Ceresa cycles of Fermat curves and Hodge theory of fundamental groups
Pre-talk

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Our goal in these slides is to go over the basic definitions that will be involved in next week’s talk.

1. Hodge structures
2. Algebraic cycles and Abel-Jacobi maps
3. Mixed Hodge structures
4. Some references
Motivation and definition

Let $X$ be a smooth projective variety over $\mathbb{C}$, considered with analytic topology. We have the de Rham isomorphism

$$H^n(X, \mathbb{C}) \cong H^n_{dR}(X),$$

where $H^n_{dR}(X)$ is the complex smooth de Rham cohomology of $X$. We have the Hodge decomposition

$$H^n_{dR}(X) = \bigoplus_{p+q=n} H^{p,q},$$

where $H^{p,q}$ is the subspace of $H^n_{dR}(X)$ consisting of cohomology classes that can be represented by differential forms of type $(p, q)$, i.e. those that locally are of the form

$$fdz_1 \wedge \cdots \wedge dz_p \wedge d\bar{z}_{p+1} \wedge \cdots \wedge d\bar{z}_n$$

(i.e. $p$ (resp. $q$) factors of $dz_i$ (resp. $d\bar{z}_i$)). We have

$$H^{q,p} = \overline{H^{p,q}},$$
where complex conjugation is by complex conjugating differential forms, which translates via de Rham’s isomorphism to the natural complex conjugation on $H^n(X, \mathbb{C})$ through the coefficients. Thus $H^n(X, \mathbb{Z})$ has the following structure: it is a finitely generated abelian group, and on

$$H^n(X, \mathbb{Z}) \otimes \mathbb{C} = H^n(X, \mathbb{C})$$

we have a decomposition $H^n(X, \mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}$ with $H^{q,p} = \overline{H^{p,q}}$.

**Definition**

An integral Hodge structure of weight $n$ consists of the following data:

(i) a finitely generated $\mathbb{Z}$-module $H_\mathbb{Z}$

(ii) a decomposition $H_\mathbb{C} := H_\mathbb{Z} \otimes \mathbb{C} = \bigoplus_{p+q=n} H^{p,q}$ (called the Hodge decomposition) of $H_\mathbb{C}$ into complex vector subspaces such that $H^{q,p} = \overline{H^{p,q}}$ (where the complex conjugation is through the second factor in $H_\mathbb{Z} \otimes \mathbb{C}$).
Examples:
(1) If $H\mathbb{Z}$ has odd rank, then the weight must be even.
(2) There is a unique Hodge structure of weight $2n$ with underlying $\mathbb{Z}$-module $\mathbb{Z}$ (with $\mathbb{Z} \otimes \mathbb{C} = \mathbb{C}$ sitting in $(n, n)$ component of the Hodge decomposition); this Hodge structure is denoted by $\mathbb{Z}(-n)$. The object $\mathbb{Z}(1)$ is called the Tate object.
(3) For $X$ a smooth complex projective variety, $H^n(X, \mathbb{Z})$ is a Hodge structure of weight $n$. We have $H^0(\ast) = \mathbb{Z}(0)$ and $H^2(P^1) = \mathbb{Z}(-1)$. 
The definition of a Hodge structure can be reformulated by replacing the Hodge decomposition with a filtration. One has the following equivalent definition:

**Definition**

An integral Hodge structure of weight $n$ consists of the following data:

(i) a finitely generated $\mathbb{Z}$-module $H_\mathbb{Z}$

(ii) a finite decreasing filtration $F^\cdot$ of the complex vector space $H_\mathbb{C} := H_\mathbb{Z} \otimes \mathbb{C}$ (called the Hodge filtration) such that for each $p$,

$$H_\mathbb{C} = F^p H_\mathbb{C} \oplus F^{n-p+1} H_\mathbb{C}.$$
The passage between the Hodge filtration and decomposition is as follows:

\[ F^p H_\mathbb{C} = \bigoplus_{p' + q' = n, p' \geq p} H^{p',q'} \]

and

\[ H^{p,q} = F^p H_\mathbb{C} \cap \overline{F^q H_\mathbb{C}}. \]

A rational Hodge structure of weight \( n \) is defined similarly, starting with a finite-dimensional rational vector space \( H_\mathbb{Q} \) rather than an abelian group. (So the data consists of \( H_\mathbb{Q} \) and a decomposition or filtration of \( H_\mathbb{C} := H_\mathbb{Q} \otimes \mathbb{C} \) as before.)

