = Mod(T^o, X^o) = X^o \odot T.$$

In particular, $Map(X, \mathbf{Hot}^o) = X^o$.

24. Categories in quasi-categories

24.1. If X is a quasi-category, we call a map $C : \Delta^o \to X$ is a *simplicial object* in X. There is then a quasi-category $[\Delta^o, X]$ of simplicial objects in X. If X is finitely complete, we shall say that a simplicial object $C : \Delta^o \to X$ is a *category* if it satisfies the *Segal conditions* . These conditions can be expressed in many ways. For example by demanding that C takes every square of the form

$$\begin{array}{ccc} [0] & \longrightarrow & [n] \\ \downarrow & & \downarrow{\scriptstyle b} \\ [m] & \xrightarrow{a} & [m+n], \end{array}$$

to a pullback square in X, where a denotes the inclusion $[m] \subseteq [m+n]$ and b the composite $[n] = [m, m+n] \subseteq [m+n]$. We denote by $Cat(X)$ the full simplicial subset of $[\Delta^o, X]$ spanned by the categories in X. We say that an arrow in $Cat(X)$ is a *functor*. If $C \in Cat(X)$, we shall put $Ob(C) = C_0$. If $Ob(C) = 1$, we shall say that C is a *monoid* (with underlying object C_1). A category C is a *groupoid* if it takes the following square

$$\begin{array}{ccc} [0] & \xrightarrow{d_0} & [1] \\ {\scriptstyle d_0}\downarrow & & \downarrow{\scriptstyle d_2} \\ [1] & \xrightarrow{d_1} & [2] \end{array}$$

to a pullback square. A groupoid C is a *group* if $Ob(C) = 1$. We denote by $Grpd(X)$ the full simplicial subset of $Cat(X)$ generated by the groupoids. The inclusion $Grpd(X) \subseteq Cat(X)$ has a right adjoint $J : Cat(X) \to Grpd(X)$ which associate to $C \in Cat(X)$ its groupoid of isomorphisms of C.

24.2. Let X be a finitely complete quasi-category. An element of $Cat^2(X) = Cat(Cat(X))$ is called a *double category* in X. A double simplicial object $\Delta^o \times \Delta^o \to X$ is a double category iff it is a category in each variable. The notion of n-fold category can be defined for every $n \geq 0$. We shall denote by $Cat^n(X)$ the quasi-category of n-fold categories in X. Let X be a finitely complete quasi-category. A monoid in X is a category $C : \Delta^o \to X$ such that $C_0 = 1$. A braided monoid in X is a 2-fold category $C : (\Delta^2)^o \to X$ such that $C_{m0} = C_{0n} = 1$. More generally, a n-fold monoid in X is a n-fold category $C : (\Delta^n)^o \to X$ such that $C_m = 1$ for all $m = (m_1, \ldots, m_n)$ such that $m_1 \cdots m_n = 0$.

24.3. Let X be a finitely complete quasi-category. A *2-category* in X is a double category $C : \Delta^o \to Cat(X)$ such that $C_0 : \Delta^o \to X$ is (essentially) constant. We shall denote by $Cat_2(X)$ the quasi-category of 2-categories in X. The quasi-category $Cat_n(X)$ of n-category objects in X can be defined by induction on $n \geq 0$. A category object $C : \Delta^o \to Cat_n(X)$ is a $(n+1)$-category iff the n-category C_0 is (essentially) constant. Here is a gobal description: a n-fold category $C : (\Delta^n)^o \to X$ is a n-category iff its restriction to the subcategory

$$Ob(\Delta)^k \times \{[0]\} \times \Delta^{n-k-1}$$

is is (essentially) constant for every $0 \leq k < n$. The inclusion $\{[0]\} \subset \Delta$ is right adjoint to the map $\Delta \to \{[0]\}$. It follows that the inclusion $i_n : \Delta^n = \Delta^n \times \{[0]\}^k \subseteq \Delta^{n+k}$ is right adjoint to the projection $p_n : \Delta^{n+k} = \Delta^n \times \Delta^k \to \Delta^n$. The pair of adjoint maps

$$p_n^* : [(\Delta^n)^o, X] \leftrightarrow [(\Delta^{n+k})^o, X] : i_n^*$$

induces a pair of adjoint maps

$$inc : Cat_n(X) \leftrightarrow Cat_{n+k}(X) : sk^n$$

The functor *inc* is fully faithful and we shall regard it as an inclusion by adopting the same notation of $C \in Cat_n(X)$ and $inc(C) \in Cat_{n+k}(X)$. The functor sk^n associates to $C \in Cat_{n+k}(X)$ its *n-skeleton* $sk^n(C) \in Cat_n(X)$.

24.4. A *monoidal category* in X is a monoid object in $Cat(X)$ or equivalently a category object in $Mon(X)$. A *braided monoidal category* is a braided monoid in $Cat(X)$ or equivalently a 3-category $C : \Delta^3 \to X$ such that $sk^2(C) = 1$. More generally, a n-fold monoidal k-category in X is a $(n+k)$-category $C \in Cat_{n+k}(X)$ such that $sk^n(C) = 1$. This is the stabilisation conjecture of Baez and Dolan proved by Hirschowitz and Simpson.

24.5. A *simplicial space* is defined to be a functor $\Delta^o \to \mathbf{S}$. A simplicial space X becomes a bisimplicial set if we put $X_{mn} = (X_m)_n$ for every $m, n \geq 0$. The category $\mathbf{S}^{(2)} = [\Delta^o, \mathbf{S}]$ of simplicial spaces admits a simplicial model structure defined by Reedy. A map of simplicial spaces $X \to Y$ is a weak equivalence for this model structure iff the map $X_m \to Y_m$ is a weak homotopy equivalence for every $m \geq 0$. The cofibrations are the monic maps. A Reedy fibrant simplicial space X is said to satisfy the *Segal condition* if the canonical map

$$X_n \to X_1 \times_{\partial_0, \partial_1} X_1 \times \cdots \times_{\partial_0, \partial_1} X_1$$

is a weak homotopy equivalence for every $n \geq 2$. A Reedy fibrant simplicial space which satisfies the Segal condition is called a *Segal space.* The Reedy model structure on $\mathbf{S}^{(2)}$ admits a localisation in which the fibrant objects are the Segal spaces [Rez]. The localised model structure is called the *model structure for Segal spaces.* The category $\mathbf{SS}$ of Segal spaces is enriched over Kan complexes. Its coherent nerve is equivalent to the quasi-category $Cat(\mathbf{Hot})$.

24.6. A *n-fold simplicial space* is defined to be a functor $(\Delta^n)^o \to \mathbf{S}$. A n-fold simplicial space X becomes a $(n+1)$-fold simplicial set if we put $X_{mr} = (X_m)_r$ for every $(m, r) \in \mathbf{N}^n \times \mathbf{N}$. The category $\mathbf{S}^{(n+1)} = [(\Delta^n)^o, \mathbf{S}]$ of n-fold simplicial spaces admits a Reedy model structure. A map of n-fold simplicial spaces $X \to Y$ is a weak equivalence for this model structure iff the map $X_m \to Y_m$ is a weak homotopy equivalence for every $m \in \mathbf{N}^n$. The cofibrations are the monic maps. A

Reedy fibrant n-fold simplicial space X is said to satisfy the *Segal condition* if it satisfies the Segal condtion in each variable. A Reedy fibrant n-fold simplicial space which satisfies the Segal condition in each variable is called a *n-fold Segal space*. The Reedy model structure on $\mathbf{S}^{(n+1)}$ admits a localisation in which the fibrant objects are the n-fold Segal spaces. The localised model structure is called the *model structure for n-fold Segal spaces*. The category $\mathbf{SS^n}$ of n-fold Segal spaces is enriched over Kan complexes. Its coherent nerve is equivalent to the quasi-category $Cat^n(\mathbf{Hot})$.

24.7. We shall say that a 2-fold Segal space $C : \Delta \times \Delta \to \mathbf{S}$ is a *Segal 2-space* if the functor $C_0 = C_{0\star} : \Delta \to\to \mathbf{S}$ is homotopicaly constant (ie. if it takes every arrow in Δ) to a homotopy equivalence). We shall denote by $\mathbf{SS}_2$ the category of Segal 2-spaces. There is a notion of Segal n-space for each $n \geq 0$. A $(n+1)$-fold Segal space $C : \Delta^o \to \mathbf{SS}_n$ is a Segal $(n+1)$-space iff the Segal n-space C_0 is homotopicaly constant. The Reedy model structure on $\mathbf{S}^{(n+1)}$ admits a localisation in which the fibrant objects are the Segal n-spaces. The localised model structure is the *model structure for Segal n-spaces*. The category $\mathbf{SS_n}$ of n-fold Segal spaces is enriched over Kan complexes. Its coherent nerve is equivalent to the quasi-category $Cat_n(\mathbf{Hot})$.

25. Absolutely exact quasi-categories

Let X be a quasi-category with pullbacks. Then the functor $Ob : Grpd(X) \to X$ has a right adjoint $Cosk^0 : X \to Grpd(X)$. The right adjoint associates to an object $b \in X$ the simplicial object $Cosk^0(b) : \Delta^o \to X$ obtained by putting $Cosk^0(b)_n = b^{[n]}$ for each $n \geq 0$. We shall say that $Cosk^0(b)$ is the *full groupoid* of b. More generally, the *equivalence groupoid $Eq(u)$* of an arrow $u : a \to b$ in X is defined to be the full groupoid of the object $u \in X/b$ (or rather its image by the canonical map $X/b \to X$). The equivalence groupoid of a pointed object $p : 1 \to b$ is the *loop group* $Eq(p) = \Omega(b)$ of b at the base point p.

25.1. Let X be a finitely complete quasi-category. We shall say that X is *absolutely exact* if the following conditions are satisfied:

- X admits surjection-mono-factorisations and these factorisations are stable under base change;
- every groupoid $C \in Grpd(X)$ is the equivalence groupoid of a surjection $u \in X$.

The quasi-category **Hot** is absolutely exact.

25.2. Let X be an absolutely exact quasi-category. Then the slice quasi-categories $b\backslash X$ and X/b are absolutely exact for any node $b \in X$. The quasi-category X^S is absolutely exact for any simplicial set S. If T is a small quasi-category with finite products, then the quasi-category of models $Mod(T, X)$ is absolutely exact. For examples, the quasi-category of monoids and of groups in X are absolutely exact.