**Remarks:**
(1) The second definition (involving filtrations) is more suitable for generalizations (see the slides on mixed Hodge structures).
(2) We usually refer to a Hodge structure of weight \( n \) by a capital English letter like \( H \), and then use the decorated versions \( H_\mathbb{Z}, H_\mathbb{Q}, \) etc. for the underlying \( \mathbb{Z} \)-module, rational vector space, etc.
Let $H$ be a Hodge structure of weight $2n$. Elements of $H_{\mathbb{Z}}$ whose images in $H_{\mathbb{C}}$ are in $(n, n)$ component of the Hodge decomposition are called Hodge classes. An element of $\xi \in H_{\mathbb{Z}}$ is a Hodge class if and only if $1 \mapsto \xi$ defines a morphism $\mathbb{Z}(-n) \to H$. Thus Hodge classes in $H$ can be thought of as morphisms $\mathbb{Z}(-n) \to H$ (and vice versa).
A slight generalization

**Definition**

An integral Hodge structure consists of the following data:

(i) a finitely generated \( \mathbb{Z} \)-module \( H_\mathbb{Z} \)

(ii) a decomposition \( H_\mathbb{C} := H_\mathbb{Z} \otimes \mathbb{C} = \bigoplus_{p,q \in \mathbb{Z}} H^{p,q} \) (called the Hodge decomposition) of \( H_\mathbb{C} \) into complex vector subspaces such that \( H^{q,p} = \overline{H^{p,q}} \) (where the complex conjugation is through the second factor in \( H_\mathbb{Z} \otimes \mathbb{C} \)).

Then a Hodge structure of weight \( n \) is a Hodge structure in which the \( (p, q) \)-component is nonzero only if \( p + q = n \).

A morphism of integral Hodge structures \( A \rightarrow B \) is a homomorphism \( A_\mathbb{Z} \rightarrow B_\mathbb{Z} \) such that its extension \( A_\mathbb{C} \rightarrow B_\mathbb{C} \) preserves the Hodge decomposition. The category of integral Hodge structures is an abelian category.
Algebraic cycles

Let $X$ be a $d$-dimensional scheme of finite type over a field. Let $Z_i(X)$ (resp. $Z^i(X)$) be the free abelian group on the set of closed subvarieties of $X$ (or equivalently, irreducible closed subsets of $X$) of dimension (resp. codimension) $i$. Elements of $Z_i(X)$ (resp. $Z^i(X)$) are called algebraic cycles of dimension (resp. codimension) $i$ on $X$. Elements of $Z_{d-1}(X) = Z^1(X)$ are called divisors. Divisors have a very long history, especially in the setting of divisors on curves. Divisors are much better understood than algebraic cycles of higher codimensions.
For a proper morphism \( f : X \to Y \), there is a pushforward map \( f_* : \mathcal{Z}_i(X) \to \mathcal{Z}_i(Y) \) defined as follows: for a closed subvariety \( Z \subset X \) of dim \( i \), one sets \( f_*(Z) = \deg(Z/f(Z)) \cdot f(Z) \), where \( \deg(Z/f(Z)) \) is the degree of the field extension \( k(Z)/k(f(Z)) \) (of rational functions on \( Z \) and \( f(Z) \)) when \( \dim(f(Z)) = \dim(Z) \) and zero otherwise. (Properness assures \( f(Z) \) is closed in \( Y \). The definition is extended to \( \mathcal{Z}_i(X) \) by linearity.)

For a flat morphism \( f : X \to Y \) of fixed relative codimension, one has the pullback map \( f^* : \mathcal{Z}^i(Y) \to \mathcal{Z}^i(X) \): for a closed subvariety \( Z \subset Y \), one sets \( f^*(Z) \) to be the sum of irreducible components of \( f^{-1}(Z) \), counted with multiplicities (see Fulton’s book).
Equivalence relations on algebraic cycles

There are various "nice" equivalence relations on $\mathcal{Z}_i(X)$. Here are three:

- Rational equivalence
- Algebraic equivalence
- Homological equivalence

These are examples of *adequate* equivalence relations. Working modulo them, one has pushforwards along proper morphisms, pullbacks along *arbitrary* morphisms, and a nice intersection theory. Roughly speaking, the subgroup $\mathcal{Z}_{i}^{rat}(X)$ (resp. $\mathcal{Z}_{i}^{alg}(X)$) of $\mathcal{Z}_i(X)$ of cycles rationally (resp. algebraically) equivalent to zero is the subgroup generated by differences of cycles that can be deformed to each other along $\mathbb{P}^1$ (resp. some curve). Being rationally trivial is the natural generalization of the notion of principality for divisors. (See Fulton for precise definitions.)
For $X$ a smooth variety over $\mathbb{C}$, the subgroup $\mathbb{Z}_i^{\text{hom}}(X)$ of homologically trivial cycles will be defined over the next few slides. We will have

$$\mathbb{Z}_i^{\text{rat}}(X) \subset \mathbb{Z}_i^{\text{alg}}(X) \subset \mathbb{Z}_i^{\text{hom}}(X),$$

but these relations are not the same in general (more on this below). The group $\mathbb{Z}_i(X)/\mathbb{Z}_i^{\text{rat}}(X)$ is called the Chow group of $i$-dimensional cycles on $X$, often denoted by $\text{CH}_i(X)$. 
Cycle class map

Let $X$ be a smooth projective variety over $\mathbb{C}$. There is a cycle class map

$$\mathcal{Z}_n(X) \longrightarrow H_{2n}(X, \mathbb{Z}).$$

Let $Z$ be a closed subvariety of dimension $n$ (hence real dimension $2n$). Sending $Z$ to the image of its fundamental class under the natural map $H_{2n}(Z, \mathbb{Z}) \longrightarrow H_{2n}(X, \mathbb{Z})$ we get (after extending linearly) the class map.

Here is an analytic description of the class map: as an element of $H^{2n}_{dR}(X) \cong H^{2n}(X, \mathbb{C})^\vee \cong H_{2n}(X, \mathbb{C})$ the class of $Z$ is simply integration over the smooth locus of $Z$ (which is a complex manifold of dimension $n$).
Let $Z_n^{\text{hom}}(X)$ be the kernel of the class map; its elements are called homologically trivial. It is easy to see that algebraically trivial cycles (and in particular, rationally trivial cycles) are homologically trivial:

$$Z_n^{\text{rat}}(X) \subset Z_n^{\text{alg}}(X) \subset Z_n^{\text{hom}}(X).$$

**Remarks:**

(1) The induced map $CH_n(X) \to H_{2n}(X, \mathbb{Z})$ is also called the class map.

(2) Often, one uses Poincaré duality to rewrite the class map as a map into cohomology, i.e. as a map $Z^n(X) \to H^{2n}(X, \mathbb{Z})$. The advantage of writing things this way is that the map gives a ring homomorphism $\bigoplus_n CH^n(X) \to \bigoplus_n H^{2n}(X, \mathbb{Z})$ (with the product on the Chow ring being the intersection product).

(3) The images of algebraic cycles under the cycle class map are Hodge cycles (as a $2n$-form on an $n$-dimensional complex manifold will be nonzero only if it is of type $(n, n)$). The celebrated Hodge conjecture predicts that after tensoring with $\mathbb{Q}$, all Hodge classes in $H_{2n}(X, \mathbb{Q})$ are in the image of the class map.