25.3. Let X be an absolutely exact quasi-category. Then the map $Sk^0 : X \to Grpd(X)$ has a left adjoint

$$B : Grpd(X) \to X.$$

If $a \in X_0$, let us denote by $Surj(a\backslash X)$ the full simplicial subset of $a\backslash X$ whose nodes are the surjections with codomain a. Let $Grpd(X, a)$ the fiber at a of the map $Ob : Grpd(X) \to X$. Then the map $Surj(a\backslash X) \to Grpd(X, a)$ which associates to a surjection $u : a \to b$ the equivalence groupoid $Eq(u)$ is an equivalence of quasi-categories. We shall denote the quasi-inverse equivalence by

$$B : Grpd(X, a) \to Surj(a\backslash X).$$

If $a = 1$, the quasi-category $Grpd(X, a)$ is the quasi-category $Grp(X)$ of group objects in X. An object of $Surj(1\backslash X)$ is a pointed connected object of X. We thus have an equivalence of quasi-categories

$$B : Grp(X) \leftrightarrow Surj(1\backslash X) : \Omega,$$

where $\Omega(b)$ is the loop group of a pointed connected object $p : 1 \to b$ in X.

25.4. If X is an absolutely exact quasi-category, then so is the quasi-category $Surj(a\backslash X)$ for every object $a \in X$. An arrow of $Surj(a\backslash X)$ is surjective (resp. monic) iff its underlying arrow in X is 0-connected (resp. a 0-cover). It follows from this that X admits n-factorisation for every $n \geq 0$. The n-factorisation of an arrow $a \to b$ is obtained from the $(n-1)$-factorisation of the diagonal $a \to a \times_b a$. More precisely, if $a \to e \to b$ is the n-factorisation of an arrow $a \to b$, then $a \to a \times_e a \to a \times_b a$ is the $(n-1)$-factorisation of the diagonal $a \to a \times_b a$. A surjection $a \to b$ is n-connected iff the diagonal $a \to a \times_b a$ is $(n-1)$-connected.

25.5. A quasi-category X is *pointed* if it contains an object $0 \in X$ which is both initial and terminal. Let X be an absolutely exact pointed quasi-category. If $C(X) \subseteq X$ is the sub-quasi-category spanned by the connected objects (connected = 0-connected) then we have an equivalence of quasi-categories,

$$B : Grp(X) \leftrightarrow C(X) : \Omega$$

This equivalence can be iterated since the quasi-category $C(X)$ is absolutely exact and pointed. We have $Grp(X) = Mod(Gr, X)$, where Gr denotes the algebraic theory of groups. We say that an element of $Grp^n(X) = Mod(Gr^{\otimes n}, X)$ is a *n-fold group*. We have $Grp^{n+1}(X) = Grp(Grp^n(X))$ for every $n \geq 0$. If $C_n(X)$ denotes the full simplicial subset of X spanned by the n-connected objects, then we have an equivalence of quasi-categories

$$B^n : Grp^n(X) \leftrightarrow C_{n-1}(X) : \Omega^n.$$

The equivalence can be constructed by induction on $n \geq 1$. We have $C_{n+1}(X) = C(C_n(X))$ and the map $\Omega^{n+1} : C_n(X) \to Grp^{n+1}(X)$ is the composite of the maps

$$C(C_{n-1}(X)) \xrightarrow{C(\Omega^n)} C(Grp^n(X)) \xrightarrow{\Omega} Grp(Grp^n(X)).$$

26. Descent theory

26.1. If X is a finitely complete quasi-category, then the map $Ob : Cat(X) \to X$ is a Grothendieck fibration. A functor $f : C \to D$ in $Cat(X)$ is a cartesian arrow iff it is *fully faithful*, that is, iff the square

$$\begin{array}{ccc} C_1 & \xrightarrow{f_1} & D_1 \\ \downarrow{\scriptstyle (s,t)} & & \downarrow{\scriptstyle (s,t)} \\ C_0 \times C_0 & \xrightarrow{f_0 \times f_0} & D_0 \times D_0, \end{array}$$

is cartesian.

26.2. Let X be an absolutely exact quasi-category. We shall say that a functor $f : C \to D$ in $Grpd(X)$ is *essentially surjective* if the morphism $tp_1 : D_1 \times_{D_0} C_0 \to D_1$ obtained from the square

$$\begin{array}{ccc} D_1 \times_{D_0} C_0 & \xrightarrow{p_2} & C_0 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle f_0} \\ D_1 & \xrightarrow{s} & D_0 \end{array}$$

is surjective, where s is the source morphism and t the target morphism. More generally, we say that a functor $f : C \to D$ in $Cat(X)$ is *essentially surjective* if the functor $J(f) : J(C) \to J(D)$ is essentially surjective. We shall say that f is a *weak equivalence* if it is fully faithful and essentially surjective.

26.3. If X is a finitely complete quasi-category. We shall say that a functor $p : E \to C$ in $Cat(X)$ is a *left fibration* if the naturality square

$$\begin{array}{ccc} E_1 & \xrightarrow{s} & E_0 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_0} \\ C_1 & \xrightarrow{s} & C_0 \end{array}$$

is cartesian, where s is the source map. The notion of right fibration is defined dually by using the target map t. We shall denote by X^C the full simplicial subset of $Cat(X)/C$ whose objects are the left fibrations $E \to C$. The pullback of a left fibration $E \to D$ along a functor $f : C \to D$ in $Cat(X)$ is a left fibration $f^*(E) \to C$. This defines a map $f^* : X^D \to X^C$. We shall say that f is a *Morita equivalence* if the map f^* is an equivalence of quasi-categories.

26.4. Let X be an absolutely exact quasi-category. Then a weak equivalence $f : C \to D$ in $Cat(X)$ is a Morita equivalence. In particular, if an arrow $u : a \to b$ in X is surjective, then the induced functor $Eq(u) \to Sk^0(b)$ is a Morita equivalence. Thus, u^* induces an equivalence of quasi-categories

$$X/b \simeq X^{Eq(u)}.$$

In particular, if $p : 1 \to b$ is a pointed connected object of X, then we have an equivalence of quasi-categories,

$$X/b \simeq X^{\Omega(b)}.$$

27. Stable quasi-categories

27.1. A *zero object* (0-object) in a quasi-category is an object which is both initial and final. A quasi-category is *pointed* if it admits a 0-object. For example, if 1 is a terminal object in a quasi-category X, then the quasi-category $1\backslash X$ is pointed. The homotopy category hoX of a pointed quasi-category X is pointed. We shall say that an arrow $x \to y$ between two objects of a quasi-category is *nul* if its image in hoX is equal to the composite $x \to 0 \to y$.

27.2. Let X be a finitely cocomplete pointed quasi-category with zero object $0 \in X$. The *smash product* of an element $x \in X$ by a finite pointed simplicial set A is the element $A \wedge x \in X$ defined by the pushout square,

$$\begin{array}{ccc} 1 \cdot x & \longrightarrow & 1 \cdot 0 \\ {\scriptstyle a \cdot x}\downarrow & & \downarrow \\ A \cdot x & \longrightarrow & A \wedge x, \end{array}$$

where $a : 1 \to A$ is the base point. This defines a map $A \wedge (-) : X \to X$. For any pair A and B of pointed simplicial sets, we have a quasi-isomorphism

$$A \wedge (B \wedge x) \simeq (A \wedge B) \wedge x)$$

which is homotopy unique. Similarly, we have a canonical quasi-isomorphism $S^0 \wedge x \simeq x$, where S^0 is the pointed 0-sphere. The *suspension* $\Sigma : X \to X$ is defined to be the map $S^1 \wedge (-) : X \to X$, where S^1 is the pointed 1-sphere. The *n-fold suspension* $\Sigma^n : X \to X$ is the map $S^n \wedge (-) : X \to X$, where S^n is the pointed n-sphere.

27.3. Dually, let X be a finitely complete pointed quasi-category with zero object $0 \in X$. The *coaction* of an element $x \in X$ by a finite pointed simplicial set A is the element $[A, x] \in X$ defined by the pullback square,

$$\begin{array}{ccc} [A, x] & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ x^A & \xrightarrow{x^a} & x^1, \end{array}$$

where $a : 1 \to A$ is the base point. This defines a map $[A, -] : X \to X$. For any pair A and B of pointed simplicial sets, we have a quasi-isomorphism

$$[A, [B, x]] \simeq [A \wedge B, x]$$

which is homotopy unique. Similarly, we have a canonical quasi-isomorphism $[S^0, x] \simeq x$. The *loop space map* $\Omega : X \to X$ is defined to be the map $[S^1, -] : X \to X$. The *n-fold loop space* $\Omega^n : X \to X$ is the map $[S^n, -] : X \to X$.

27.4. The opposite X^o of a finitely cocomplete pointed quasi-category X is finitely complete, We have the formulas

$$(A \wedge x)^o = [A^o, x^o] \simeq [A, x^o],$$

since A and A^o are weakly homotopy equivalent. In particular, we have $(\Sigma \cdot x)^o \simeq \Omega(x^o)$. The map $A \wedge (-) : X \to X$ is left adjoint to the map $[A, -] : X \to X$ if X is finitely bicomplete.

27.5. Let $\mathbf{LP}$ be the category of locally presentable quasi-categories and cocontinuous maps, and let $\mathbf{LP}_\bullet$ be the full subcategory of $\mathbf{LP}$ spanned by the pointed locally presentable quasi-categories. Then the inclusion functor $\mathbf{LP}_\bullet \subset \mathbf{LP}$ has a (pseudo) left adjoint which associates to $X \in \mathbf{LP}$ the quasi-category $X_\bullet = 1\backslash X$, where 1 denotes the terminal object of X. The canonical map $X \to 1\backslash X$ associates to $x \in X$, the arrow $1 \to 1 \sqcup x$.

27.6. Let X be a pointed quasi-category with finite coproducts and products. For any pair of objects $x, y \in X$ there is a canonical arrow $x \sqcup y \to x \times y$ well defined in hoX; we shall say that X is *semi-additive* if the arrow is quasi-invertible for any pair x and y. In this case, the coproduct (resp. the product) of x and y can be written as a *direct sum* $x \oplus y$. The homotopy category of a semi-additive quasi-category X is semi-additive. If X is semi-additive, then the set $hoX(x, y)$ has the structure of a commutative monoid for any pair of objects $x, y \in X$. We shall say that X is *additive* if the monoid $hoX(x, y)$ is a group for any pair $x, y \in X$.

27.7. We shall say that a finitely bicomplete pointed quasi-category is *stable* if the suspension map $\Sigma : X \to X$ is an equivalence of quasi-categories, or equivalently if the loop space map $\Omega : X \to X$ is an equivalence. A stable quasi-category is additive.

27.8. We shall say that a stable quasi-category X is *exact* if every arrow $a \to b$ in X is the fiber of an arrow $b \to b'$ as well as the cofiber of an arrow $a' \to a$. An exact stable quasi-category is absolutely exact. For every object $a \in X$, the map $Fib : a\backslash X \to X/a$ which associates to an arrow $f : a \to b$ its fiber $ker(f) \to a$ is an equivalence of quasi-categories. The (pseudo-)inverse equivalence is the map $Cofib : X/a \to a\backslash X$ which associates to an arrow $g : e \to a$ its cofiber $a \to coker(f)$. The homotopy category hoX of an exact stable quasi-category X is triangulated.

27.9. Let $\mathbf{SLP}$ be the subcategory of $\mathbf{LP}_\bullet$ spanned by the stable locally presentable quasi-categories. Then the inclusion functor $\mathbf{SLP} \subset \mathbf{LP}_\bullet$ has a (pseudo) left adjoint

$$St : \mathbf{LP}_\bullet \to \mathbf{SLP}$$

which associates to a pointed quasi-category X the stable quasi-category $St(X)$ generated by X. If the loop map $\Omega : X \to X$ preserves directed colimits, the quasi-category $St(X)$ can be constructed as the (homotopy) projective limit of the sequence of quasi-categories

$$X \xleftarrow{\Omega} X \xleftarrow{\Omega} X \xleftarrow{\Omega} X \longleftarrow \cdots$$

The canonical map $X \to St(X)$ associates to $x \in X$ the sequence $(Q\Sigma^n x : n \geq 0)$, where $Qx = \Omega^\infty \Sigma^\infty x$ denotes the colimit in X of the sequence

$$x \to \Omega\Sigma x \to \Omega^2\Sigma^2 x \to \cdots .$$

27.10. The stable quasi-category $St(\mathbf{Hot}_\bullet)$ is the *quasi-category of spectra*,

$$\mathbf{Spec} = St(\mathbf{Hot}_\bullet).$$

The stable quasi-category **Spec** is exact.

27.11. The *sphere spectrum* $S = QS^0 \in \mathbf{Spec}$ is defined to be the image of the 0-sphere $S^0 \in \mathbf{Hot}_\bullet$ by the canonical map $Q : \mathbf{Hot}_\bullet \to \mathbf{Spec}$. The stable quasi-category **Spec** is freely generated by the sphere spectrum. This means that for any stable quasi-category X and any object $a \in X$, there is a cocontinuous map $f : \mathbf{Spec} \to X$ such that $f(S) = a$ and that f is unique up to a unique invertible 2-cell.

27.12. If X is a stable locally presentable quasi-category, then so is $X \otimes Y$ for any locally presentable quasi-category Y. Similarly, for the quasi-categories $Map(Y, X)$ for any locally presentable quasi-category Y and the quasi-category $Mod(T, X)$ for any algebraic theory T. It follows that the category **SLP** of stable locally presentable quasi-categories is closed (pseudo-)monoidal. The unit object for the tensor product is the quasi-category of spectra **Spec**.

28. ∞-TOPOS

28.1. Recall that a finitely complete quasi-category X is said to be *locally cartesian closed* if the quasi-category X/a is cartesian closed for every object $a \in X$. A finitely complete quasi-category X is locally cartesian closed iff the base change map $f^* : X/b \to X/a$ has a right adjoint $f_* : X/a \to X/b$ for any map $f : a \to b$ in X. A locally presentable quasi-category X is locally cartesian closed, iff the pullback map $f^* : X/b \to X/a$ is cocontinuous for any map $f : a \to b$ in X.

28.2. We shall say that a locally presentable quasi-category X is an ∞-*topos* if the following conditions are satisfied:

- X is locally cartesian closed;
- X is absolutely exact;
- The canonical map

$$X/\sqcup a_i \to \prod_i X/a_i$$

is an equivalence for any family of objects $(a_i : i \in I)$ in X.

28.3. The quasi-category **Hot** is the primary example of an ∞-topos. If X is an ∞-*topos*, then so is the quasi-categorie X/a for any object $a \in X$ and the quasi-category X^A for any simplicial set A. In particular, the quasi-category $\mathcal{P}(A)$ is an ∞-topos for any simplicial set A.

28.4. A *geometric morphism* $g : X \to Y$ between ∞-topoi is an adjoint pair of maps

$$g^* : Y \leftrightarrow X : g_*$$

in which g^* is a cartesian map. We call g^* the *inverse image part* of g and g_* the *direct image part* of g. If f and $g : X \to Y$ are geometric morphisms, a *geometric transformation* $\alpha : f \to g$ is a pair of adjoint 2-cells $\alpha^* : f^* \to g^*$ and $\alpha_* : g_* \to f^*$. We shall denote by $\mathbf{Top}^\infty$ the 2-category of ∞-topoi and geometric morphisms. An *geometric map* $Y \to X$ between ∞-toposes is a cocontinuous cartesian map $Y \to X$. Every geometric map $Y \to X$ is the inverse image part of a geometric morphism $X \to Y$ which is unique up to a unique invertible geometric transformation. The 2-category $\mathbf{Top}^\infty$ is equivalent to the opposite of the 2-category of topoi and geometric maps (beware that the duality preserves the direction of 2-cells).

28.5. If $u : A \to B$ is a map of simplicial sets, then the pair of adjoint maps $u^* : \mathcal{P}(B) \to \mathcal{P}(A) : u_*$ is a geometric morphism $\mathcal{P}(A) \to \mathcal{P}(B)$. If X is an ∞-topos, then the pair of adjoint maps $f^* : X/b \to X/a : f_*$ is a geometric morphism $X/a \to X/b$ for any map $f : a \to b$ in X.

28.6. Let $\mathcal{E}$ be a Grothendieck topos. Then the category $[\Delta^o, \mathcal{E}]$ of simplicial sheaves on $\mathcal{E}$ has a simplicial model structure introduced by the author. The coherent nerve of the category of fibrant objects is an ∞-topos $\mathcal{E}^\infty$. This defines a functor

$$(-)^\infty : \mathbf{Top}^0 \to \mathbf{Top}^\infty,$$

where $\mathbf{Top}^0$ is the category of Grothendieck toposes. The functor has a left adjoint constructed as follows. If X is an ∞-topos, an object $x \in X$ is said to be *discrete* if the diagonal $x \to x \times x$ is monic. Let $Dis(X)$ be the full simplicial subset of X spanned by the discrete objects. The inclusion $Dis(X) \subseteq X$ has a left adjoint $\pi_0 : X \to Dis(X)$. The canonical map $Dis(X) \to hoDis(X)$ is an equivalence of quasi-categories. We shall identify $Dis(X)$ with $hoDis(X)$. The category $Dis(X)$ is a Grothendieck topos. A geometric map $X \to Y$ induces a geometric map $Dis(X) \to Dis(Y)$. This defines a 2-functor

$$Dis : \mathbf{Top}^\infty \to \mathbf{Top}^0,$$

which is right adjoint to the functor $\mathcal{E} \mapsto \mathcal{E}^\infty$.

28.7. If X is an ∞-topos, we shall say that a reflexive sub quasi-category $Y \subseteq X$ is a *sub-topos* if the reflection $X \to Y$ is cartesian (i.e. preserves finite limits). We shall say that a set Σ of arrows in X is a *Grothendieck topology* if the quasi-category of Σ-local objects $X^\Sigma \subseteq X$ is a sub-topos.

28.8. Let $f : X \to Y$ be a geometric morphism between ∞-toposes. If Σ is Grothendieck topology on Y, then $f_*(X) \subseteq Y^\Sigma$ iff f^* take every quasi-isomorphism in Then

in X is a *Grothendieck topology* if the quasi-category of Σ-local objects $X^\Sigma \subseteq X$ is a sub-topos.

, we shall say that a reflexive sub quasi-category $Y \subseteq X$ is a *sub-topos* if the reflection $X \to Y$ is cartesian (i.e. preserves finite limits). We shall say that a set Σ of arrows in X is a *Grothendieck topology* if the quasi-category of Σ-local objects $X^\Sigma \subseteq X$ is a sub-topos.

Let A be a simplicial set and let $\Sigma \subseteq \mathcal{P}(A)$ be a Grothendieck topology. For each object $a \in A$ can be obtained by specifying for each object $a \in A$ a set Σ_a of arrows with codomain $a = y_A(a)$ in $\mathcal{P}(A)$. The family $\Sigma = (\Sigma_a : a \in A_0)$ is said to be *closed under base change* if we have $f^*(\Sigma_b) \subset \Sigma_a$ for every arrow $f : a \to b$ in A. The author does not know of More generally, we shall say that a map $f : x \to y$ in $\mathcal{P}(A)$ is *fibrewise in* Σ if the map $u * (x) \to a$ belongs to Σ_a for every $a \in A_0$ and every map $a \to y$ in $\mathcal{P}(A)$. Let $\Sigma' \subseteq \mathcal{P}(A)$ be the class of maps fibrewise in Σ. whose fibers belongs to Σ. The class $\overline{\Sigma}$ is by construction closed under base change. We shall say that Σ is closed under Let us suppose that Σ is closed under composition and that it has the right cancellation property. Let us suppose that Σ satisfies

The set $\Sigma = \sqcup_a \Sigma_a$ is said to be *closed under base change* if we have $f^*(\Sigma_b) \subset \Sigma_a$ for every arrow $f : a \to b$ in A.

We shall say that Σ is cl

In this case, the set $\Sigma = \sqcup_a \Sigma_a$ is a Grothendieck topology in $\mathcal{P}(A)$. Every subtopos of $\mathcal{P}(A)$ is of the form $\mathcal{P}(A)^\Sigma$ for a Grothendieck topology Σ of this form. The pair (A, Σ) is called a *site*. A prestack $s \in \mathcal{P}(A)$ is called a *stack* if it is a Σ-local object. Every ∞-topos is equivalent to a quasi-category of stacks on a site.

28.9. If A is a simplicial set, a Grothendieck topology on $\mathcal{P}(A)$ can be obtained by specifying for each object $a \in A$ a set Σ_a of arrows with codomain $a = y_A(a)$ in $\mathcal{P}(A)$. The family $(\Sigma_a : a \in A_0)$ is said to be *closed under base change* if we have $f^*(\Sigma_b) \subseteq \Sigma_a$ for every arrow $f : a \to b$ in A. In this case, the set $\Sigma = \sqcup_a \Sigma_a$ is a Grothendieck topology in $\mathcal{P}(A)$. Every sub-topos of $\mathcal{P}(A)$ is of the form $\mathcal{P}(A)^\Sigma$ for a Grothendieck topology Σ of this form. The pair (A, Σ) is called a *site*. A prestack $s \in \mathcal{P}(A)$ is called a *stack* if it is a Σ-local object. Every ∞-topos is equivalent to a quasi-category of stacks on a site.