(4) There are also cycle class maps into other cohomology theories (e.g. étale).
As we already pointed out,

\[ \mathcal{Z}^\text{rat}_n(X) \subset \mathcal{Z}^\text{alg}_n(X) \subset \mathcal{Z}^\text{hom}_n(X). \]

Already for divisors on curves, \( \mathcal{Z}^\text{rat}_{\dim(X)-1}(X) \) (= principal divisors) is strictly smaller than the other two groups, which coincide and are equal to the subgroup of degree zero divisors. For divisors in general, algebraic and homological equivalence coincide after tensoring with \( \mathbb{Q} \) (a theorem of Matsusaka). But for algebraic cycles of dimensions \( 0 < n < \dim(X) - 1 \), by a famous theorem of Griffiths algebraic and homological equivalence are not the same in general. The first explicit example of an algebraic cycle which is homologically trivial but algebraically nontrivial was given by B. Harris in 1982.

A useful tool in trying to distinguish between the three equivalence relations above is the Abel-Jacobi map.
The classical Abel-Jacobi map is for a smooth complex projective curve $C$; it goes from $CH_0(C)^{\text{hom}} := \mathcal{Z}^{\text{hom}}_0(C)/\mathcal{Z}^{\text{rat}}_0(C)$ (the space of degree zero divisors on $C$ modulo principal divisors) to a compact complex torus constructed using holomorphic differential forms and integral homology of $C$. Given $p, q \in C$, choose a path $\gamma$ from $p$ to $q$. We have a linear map

$$\int_{\gamma} : \Omega^1_{\text{hol}}(C) \longrightarrow \mathbb{C}$$

where $\Omega^1_{\text{hol}}(C)$ is the space of holomorphic 1-forms on $C$. The choice of $\gamma$ results in an ambiguity in $H_1(C, \mathbb{Z})$ (considered as a subgroup of $\Omega^1_{\text{hol}}(C)^{\vee}$).
We get a map

\[ \mathcal{Z}_0^{\text{hom}}(C) \longrightarrow \frac{\Omega^1_{\text{hol}}(C)^\vee}{H_1(C, \mathbb{Z})} \quad q - p \mapsto \int_p^q. \]

This map sends the subgroup \( \mathcal{Z}_0^{\text{rat}}(C) \) to zero. The classical Abel-Jacobi map is the induced map

\[ CH_0^{\text{hom}}(C) \longrightarrow \frac{\Omega^1_{\text{hol}}(C)^\vee}{H_1(C, \mathbb{Z})}, \]

which by the theorem of Abel and Jacobi is an isomorphism.
Let us go back to abstract Hodge structure for a moment. Let $H$ be an integral Hodge structure of weight $n$ with underlying $\mathbb{Z}$-module $H_{\mathbb{Z}}$. One defines the dual Hodge structure $H^\vee$ as the integral Hodge structure of weight $-n$ on $H_{\mathbb{Z}}^\vee (= \text{Hom}_{\mathbb{Z}}(H_{\mathbb{Z}}, \mathbb{Z}))$ with the Hodge filtration defined by

$$F^p(H_{\mathbb{Z}}^\vee \otimes \mathbb{C}) = F^pH_{\mathbb{C}}^\vee := \{ f \in H_{\mathbb{C}}^\vee : f(F^{-p+1}H_{\mathbb{C}}) = 0 \}.$$ 

(In terms of the Hodge decomposition, this translates to $(H^\vee)^{p,q} = (H^{-p,-q})^\vee$.)