28.10. If A is a simplicial set, then every set of arrows $\Sigma \subseteq \mathcal{P}(A)$ generates a Grothendieck topology $\Sigma' = \sqcup_a \Sigma'_a$, where Σ'_a is a set of arrows with codomain a representable $a \in A_0$. By construction, an arrow $u : c \to a$ belongs to Σ'_a iff there exists a pull back square

$$\begin{array}{ccc} c & \longrightarrow & e \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ a & \longrightarrow & b, \end{array}$$

with $g \in \Sigma$. If X is an ∞-topos, a geometric map $\mathcal{P}(A) \to X$ inverts every arrow in Σ iff it inverts every arrow in Σ'.

28.11. If A is a simplicial set, then the quasi-category $\mathcal{P}(A)$ is equivalent to the coherent nerve of the simplicial category $\mathcal{R}(A)$ of fibrant objects of the model category $(\mathcal{S}/A, W_r)$ by **??**. If Σ is a set of arrows in $\mathcal{S}/A$, let us denote by $(\mathcal{S}/A, W_r(\Sigma))$ the model structure obtained by localising the model structure $(\mathcal{S}/A, W_r)$ with respect to Σ. We shall say that Σ is a *quasi-topology* if every homotopy pullback square in the model category $(\mathcal{S}/A, W_r)$ is also a homotopy pullback square in the model category $(\mathcal{S}/A, W_r(\Sigma))$. For each vertex $a \in A$, let us choose a factorisation of the map $a : 1 \to A$ as a right cofibration $1 \to Ra$ followed by a right fibration $Ra \to A$. Suppose that we have a set Σ_a of right fibrations with codomain Ra for each $a \in A_0$. We shall say that the family $(\Sigma_a : a \in A_0)$ is *closed under base change* if we have $f^*(\Sigma_b) \subseteq \Sigma_a$ for every map $f : Ra \to Rb$ in $\mathcal{S}/A$.

28.12. Let X and Y be ∞-toposes. We call a geometric morphism $f : X \to Y$ an *embedding* if direct image part f_* is fully faithful. In this case the essential image of f_* is a sub-topos $Z \subseteq Y$ and the induced map $X \to Z$ is an equivalence of quasi-categories.

28.13. Let X and Y be ∞-toposes. We call a geometric morphism $f : X \to Y$ a *surjection* if the map $f^* : Y \to X$ is conservative. Every geometric morphism can be factored as a surjection followed by an embedding and this factorisation is unique up to equivalence.

28.14. Recall that if X is a bicomplete quasi-category and A is a simplicial set, then every map $f : A \to X$ has a cocontinuous extension $\bar{f} : \mathcal{P}(A) \to X$. A locally presentable quasi-category X is an ∞-topos iff for any small cartesian quasi-category T, the cocontinuous extension $\bar{f} : \mathcal{P}(T) \to X$ of any cartesian map $f : T \to X$ is cartesian.

28.15. Every simplicial set A generates freely a cartesian quasi-category $A \to C(A)$ by **??**. Similarly, every simplicial set A generates freely an ∞-topos $i : A \to Top(A)$. This means that If X is an ∞-topos, then every map $f : A \to X$ has a geometric extension $f' : T(A) \to X$ along i and that this extension is unique up to a unique invertible 2-cell. By construction, $Top(A) = \mathcal{P}(C(A))$. The map $i : A \to Top(A)$ is obtained by composing the canonical map $A \to C(A)$ with the Yoneda map $C(A) \to \mathcal{P}(C(A))$.

28.16. A *geometric sktech* is a pair (A, Σ), where Σ is a set of arrows in $Top(A)$. A *geometric model* of (A, Σ) with values in an ∞-topos X is a map $f : A \to X$ whose geometric extension $f' : T(A) \to X$ takes every arrow in Σ to a quasi-isomorphism in X. We shall denote by $Mod(A/\Sigma, X)$ the full simplicial subset of X^A which is spanned by the models $A \to X$.

28.17. Every geometric sktech has a *universal geometric model* $u : A \to Top(A/\Sigma)$. The universality means that for every ∞-topos X and every model $f : A \to X$ there is a geometric map $f' : Top(A/\Sigma) \to X$ such that $f'u = f$ and moreover that f' is unique up to a unique invertible 2-cell. We shall say that $Top(A/\Sigma)$ is the *classyfying topos* of (A, Σ). The ∞-topos $Top(A/\Sigma)$ is a sub-topos of the topos $Top(A)$. We have $Top(A/\Sigma) = Top(A)^{\Sigma'}$, where $\Sigma' \subset Top(A)$ is the Grothendieck topology generated by Σ.

29. Higher quasi-categories

29.1. Let X be a quasi-category with pullbacks. We say that a category $C \in Cat(X)$ satisfies the *Rezk condition*, or that it is *reduced*, if the groupoid $J(C)$ is essentially constant, that is, if the canonical functor $Sk^0(C_0) \to J(C)$ is quasi-invertible. We shall denote by $RCat(X)$ the quasi-category of reduced categories in X.

An ordinary category $C \in Cat(\mathbf{Set})$ is reduced iff its groupoid of isomorphism is discrete, that is, if every isomorphism of C is an identity.

29.2. Let X be an absolutely exact quasi-category. Then the inclusion $RCat(X) \subset Cat(X)$ has a left adjoint

$$R : Cat(X) \to RCat(X)$$

which associates to a category $C \in Cat(X)$ its *reduction* RC. By construction, $(RC)_n = BJ(C^{[n]})$ for every $n \geq 0$, where $C^{[n]}$ is the internal category of functor $[n] \to C$. The canonical map $C \to RC$ is a weak equivalence of categories (it is fully faithful and essentially surjective). It is thus a Morita equivalence, since the quasi-category X is absolutely exact.

29.3. The *box product* $A \Box B$ of two simplicial sets A and B is the bisimplicial set obtained by putting

$$(A \Box B)_{mn} = A_m \times B_n$$

for every $m, n \geq 0$. The functor $(A, B) \mapsto A \Box B$ is divisible on both sides. In particular, the functor $A \Box (-) : \mathbf{S} \to \mathbf{S}^{(2)}$ admits a right adjoint $A \backslash (-) : \mathbf{S}^{(2)} \to \mathbf{S}$ for every simplicial set A. Let J be the groupoid generated by one isomorphism $0 \to 1$. A Segal space X is said to be *complete* by Rezk if the map

$$1 \backslash X \longrightarrow J \backslash X$$

obtained from the map $J \to 1$ is a weak homotopy equivalence. The Segal space model structure on $\mathbf{S}^{(2)}$ admits a localisation in which the fibrant objects are the complete Segal spaces [Rez]. The localised model structure is called the *model structure for complete Segal spaces*. The category of complete Segal spaces is enriched over Kan complexes. Its coherent nerve is equivalent to the quasi-category $RCat(\mathbf{Hot})$.

29.4. Let $i_1 : \Delta \to \Delta \times \Delta$ be the functor defined by putting $i_1([n]) = ([n],[0])$ for every $n \geq 0$. If X is a bisimplicial set, then $i_1^*(X)$ is the first row of X. The functor i_1 is right adjoint to the first projection $p_1 : \Delta \times \Delta \to \Delta$. The pair of adjoint functors

$$p_1^* : \mathbf{S} \leftrightarrow \mathbf{S}^{(2)} : i_1^*.$$

is a Quillen equivalence between the model category for quasi-categories and the model category for complete Segal spaces [JT2]. The equivalence is not simplicial, since the model category for quasi-categories is not simplicial. The category $\mathbf{QCat}$ is enriched over Kan complexes if we put $Hom^J(X,Y) = J(Y^X)$ for $X, Y \in \mathbf{QCat}$. Let a, b and $c : \Delta \to \mathbf{S}^{(2)}$ be the functors defined by putting $a([n]) = \Delta[n]'\Box 1$, $b([n]) = \Delta[n]'\Box\Delta[n]$ and $c([n]) = 1\Box\Delta[n]$ for every $n \geq 0$. If X and Y are complete Segal spaces, let us put

$$Hom_a(X,Y) = a^!(Y^X),\ Hom_b(X,Y) = b^!(Y^X) \text{ and } Hom_c(X,Y) = c^!(Y^X),$$

where Y^X is the exponentiation of Y by X in the topos $\mathbf{S}^{(2)}$. This defines three simplicial enrichements of the category $\mathbf{CSS}$ of complete Segal spaces. The projections

$$Hom_a(X,Y) \longleftarrow Hom_b(X,Y) \longrightarrow Hom_c(X,Y)$$

obtained from the inclusions $a([n]) \subseteq b([n]) \supseteq c([n])$ are homotopy equivalences. Hence the three enrichements are equivalent in the Dwyer-Kan-Bergner model structure of $\mathbf{SCat}$ (we are neglecting the fact that the simplicial category $\mathbf{CSS}$ is big). The functor $i^* : \mathbf{CSS} \to \mathbf{QCat}$ induces a Dwyer-Kan equivalence

$$(\mathbf{CSS}, Hom_a) \to (\mathbf{QCat}, Hom^J).$$

It follows from these considerations that the coherent nerve of $\mathbf{CSS}$ is naturally equivalent to the quasi-category $\mathbf{Hot_1}$. By combining these results, we obtain an equivalence of quasi-categories

$$RCat(\mathbf{Hot}) \simeq \mathbf{Hot_1}.$$

29.5. Let X be an absolutly exact quasi-category. We say that a double category $C : \Delta^o \times \Delta^o \to X$ is *reduced* if it is a reduced category in each variable. There is a notion of reduced n-fold category for every $n \geq 0$. We shall denote by $RCat^n(X)$ the quasi-category of reduced n-fold categories in X. We have $RCat^{n+1}(X) = RCat(RCat^n(X))$. The inclusion $RCat^n(X) \subseteq Cat^n(X)$ has a left adjoint

$$R : Cat^n(X) \to RCat^n(X).$$

. We shall say that a n-category $C \in Cat_n(X)$ is *reduced* if it is reduced as a n-fold category. We shall denote by $RCat_n(X)$ the quasi-category of reduced n-categories in X. The inclusion $RCat_n(X) \subseteq Cat_n(X)$ has a left adjoint

$$R : Cat_n(X) \to RCat_n(X).$$

29.6. We say that a n-fold Segal space $C : \Delta^n \to \mathbf{S}$ is *reduced* if it is reduced in each variable. We shall denote by $\mathbf{RSS^n}$ the category of reduced n-fold Segal spaces. The model structure for n-fold Segal spaces admits a localisation in which the fibrant objects are the reduced n-fold Segal spaces. The coherent nerve of the simplicial category $\mathbf{RSS}^n$ is equivalent to the quasi-category $RCat^n(\mathbf{Hot})$. We say that a Segal n-space $C : \Delta^n \to \mathbf{S}$ is *reduced* if it is reduced as a n-fold Segal space. We shall denote by $\mathbf{RSS}_n$ the category of reduced Segal n-spaces. The model category of Segal n-spaces admits a localisation in which the fibrant objects are the reduced Segal n-spaces. The localised model structure is called the *model structure for reduced Segal n-spaces*. The category $\mathbf{RSS_n}$ is enriched over Kan complexes. Its coherent nerve is equivalent to the quasi-category $RCat_n(\mathbf{Hot})$.

29.7. For each $n \geq 0$, let us put

$$\mathbf{Hot}_n = RCat_n(\mathbf{Hot}).$$

We shall say that an object of $\mathbf{Hot}_n$ is a *quasi-n-category*. The quasi-category $\mathbf{Hot}_n$ is cartesian closed. If $n \leq m$, the inclusion $\mathbf{Hot}_n \subseteq \mathbf{Hot}_m$ is full and preserves exponentiation. It has a left adjoint λ_n and a right adjoint ρ_n.

29.8. The sequence of inclusion

$$\mathbf{Hot} = \mathbf{Hot}_0 \subset \mathbf{Hot}_1 \subset \mathbf{Hot}_2 \subset \cdots$$

has a colimit $\mathbf{Hot}_\omega$ in $\mathbf{LP}$. We shall say that an object of $\mathbf{Hot}_\omega$ is a *quasi-ω-category*. The quasi-category $\mathbf{Hot}_\omega$ can be constructed as the (homotopy) projective limit of the sequence of quasi-categories

$$\mathbf{Hot}_0 \xleftarrow{\rho_0} \mathbf{Hot}_1 \xleftarrow{\rho_1} \mathbf{Hot}_2 \xleftarrow{\rho_2} \mathbf{Hot}_3 \longleftarrow \cdots .$$

In other words, a quasi-ω-category is a sequence $x = (x_n)$ of objects $x_n \in \mathbf{Hot}_n$ connected by a sequence of maps

$$x_0 \to x_1 \to x_2 \to \cdots$$

such that $\rho_n(x_{n+1}) = x_n$ for every $n \geq 0$. Each inclusion $\mathbf{Hot}_n \subset \mathbf{Hot}_\omega$ has a left adjoint λ_n and a right adjoint ρ_n. The left adjoint λ_n associates to an object $x = (x_n) \in \mathbf{Hot}_\omega$ the colimit in $\mathbf{Hot}_n$ of the sequence

$$x_n \to \lambda_n x_{n+1} \to \lambda_n x_{n+2} \to \cdots .$$

Notice the contravariant sequence

$$\lambda_0(x) \leftarrow \lambda_1(x) \leftarrow \lambda_2(x) \leftarrow .$$

The quasi-category $\mathbf{Hot}_\omega$ is cartesian closed. If an object $x = (x_n) \in \mathbf{Hot}_\omega$ and $y = (y_n) \in \mathbf{Hot}_\omega$, then y^x is represented by the sequence

$$y_0^{\lambda_0 x} \to y_1^{\lambda_1 x} \to y_2^{\lambda_2 x} \to \cdots .$$

Each inclusion $\mathbf{Hot}_n \subset \mathbf{Hot}_\omega$ preserves exponentiation.

30. Theta-categories

30.1. We begin by recalling the duality between the category Δ and the category of intervals. An *interval* I is a linearly ordered set with a first and last elements respectively denoted 0 and 1, or $\bot$ and $\top$. If $0 \neq 1$ the interval is said to be *strict*. A *morphism* between two intervals is an order preserving map $f : I \to J$ such that $f(0) = 0$ and $f(1) = 1$. We shall denote by $\mathcal{D}_1$ the category of finite strict intervals (it is the category of finite 1-*disk*). The category $\mathcal{D}_1$ is dual to the category Δ. The duality functor $(-)^* : \Delta^o \to \mathcal{D}_1$ associates to $[n]$ the set $[n]^* = \Delta([n], [1]) = [n+1]$ equipped with the pointwise ordering. The inverse functor $\mathcal{D}_1^o \to \Delta$ associates to an interval $I \in D_1$ the set $I^* = \mathcal{D}_1(I, [1])$ equipped with the pointwise ordering. A simplicial set is usually defined to be a contravarint functors $\Delta^o \to \mathbf{Set}$. The duality implies that it can be defined as a covariant functor $\mathcal{D}_1 \to \mathbf{Set}$. We can extends the notion of simplicial set by extending the notion of interval with the notion of n-disk.

30.2. The euclidian ball $B^n = \{x \in \mathbf{R}^n :\| x \|\leq 1\}$ is the main geometric example of an n-disk. Observe that the fiber $p^{-1}(x)$ of the projection $p = p_1 : B^n \to [-1, 1]$ is an $(n-1)$-disks except when $x = \pm 1$ in which case $p^{-1}(x)$ is a point. There is a complementary view with the projection $q : B^n \to B^{n-1}$. The fiber $q^{-1}(x)$ is a 1-disk except when $x \in \partial B^{n-1}$ in which case $q^{-1}(x)$ is a point; there are two canonical sections $i_0, i_1 : B^{n-1} \to B^n$ obtained by selecting the bottom and the top elements in each fiber; the image of i_0 is the bottom hemisphere and the image of i_1 the top hemisphere; observe that $i_0(x) = i_1(x)$ iff $x \in \partial B^{n-1}$. The map $q : B^n \to B^{n-1}$ is an example of bundle of intervals. In general, a *bundle of intervals* over a set B is an interval object in the category $\mathbf{Set}/B$. More explicitly, it is a map $p : X \to B$ whose fibers have an interval structure. The map p has two canonical sections $i_0, i_1 : B \to E$ obtained by selecting the bottom and the top elements in each fiber. We shall say that the equaliser of i_0 and i_1 is the *singular set* of the bundle. If we order the coordinates in R^n we obtain a sequence of bundles of intervals:

$$1 \leftarrow B^1 \leftarrow B^2 \leftarrow \cdots B^{n-1} \leftarrow B^n.$$

Observe that $\partial B^{k+1} = i_0(B^k) \cup i_1(B^k)$.

30.3. A *n-disk* D is defined to be a sequence of length n of bundles of intervals

$$1 = D_0 \leftarrow D_1 \leftarrow D_2 \leftarrow \cdots D_{n-1} \leftarrow D_n$$

in which the singular set of the projection $p : D_{k+1} \to D_k$ is equal to $i_0(D_{k-1}) \cup i_1(D_{k-1})$ for every $0 \leq k < n$. If $k = 0$ this condition means that D_1 is a strict interval.

30.4. It follows from the definition that $i_0 i_0 = i_1 i_0$ and $i_0 i_1 = i_1 i_1$. We define the *boundary* ∂D_k to be $i_0(D_{k-1}) \cup i_1(D_{k-1})$ and the *interior* $int(D_k)$ to be $D_k \backslash \partial D_k$. By convention $\partial D_0 = \emptyset$. A *planar tree* T of height $\leq n$ is defined to be a sequence of maps

$$1 = T_0 \leftarrow T_1 \leftarrow T_2 \leftarrow \cdots \leftarrow T_n$$

with linearly ordered fibers. The interior of a n-disk D is a planar tree of height $\leq n$,

$$1 = \leftarrow int(D_1) \leftarrow int(D_2) \leftarrow \cdots int(D_{n-1}) \leftarrow int(D_n).$$

Every planar tree T of height $\leq n$ is the interior of a unique n-disk $\bar{T}$. The *size* $\mid D \mid$ of a disk D is defined to be the number of edges of the tree $int(D)$. We have

$$\mid D \mid = \sum_{k=1}^{n} \mathrm{Card}(int(D_k)).$$

30.5. A *morphism* $D \to D'$ between n-disks is defined to be a sequence of maps $f_n : S_n \to T_n$ commuting with the projections

$$\begin{array}{ccccccccc} 1 & \longleftarrow & D_1 & \longleftarrow & D_2 & \longleftarrow \cdots & D_{n-1} & \longleftarrow & D_n \\ & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_{n-1}} & & \downarrow{\scriptstyle f_n} \\ 1 & \longleftarrow & D'_1 & \longleftarrow & D'_2 & \longleftarrow \cdots & D'_{n-1} & \longleftarrow & D'_n \end{array}$$

and inducing a map of intervals between the fibers. We shall denote by $\mathcal{D}_n$ the category of finite n-disks. Let $\mathcal{B}^n$ be the euclidian n-disk,

$$1 \leftarrow B^1 \leftarrow B^2 \leftarrow \cdots B^{n-1} \leftarrow B^n.$$

If $D \in \mathcal{D}_n$, then the set $hom(D, \mathcal{B}^n)$ is an euclidian ball of dimension $\mid D \mid$.

30.6. The following neat description of $hom(D, \mathcal{B}^n)$ is due to Clemens Berger. The order relation on fibers of the planar tree $T = int(D)$ can be transported to the edges. The topological space $hom(D, \mathcal{B}^n)$ is homeomorphic to the space of maps $f : edges(T) \to [-1, 1]$ satisfying the following conditions

- $\sum_{e \in C} f(e)^2 \leq 1$ for every maximal chain C connecting a leaf to the root;
- $f(e) \leq f(e')$ for two edges $e \leq e'$ with the same target.

We associate to f a map of n-disks $f' : D \to \mathcal{B}^n$ by putting $f'(x) = (f(e_1), \cdots, f(e_k))$ for every $x \in T_k$, where $(e_1, \cdots, e_k)$ is the sequence of edges in chain connecting the root to the vertex x.

30.7. The category Θ_n is by definition the opposite of $\mathcal{D}_n$. We shall write $D = C^\star$ and $C = {}^*D$ for an object $C \in \Theta_n$ and the opposite disk $D \in \mathcal{D}_n$. We say that an object $C \in \Theta_n$ is a Θ_n-*cell*. The *dimension* of C is defined to be $\mid C^\star \mid$. A Θ_n-*set* is defined to be a functor

$$X : \Theta_n^o \to \mathbf{Set},$$

or equivalently a functor $X : \mathcal{D}_n \to \mathbf{Set}$. We shall denote by $\hat{\Theta}_n$ the category of Θ_n-sets of order n. We shall use the Yoneda functor $\Theta_n \to \hat{\Theta}_n$ to identify Θ_n with a full subcategory of $\hat{\Theta}_n$. Consider the functor $R : \Theta_n \to \mathbf{Top}$ defined by putting $R(C) = Hom(C^*, \mathcal{B}^n)$, where **Top** denotes the category of compactly generated spaces. Its left Kan extension $R : \hat{\Theta}_n \to \mathbf{Top}$ preserves finite limits. We call $R(X)$ the *geometric realisation* of a Θ_n-set X.