Using this (and by the duality between homology and cohomology) we get a Hodge structure of weight $-n$ on the homology $H_n(X, \mathbb{Z})$ of a smooth complex projective variety.
For a integral Hodge structure $H$ of odd weight $2n − 1$, the (Griffiths) intermediate Jacobian of $H$ is defined to be

$$JH := \frac{H^\mathbb{C}}{F^n H^\mathbb{C} + H_\mathbb{Z}}.$$

This is a compact complex torus. In the case of $H = H^1(X)$ with $X$ a smooth complex projective curve, this is an abelian variety (namely, the Jacobian of $X$). But in general, even in the case of $H = H^{2n-1}(X)$ for a smooth complex projective variety, $JH$ need not be algebraic.
We now define Griffiths’ Abel-Jacobi map. Let $X$ be a smooth projective variety over $\mathbb{C}$. The Abel-Jacobi map of Griffiths is a map

$$AJ : CH_n^{\text{hom}}(X) \longrightarrow JH_{2n+1}(X)$$

defined as follows. We have

$$JH_{2n+1}(X) = H_{2n+1}(X, \mathbb{C})/ (F^{-n}H_{2n+1}(X, \mathbb{C}) + H_{2n+1}(X, \mathbb{Z})).$$

Note that $F^{-n}H_{2n+1}(X, \mathbb{C})$ is the subspace of $H_{2n+1}(X, \mathbb{C})$ which vanishes on $F^{n+1}H^{2n+1}(X, \mathbb{C})$. So we can identify

$$JH_{2n+1}(X) = (F^{n+1}H^{2n+1}(X, \mathbb{C}))^\vee / H_{2n+1}(X, \mathbb{Z}).$$

(In particular, if $X = C$ is a curve, $JH_1(C)$ is just $\frac{\Omega^1_{\text{hol}}(C)^\vee}{H_1(C, \mathbb{Z})}$: the codomain of the classical Abel-Jacobi map.)
Now given $Z \in Z_n^{\text{hom}}(X)$, let $\Gamma$ be an integral topological $(2n + 1)$-chain whose boundary is $Z$. Given a cohomology class in $F^{n+1}H^{2n+1}(X, \mathbb{C})$, choose a representative $\omega$ that is a (closed) smooth $(2n + 1)$-form on $X$ of holomorphy degree $\geq n + 1$ (i.e. locally with at least $n + 1$ factors of $dz_i$’s). Consider

$$F^{n+1}H^{2n+1}(X, \mathbb{C}) \longrightarrow \mathbb{C} \quad [\omega] \mapsto \int_{\Gamma} \omega.$$  

This is well-defined: if $\omega'$ is another choice of representative (satisfying the holomorphy degree condition), $\omega - \omega'$ is exact and of holomorphy degree $\geq m + 1$; by Hodge theory, we have $\omega - \omega' = d\nu$ for some $2n$-form of holomorphy degree $\geq n + 1$, so that by Stokes’ theorem

$$\int_{\Gamma} (\omega - \omega') = \int_{Z} \nu = 0,$$

as $Z$ has complex dimension $n$ and hence $\nu$ vanishes on $Z$. 
If $\Gamma'$ is another integral chain such that $\partial \Gamma' = \partial \Gamma = Z$, then $\int \Gamma$ and $\int \Gamma'$ differ by an element of $H_{2n+1}(X, \mathbb{Z})$, so that we get a well-defined element

$$\left[ \int \partial^{-1}Z \right] \in (F^{n+1}H^{2n+1}(X, \mathbb{C}))^\vee / H_{2n+1}(X, \mathbb{Z}).$$

(Here $\partial^{-1}Z$ is any integral chain whose boundary is $Z$.) Thus we get a well-defined map

$$Z_n^{\text{hom}}(X) \longrightarrow JH_{2n+1}(X) \quad Z \mapsto \left[ \int_{\partial^{-1}(Z)} \right].$$

One can show that this map vanishes on $Z_n^{\text{rat}}(X)$. The induced map

$$AJ : CH_n^{\text{hom}}(X) \longrightarrow JH_{2n+1}(X)$$

is the Abel-Jacobi map of Griffiths.
Remarks:

1. In the case of $CH^0_{\text{hom}}(C)$ for a curve, the construction just gives the classical Abel-Jacobi map.

2. Often one uses Poincaré duality and considers the Abel-Jacobi map as a map into the intermediate Jacobian of $H^{2\dim(X)-2n-1}(X)$. In our talk next week though the Abel-Jacobi map will be homological (defined as above).