30.8. For each $0 \leq k \leq n$, let us denote by e_k the n-disk whose interior is the tree with a single chain of k edges. The geometric realisation of the cell $b^k = {}^*e_k$ is the euclidian n-ball. There is a unique map of disks $e_{k-1} \to e_k$, hence also a unique map of cells $b^k \to b^{k-1}$. The sequence

$$1 = b^0 \leftarrow b^1 \leftarrow b^2 \leftarrow \cdots b^n$$

has the structure of a n-disk b in the topos $\hat{\Theta}_n$. It is the generic n-disk in the sense of classifying topos.

30.9. The *composite* $D \circ E$ of a n-disk D with a m-disk E is the $m+n$ disk

$$1 = D_0 \leftarrow D_1 \leftarrow \cdots \leftarrow D_n \leftarrow (D_n, \partial D_n) \times E_1 \leftarrow \cdots \leftarrow (D_n, \partial D_n) \times E_m,$$

where $(D_n, \partial D_n) \times E_k$ is defined by the pushout square

$$\begin{array}{ccc} \partial D_n \times E_k & \longrightarrow & D_n \times E_k \\ \downarrow & & \downarrow \\ E_k & \longrightarrow & (D_n, \partial D_n) \times E_k. \end{array}$$

This composition operation is associative.

30.10. The category $\mathbf{S}^{(n)} = [(\Delta^n)^o, \mathbf{Set}]$, contains n intervals

$$I_k = 1 \Box 1 \Box \cdots 1 \Box I \Box 1 \cdots 1 \Box 1,$$

one for each $0 \leq k \leq n$. It thus contain a n-disk $I^{(n)} : I_1 \circ I_2 \circ \cdots \circ I_n$. Hence there is a geometric morphism

$$(\rho^*, \rho_*) : \mathbf{S}^{(n)} \to \hat{\Theta},$$

such that $\rho^*(b) = I^{(n)}$. We shall say that a map of Θ_n-sets $f : X \to Y$ is a *weak categorical equivalence* if the map $\rho^*(f) : \rho^*(X) \to \rho^*(Y)$ is a weak equivalence in the model structure for reduced Segal n-spaces. The category $\hat{\Theta}_n$ admits a model structure in which the weak equivalences are the weak categorical equivalences and the cofibrations are the monomorphisms. We shall say that a fibrant object is a *Θ_n-category*. The model structure is cartesian closed and left proper. We call it the *model structure for Θ_n-categories*. We denote by $\Theta_n\mathbf{Cat}$ the category of Θ_n-categories. The pair of adjoint functors

$$\rho^* : \hat{\Theta}_n \to \mathbf{S}^{(n)} : \rho_*$$

is a Quillen equivalence between the model structure for Θ_n-categories and the model structure for reduced Segal n-spaces.

31. Appendix

31.1. We fix some notations about simplicial sets. We shall denote by Δ the category of finite non-empty ordinals and order preserving maps. It is standard to denote the ordinal $n+1 = \{0, \ldots, n\}$ by $[n]$. A map $u : [m] \to [n]$ can be specified by listing its values $(u(0), \ldots, u(m))$. We shall denote by $d_i : [n-1] \to [n]$ the injection which omits $i \in [n]$ and by $s_i : [n] \to [n-1]$ the surjection which repeats $i \in [n-1]$.

31.2. We shall denote by $\mathbf{S}$ the category $[\Delta^o, \mathbf{Set}]$ of simplicial sets. If X is a simplicial set, it is standard to denote $X([n])$ by X_n. We often denote the map $X(d_i) : X_n \to X_{n-1}$ by ∂_i and the map $X(s_i) : X_{n-1} \to X_n$ by σ_i. An element of X_n is called a *n-simplex*; a 0-simplex is called a *vertex* and a 1-simplex an *arrow*. For each $n \geq 0$, the simplicial set $\Delta(-, [n])$ is called the *combinatorial simplex* of dimension n and denoted by $\Delta[n]$. The simplex $\Delta[1]$ is called the *combinatorial interval* and we shall denote it by I. The simplex $\Delta[0]$ is the terminal object of the category $\mathbf{S}$ and we shall denote it by 1. By the Yoneda lemma, for every $X \in \mathbf{S}$ the evaluation map $x \mapsto x(1_{[n]})$ defines a bijection between the maps $\Delta[n] \to X$ and the elements of X_n for each $n \geq 0$; we shall identify these two sets by adopting the same notation for a map $x : \Delta[n] \to X$ and the simplex $x(1_{[n]}) \in X_n$. If $u : [m] \to [n]$ we shall denote the simplex $X(u)(x) \in X_m$ as a composite $xu : \Delta[m] \to X$. If $n > 0$ and $x \in X_n$ the simplex $\partial_i(x) = xd_i : \Delta[n-1] \to X$ is called the *i-th face* of x. If $f \in X_1$ we shall say that the vertex $a = \partial_1(f) = fd_1$ is the *source* of the arrow f and that $b = \partial_0(f) = fd_0$ is its *target*. We shall write $f : a \to b$ to indicate that $a = \partial_1(f)$ and that $b = \partial_0(f)$. If $a \in X_0$, we shall denote the (degenerate) arrow as_0 as a *unit* $1_a : a \to a$.

31.3. Let $\tau : \Delta \to \Delta$ be the automorphism of the category Δ which reverses the order of each ordinal. If $u : [m] \to [n]$ is a map in Δ, then $\tau(u)$ is the map $u^o : [m] \to [n]$ given by $u^o(i) = n - f(m-i)$. The *opposite* X^o is a simplicial set X is obtained by composing the (contravariant) functor $X : \Delta \to \mathbf{Set}$ with the functor τ. We shall distinguish between the simplicies of X and X^o by writing $x^o \in X^o$ for each $x \in X$, with the convention that $x^{oo} = x$. If $f : a \to b$ is an arrow in X, then $f^o : b^o \to a^o$ is an arrow in X^o.

31.4. If X is a simplicial set, we say that a subfunctor $A \subseteq X$ is a *simplicial subset* of X. If $n > 0$ and $i \in [n]$ the image of the map $d_i : \Delta[n-1] \to \Delta[n]$ is denoted $\partial_i\Delta[n] \subset \Delta[n]$. The *simplicial sphere* $\partial\Delta[n] \subset \Delta[n]$ is the union the faces $\partial_i\Delta[n]$ for $i \in [n]$; by convention $\partial\Delta[0] = \emptyset$. If $n > 0$, we shall say that a map $x : \partial\Delta[n] \to X$ is a *simplicial sphere in* X; such a map is determined by the sequence of its faces $(x_0, \ldots, x_n) = (xd_0, \ldots, xd_n)$. A simplicial sphere $\partial\Delta[2] \to X$ is called a *triangle*. Every n-simplex $y : \Delta[n] \to X$ has a *boundary* $\partial y = (\partial_0 y, \ldots, \partial_n y) = (yd_0, \ldots, yd_n)$ obtained by restricting y to $\partial\Delta[n]$. If $\partial y = x$ we shall say that the simplex y *fills* the simplicial sphere x. We shall say that a simplicial sphere $x : \partial\Delta[n] \to X$ *commutes* if it can be filled.

31.5. If $n > 0$ and $k \in [n]$, the *horn* $\Lambda^k[n] \subset \Delta[n]$ is defined to be the union of the faces $\partial_i\Delta[n]$ with $i \neq k$. A map $x : \Lambda^k[n] \to X$ is called a *horn in* X; it is determined by a lacunary sequence of faces $(x_0, \ldots, x_{k-1}, *, x_{k+1}, \ldots, x_n)$. A *filler* for x is a simplex $\Delta[n] \to X$ which extends x.

31.6. A pair $(\mathcal{A}, \mathcal{B})$ of classes of maps in a category $\mathcal{E}$ is called a *factorisation system* if the following two conditions are satisfied:

- the classes $\mathcal{A}$ and $\mathcal{B}$ are closed under composition and contain the isomorphisms;
- every map $f : A \to B$ admits a factorisation $f = pu : A \to E \to B$ with $u \in \mathcal{A}$ and $p \in \mathcal{B}$, and this factorisation is unique up to unique isomorphism.

In this definition, the uniqueness of the factorisation of a map $f : A \to B$ means that for any other factorisation $f = qv : A \to F \to B$ with $v \in \mathcal{A}$ and $q \in \mathcal{B}$, there exists a unique isomorphism $i : E \to F$ such that $iu = v$ and $qi = p$.

31.7. The left class $\mathcal{A}$ of a factorisation system has the right cancellation property and the right class has the left cancellation property. Let us define these notions. We say that a class of maps $\mathcal{M}$ in a category $\mathcal{E}$ has the *right cancellation property* if the implication

$$vu \in \mathcal{M} \text{ and } u \in \mathcal{M} \ \Rightarrow \ v \in \mathcal{M}$$

is true for any pair of maps $u : A \to B$ and $v : B \to C$. Dually, we say that $\mathcal{M}$ has the *left cancellation property* if the implication

$$vu \in \mathcal{M} \text{ and } v \in \mathcal{M} \ \Rightarrow \ u \in \mathcal{M}$$

is true.

31.8. Recall that an arrow $u : A \to B$ in a category $\mathcal{E}$ is said to have the *left lifting property* with respect to another arrow $f : X \to Y$, or that f has the *right lifting property* with respect to u, if every commutative square

$$\begin{array}{ccc} A & \xrightarrow{x} & X \\ {\scriptstyle u}\downarrow & \nearrow & \downarrow{\scriptstyle f} \\ B & \xrightarrow[y]{} & Y \end{array}$$

has a diagonal filler $d : B \to X$ (that is, $du = x$ and $fd = y$). We shall denote this relation by $u \pitchfork f$. If the diagonal filler is unique we shall write $u \perp f$ and say that u is *left orthogonal* to f, ot that f is *right orthogonal* to u. If $\mathcal{A}$ and $\mathcal{B}$ are two classes of maps in $\mathcal{E}$, we shall write $\mathcal{A} \pitchfork \mathcal{B}$ to indicate that we have $u \pitchfork f$ for every $u \in \mathcal{A}$ and $f \in \mathcal{B}$. For any class of maps $\mathcal{M} \subseteq \mathcal{E}$, we shall denote by ${}^{\pitchfork}\mathcal{M}$ (resp. $\mathcal{M}^{\pitchfork}$) the class of maps having the left lifting property (resp. right lifting property) with respect to every map in $\mathcal{M}$. Each class ${}^{\pitchfork}\mathcal{M}$ and $\mathcal{M}^{\pitchfork}$ contains the isomorphisms and is closed under composition.

31.9. A pair $(\mathcal{A}, \mathcal{B})$ of classes of maps in a category $\mathcal{E}$ is called a *weak factorisation system* if the following two conditions are satisfied:

- every map $f \in \mathcal{E}$ admits a factorisation $f = pu$ with $u \in \mathcal{A}$ and $p \in \mathcal{B}$;
- $\mathcal{A} = {}^{\pitchfork}\mathcal{B}$ and $\mathcal{A}^{\pitchfork} = \mathcal{B}$.

We say that $\mathcal{A}$ is the *left class* and that $\mathcal{B}$ is the *right class* of the weak factorisation system. The intersection $\mathcal{A} \cap \mathcal{B}$ is the class of isomorphisms.

Every factorisation system is a weak factorisation system.

31.10. We shall say that a map in a topos is a trivial fibration if it has the right lifting property with respect to every monomorphism. This terminology is non-standard but useful. The trivial fibrations often coincide with the acyclic fibrations (which can be defined in any model category). A map of simplicial sets is a trivial fibration iff it has the right lifting property with respect to the inclusion $\delta_n : \partial\Delta[n] \subset \Delta[n]$ for every $n \geq 0$. If $\mathcal{A}$ be the class monomorphisms in a topos and $\mathcal{B}$ is the class of trivial fibrations, then the pair $(\mathcal{A}, \mathcal{B})$ is a weak factorisation system. An object X in a topos is said to be *injective* if the map $X \to 1$ is a trivial fibration.

31.11. Let $\mathcal{E}$ be a cocomplete category. If $\alpha = \{i : i < \alpha\}$ is a non-zero ordinal, we shall say that a functor $C : \alpha \to \mathcal{E}$ is an α-*chain* if the canonical map

$$\varinjlim_{i<j} C(i) \to C(j)$$

is an isomorphism for every non-zero limit ordinal $j < \alpha$. The *composite* of C is the canonical map

$$C(0) \to \varinjlim_{i<\alpha} C(i).$$

We shall say that a subcategory $\mathcal{A} \subseteq \mathcal{E}$ is closed under *transfinite composition* if the composite of any α-chain $C : \alpha \to \mathcal{E}$ with values in $\mathcal{A}$ belongs to $\mathcal{A}$.

31.12. Let $\mathcal{E}$ be a cocomplete category. We shall say that a class of maps $\mathcal{A} \subseteq \mathcal{E}$ is *saturated* if it satisfies the following conditions:

- $\mathcal{A}$ contains the isomorphisms and is closed under composition ;
- $\mathcal{A}$ is closed under transfinite composition;
- $\mathcal{A}$ is closed under cobase change and retract;

Every class of maps $\Sigma \subseteq \mathcal{E}$ is contained in a smallest saturated class called the saturated class *generated* by Σ.

31.13. The following theorem is a special case of a classical result. If Σ is a set of maps in a locally presentable category, then the pair $(\overline{\Sigma}, \Sigma^{\pitchfork})$ is a weak factorisation system, where $\overline{\Sigma}$ denotes the saturated class generated by Σ.

31.14. We shall say that a class $\mathcal{W}$ of maps in a category $\mathcal{E}$ has the *"three for two"* property if the following condition is satisfied:

- If two of three maps $u : A \to B$, $v : B \to C$ and $vu : A \to C$ belongs to $\mathcal{W}$, then so does the third.

31.15. Let $\mathcal{E}$ be a finitely bicomplete category. We shall say that a triple $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ of classes of maps in $\mathcal{E}$ is a *model structure* if the following conditions are satisfied:

- $\mathcal{W}$ has the "three for two" property;
- the pairs $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ are weak factorisation systems.

31.16. A *model category* is a category $\mathcal{E}$ equipped with a model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$. The class $\mathcal{W}$ of a model structure contains the isomorphisms and it is closed under retracts, see [JT1] for a proof. We shall say that a model structure is *trivial* if $\mathcal{W}$ is the class of isomorphisms. A map in $\mathcal{W}$ is said to be *acyclic* or to be a *weak equivalence*. A map in $\mathcal{C}$ is called a *cofibration* and a map in $\mathcal{F}$ a *fibration*. A map in $\mathcal{C} \cap \mathcal{W}$ is called an *acyclic cofibration* and a map in $\mathcal{F} \cap \mathcal{W}$ an *acyclic fibration*. An object $X \in \mathcal{E}$ is *fibrant* if the map $X \to \top$ is a fibration, where $\top$ is the terminal object of $\mathcal{E}$. Dually, an object $A \in \mathcal{E}$ is *cofibrant* if the map $\perp \to A$ is a cofibration, where $\perp$ is the initial object of $\mathcal{E}$.

31.17. A model structure is said to be *left proper* if the cobase change of a weak equivalence along a cofibration is a weak equivalence. Dually, a model structure is said to be *right proper* if the base change of a weak equivalence along a fibration is a weak equivalence. A model structure is *proper* if it is both left and right proper.

31.18. If $\mathcal{E}$ is a model category, then so is the slice category $\mathcal{E}/B$ for each object $B \in \mathcal{E}$. By definition, a map in $\mathcal{E}/B$ is a weak equivalence (resp. a cofibration , resp. a fibration) iff the underlying map in $\mathcal{E}$ is a weak equivalence (resp. a cofibration , resp. a fibration). Dually, each category $B\backslash\mathcal{E}$ is a model category.

31.19. The *homotopy category* of a model category $\mathcal{E}$ is defined to be the category of fractions $Ho(\mathcal{E}) = \mathcal{W}^{-1}\mathcal{E}$. We shall denote by $[u]$ the image of a map $u \in \mathcal{E}$ by the canonical functor $\mathcal{E} \to Ho(\mathcal{E})$. A map $u : A \to B$ is a weak equivalence iff $[u]$ invertible in $Ho(\mathcal{E})$ by [Q].

31.20. We shall denote by $\mathcal{E}_f$ (resp. $\mathcal{E}_c$) the full sub-category of fibrant (resp. cofibrant) objects of a model category $\mathcal{E}$. We shall put $\mathcal{E}_{fc} = \mathcal{E}_f \cap \mathcal{E}_c$. A *fibrant replacement* of an object $X \in \mathcal{E}$ is a weak equivalence $X \to RX$ with codomain a fibrant object. Dually, a *cofibrant replacement* of X is a weak equivalence $LX \to X$ with domain a cofibrant object. Let us put $Ho(\mathcal{E}_f) = \mathcal{W}_f^{-1}\mathcal{E}_f$ where $\mathcal{W}_f = \mathcal{W} \cap \mathcal{E}_f$ and similarly for $Ho(\mathcal{E}_c)$ and $Ho(\mathcal{E}_{fc})$. Then the diagram of inclusions

$$\begin{array}{ccc} \mathcal{E}_{fc} & \longrightarrow & \mathcal{E}_f \\ \downarrow & & \downarrow \\ \mathcal{E}_c & \longrightarrow & \mathcal{E} \end{array}$$

induces a diagram of equivalences of categories

$$\begin{array}{ccc} Ho(\mathcal{E}_{fc}) & \longrightarrow & Ho(\mathcal{E}_f) \\ \downarrow & & \downarrow \\ Ho(\mathcal{E}_c) & \longrightarrow & Ho(\mathcal{E}). \end{array}$$

31.21. Recall from [Ho] that a cocontinuous functor $F : \mathcal{U} \to \mathcal{V}$ between two model categories is said to be a *left Quillen functor* if it takes a cofibration to a cofibration and an acyclic cofibration to an acyclic cofibration. A left Quillen functor takes a weak equivalence between cofibrant objects to a weak equivalence. Dually, a continuous functor $G : \mathcal{V} \to \mathcal{U}$ between two model categories is said to be a *right Quillen functor* if it takes a fibration to a fibration and an acyclic fibration to an acyclic fibration. A right Quillen functor takes a weak equivalence between fibrant objects to a weak equivalence.

31.22. A left Quillen functor $F : \mathcal{U} \to \mathcal{V}$ induces a functor $F_c : \mathcal{U}_c \to \mathcal{V}_c$ hence also a functor $Ho(F_c) : Ho(\mathcal{U}_c) \to Ho(\mathcal{V}_c)$. Its *left derived functor* is a functor

$$F^L : Ho(\mathcal{U}) \to Ho(\mathcal{V})$$

for which the following diagram of functors commutes up to isomorphism,

$$\begin{array}{ccc} Ho(\mathcal{U}_c) & \xrightarrow{Ho(F_c)} & Ho(\mathcal{V}_c) \\ \downarrow & & \downarrow \\ Ho(\mathcal{U}) & \xrightarrow{F^L} & Ho(\mathcal{V}), \end{array}$$

The functor F^L is unique up to a canonical isomorphism. It can be computed as follows. For each object $A \in \mathcal{U}$, we can choose a cofibrant replacement $\lambda_A : LA \to A$, with λ_A an acyclic fibration. We can then choose for each arrow $u : A \to B$ an arrow $L(u) : LA \to LB$ such that $u\lambda_A = \lambda_B L(u)$,

$$\begin{array}{ccc} LA & \xrightarrow{\lambda_A} & A \\ {\scriptstyle L(u)}\downarrow & & \downarrow{\scriptstyle u} \\ LB & \xrightarrow{\lambda_B} & B. \end{array}$$

Then

$$F^L([u]) = [F(L(u))] : FLA \to FLB.$$

31.23. Dually, a right Quillen functor $G : \mathcal{V} \to \mathcal{U}$ induces a functor $G_f : \mathcal{V}_f \to \mathcal{U}_f$ hence also a functor $Ho(G_f) : Ho(\mathcal{V}_f) \to Ho(\mathcal{U}_f)$. Its *right derived functor* is a functor

$$G^R : Ho(\mathcal{V}) \to Ho(\mathcal{U})$$

for which the following diagram of functors commutes up to a canonical isomorphism,

$$\begin{array}{ccc} Ho(\mathcal{V}_f) & \xrightarrow{Ho(G_f)} & Ho(\mathcal{U}_f) \\ \downarrow & & \downarrow \\ Ho(\mathcal{V}) & \xrightarrow{G^R} & Ho(\mathcal{U}). \end{array}$$

The functor G^R is unique up to a canonical isomorphism. It can be computed as follows. For each object $X \in \mathcal{V}$ let us choose a fibrant replacement $\rho_X : X \to RX$, with ρ_X an acyclic cofibration. We can then choose for each arrow $u : X \to Y$ an arrow $R(u) : RX \to RY$ such that $R(u)\rho_X = \rho_Y u$,

$$\begin{array}{ccc} X & \xrightarrow{\rho_X} & RX \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle R(u)} \\ Y & \xrightarrow{\rho_Y} & RY. \end{array}$$

Then

$$G^R([u]) = [G(R(u))] : GRX \to GRY.$$

31.24. Let $F : \mathcal{U} \leftrightarrow \mathcal{V} : G$ be an adjoint pair of functors between two model categories. Then the following two conditions are equivalent:

- F is a left Quillen functor;
- G is a right Quillen functor.

When these conditions are satisfied, the pair (F, G) is said to be a *Quillen pair*. In this case, we obtain an adjoint pair of functors

$$F^L : Ho(\mathcal{U}) \leftrightarrow Ho(\mathcal{V}) : G^R.$$

If $A \in \mathcal{U}$ is cofibrant, the adjunction unit $A \to G^R F^L(A)$ is obtained by composing the maps $A \to GFA \to GRFA$, where $FA \to RFA$ is a fibrant replacement of FA. If $X \in \mathcal{V}$ is fibrant, the adjunction counit $F^L G^R(X) \to X$ is obtained by composing the maps $FLGX \to FGX \to X$, where $LGX \to GX$ is a cofibrant replacement of GX.

31.25. A Quillen pair (F, G) is said to be a *Quillen equivalence* if the adjoint pair (F^L, G^R) is an equivalence of categories.

31.26. We shall say that a Quillen pair $F : \mathcal{U} \leftrightarrow \mathcal{V} : G$ is a *homotopy localisation* $\mathcal{U} \to \mathcal{V}$ if the right derived functor G^R is full and faithful. Dually, we shall say that (F, G) is a *homotopy colocalisation* $\mathcal{V} \to \mathcal{U}$ if the left derived functor F^L is full and faithful.

31.27. Let $F : \mathcal{U} \leftrightarrow \mathcal{V} : G$ be a homotopy localisation beween two model categories. We shall say that an object $X \in \mathcal{U}$ is *local* (with respect to the the pair (F, G)) if it belongs to the essential image of the right derived functor $G^R : Ho(\mathcal{V}) \to Ho(\mathcal{U})$.

31.28. Let $\mathcal{M}_i = (\mathcal{C}_i, \mathcal{W}_i, \mathcal{F}_i)$ $(i = 1, 2)$ be two model structures on a category $\mathcal{E}$. If $\mathcal{C}_1 = \mathcal{C}_2$ and $\mathcal{W}_1 \subseteq \mathcal{W}_2$, we shall say that the model structure $\mathcal{M}_2$ is a *Bousfield localisation* of the model structure $\mathcal{M}_1$.

31.29. Let $\odot : \mathcal{E}_1 \times \mathcal{E}_2 \to \mathcal{E}_3$ be a functor of two variables with values in a finitely cocomplete category $\mathcal{E}_3$. If $u : A \to B$ is map in $\mathcal{E}_1$ and $v : S \to T$ is a map in $\mathcal{E}_2$, we shall denote by $u \odot' v$ the map

$$A \odot T \sqcup_{A \odot S} B \odot S \longrightarrow B \odot T$$

obtained from the commutative square

$$\begin{array}{ccc} A \odot S & \longrightarrow & B \odot S \\ \downarrow & & \downarrow \\ A \odot T & \longrightarrow & B \odot T. \end{array}$$

This defines a functor of two variables

$$\odot' : \mathcal{E}_1^I \times \mathcal{E}_2^I \to \mathcal{E}_3^I,$$

where $\mathcal{E}^I$ denotes the category of arrows of a category $\mathcal{E}$.

31.30. [Ho] We shall say that a functor of two variables $\odot : \mathcal{E}_1 \times \mathcal{E}_2 \to \mathcal{E}_3$ between three model categories is a *left Quillen functor* if it is concontinuous in each variable and the following conditions are satisfied:

- $u \odot' v$ is a cofibration if $u \in \mathcal{E}_1$ and $v \in \mathcal{E}_2$ are cofibrations;
- $u \odot' v$ is an acyclic cofibration if $u \in \mathcal{E}_1$ and $v \in \mathcal{E}_2$ are cofibrations and one of the maps u or v is acyclic.

Dually, we shall say that the functor of two variables $\odot$ is a *right Quillen functor* if the opposite functor $\odot^o : \mathcal{E}_1^o \times \mathcal{E}_2^o \to \mathcal{E}_3^o$ is a left Quillen functor.

31.31. [Ho] A model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ on monoidal closed category $\mathcal{E} = (\mathcal{E}, \otimes)$ is said to be *monoidal* if the tensor product $\otimes : \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ is a left Quillen functor of two variables and if the unit object of the tensor product is cofibrant.

31.32. A model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ on a category $\mathcal{E}$ is said to be *cartesian* if the cartesian product $\times : \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ is a left Quillen functor of two variables and if the terminal object 1 is cofibrant.

31.33. We say that a functor of two variables $\odot : \mathcal{E}_1 \times \mathcal{E}_2 \to \mathcal{E}_3$ is *divisible on the left* if the functor $A \odot (-) : \mathcal{E}_2 \to \mathcal{E}_3$ admits a right adjoint $A \backslash (-) : \mathcal{E}_3 \to \mathcal{E}_2$ for every object $A \in \mathcal{E}_1$. In this case we obtain a functor of two variables $(A, X) \mapsto A \backslash X$,

$$\mathcal{E}_1^o \times \mathcal{E}_3 \to \mathcal{E}_2,$$

called the *left division functor*. Dually, we say that $\odot$ is *divisible on the right* if the functor $(-) \odot B : \mathcal{E}_1 \to \mathcal{E}_3$ admits a right adjoint $(-)/B : \mathcal{E}_3 \to \mathcal{E}_1$ for every object $B \in \mathcal{E}_2$. In this case we obtain a functor of two variables $(X, B) \mapsto X/B$,

$$\mathcal{E}_3 \times \mathcal{E}_2^o \to \mathcal{E}_1,$$

called the *right division functor*.

31.34. If a functor of two variables $\odot : \mathcal{E}_1 \times \mathcal{E}_2 \to \mathcal{E}_3$ is divisible on both sides, then so are the left division functor $\mathcal{E}_1^o \times \mathcal{E}_3 \to \mathcal{E}_2$ and the right division functor $\mathcal{E}_3 \times \mathcal{E}_2^o \to \mathcal{E}_1$. This is called a tensor-hom-cotensor situation in **??**. There is then a bijection between the following three kinds of maps

$$A \odot B \to X, \qquad B \to A\backslash X, \qquad A \to X/B.$$

Hence the contravariant functors $A \mapsto A\backslash X$ and $B \mapsto B\backslash X$ are mutually right adjoint.

31.35. Suppose the category $\mathcal{E}_2$ is finitely complete and that the functor $\odot : \mathcal{E}_1 \times \mathcal{E}_2 \to \mathcal{E}_3$ is divisible on the left. If $u : A \to B$ is map in $\mathcal{E}_1$ and $f : X \to Y$ is a map in $\mathcal{E}_3$, we denote by $\langle u\backslash\ f\rangle$ the map

$$B\backslash X \to B\backslash Y \times_{A\backslash Y} A\backslash X$$

obtained from the commutative square

$$\begin{array}{ccc} B\backslash X & \longrightarrow & A\backslash X \\ \downarrow & & \downarrow \\ B\backslash Y & \longrightarrow & A\backslash Y. \end{array}$$

The functor $f \mapsto \langle u\backslash f\rangle$ is right adjoint to the functor $v \mapsto u \odot' v$ for every map $u \in \mathcal{E}_1$. Dually, suppose that the category $\mathcal{E}_1$ is finitely complete and that the functor $\odot$ is divisible on the right. If $v : S \to T$ is map in $\mathcal{E}_2$ and $f : X \to Y$ is a map in $\mathcal{E}_3$, we denote by $\langle f/v\rangle$ the map

$$X/T \to Y/T \times_{Y/S} X/S$$

obtained from the commutative square

$$\begin{array}{ccc} X/T & \longrightarrow & X/S \\ \downarrow & & \downarrow \\ Y/T & \longrightarrow & Y/S. \end{array}$$

the functor $f \mapsto \langle f/v\rangle$ is right adjoint to the functor $u \mapsto u \odot' v$ for every map $v \in \mathcal{E}_2$.

31.36. Let $\odot : \mathcal{E}_1 \times \mathcal{E}_2 \to \mathcal{E}_3$ be a functor of two variables divisible on both sides, where $\mathcal{E}_i$ is a finitely bicomplete category for $i = 1, 2, 3$. If $u \in \mathcal{E}_1$, $v \in \mathcal{E}_2$ and $f \in \mathcal{E}_3$, then

$$(u \odot' v) \pitchfork f \quad \Longleftrightarrow \quad u \pitchfork \langle f/v\rangle \quad \Longleftrightarrow \quad v \pitchfork \langle u\backslash f\rangle.$$

31.37. Let $\odot : \mathcal{E}_1 \times \mathcal{E}_2 \to \mathcal{E}_3$ be a functor of two variables divisible on each side between three model categories. Then the functor $\odot$ is a left Quillen functor iff the corresponding left division functor $\mathcal{E}_1^o \times \mathcal{E}_3 \to \mathcal{E}_2$ is a right Quillen functor iff the the corresponding right division functor $\mathcal{E}_1^o \times \mathcal{E}_3 \to \mathcal{E}_2$ is a right Quillen functor.

31.38. Let $\mathcal{E}$ be a *symmetric* monoidal closed category. Then the objects X/A and $A\backslash X$ are canonicaly isomorphic; we can identify them by adopting a common notation, for example $[A, X]$. Similarly, the maps $\langle f/u\rangle$ and $\langle u\backslash f\rangle$ are canonicaly isomorphic; we shall identify them by adopting a common notation, for example $\langle u, f\rangle$. A model structure on $\mathcal{E}$ is monoidal iff the following two conditions are satisfied:

- if u is a cofibration and f is a fibration, then $\langle u, f\rangle$ is a fibration which is acyclic if in addition u or f is acyclic;
- the unit object is cofibrant.

31.39. This is it.

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