3. One can define the Abel-Jacobi map more geometrically. See Jannsen’s book *Mixed Motives and Algebraic K-theory*, for instance.

4. Recall that in the classical case of 0-cycles on curves, the Abel-Jacobi map is an isomorphism. In the general case for complex varieties, however, it is known that the kernel of the Abel-Jacobi map can be “huge” (a theorem of Mumford). For varieties over number fields though, Bloch and Beilinson conjecture the Abel-Jacobi map to be injective after tensoring with $\mathbb{Q}$. 
The notion of a mixed Hodge structure was defined by Deligne in 1970’s (Theorie de Hodge, I-III, Publ. Math. IHES) to generalize Hodge theory to the setting of arbitrary complex varieties. Before we define what a mixed Hodge structure is, recall that if $W$ is an increasing filtration on a vector space $V$ over a field $F$, one defines $Gr^W_n(V) := V_n/V_{n-1}$. If $K$ is a field extension of $F$, the filtration $W$ extends to a filtration on $V_K := V \otimes K$ in an obvious way; we denote the filtration on $V_K$ also by $W$, and identify $Gr^W_n(V_K) = Gr^W_n(V) \otimes K$. 

The category of mixed Hodge structures
Definition

An integral mixed Hodge structure consists of the data of

(i) a finitely generated $\mathbb{Z}$-module $V_{\mathbb{Z}}$

(ii) a finite increasing filtration $W_\cdot$ on $V_{\mathbb{Q}}$ (called the weight filtration)

(iii) a finite decreasing filtration $F_\cdot$ of $V_{\mathbb{C}}$ (again called the Hodge filtration)

such that for each $n$, the rational vector space $Gr_n V_{\mathbb{Q}}$ equipped with the filtration on its complexification defined by

$$F^p (Gr_n^W V_{\mathbb{C}}) := \frac{(F^p V_{\mathbb{C}} \cap W_n V_{\mathbb{C}}) + W_{n-1} V_{\mathbb{C}}}{W_{n-1} V_{\mathbb{C}}}$$

forms a rational Hodge structure of weight $n$. 
Example: $H^n(X, \mathbb{Z})$ for any complex variety (a theorem of Deligne). One way to describe the weight and Hodge filtrations here is by using the complex of smooth differential forms on $X$ with at most logarithmic singularity at infinity.
Rational mixed Hodge structures are defined similarly, starting with a finite-dimensional rational vector space rather than an abelian group. Morphisms of (rational or integral) mixed Hodge structures are defined in the obvious way. The category of mixed Hodge structures is an abelian category. (Recall that the category of filtered vector spaces is not abelian; the key difference here is that the presence of two filtrations forces morphisms of mixed Hodge structures to be strict with respect to the filtrations (i.e. $f(W_nA) = f(A) \cap W_nB$ and $f(F^pA) = f(A) \cap F^pB$ for a morphism $f : A \to B$).)

Every Hodge structure can be considered as a mixed Hodge structure in a natural way. The category of Hodge structures can be identified as a full subcategory of the category of mixed Hodge structures.
Tensor products and internal Hom spaces

Let $A$ and $B$ be mixed Hodge structures. The tensor product $A \otimes B$ is defined by taking the tensor product of the underlying $\mathbb{Z}$-modules and filtrations (in the standard way, i.e. by $W_n(A \otimes B) = \sum_{r+s=n} W_r A \otimes W_s B$).

We denote $A(n) := A \otimes \mathbb{Z}(n)$. The object $A(n)$ has the same underlying $\mathbb{Z}$-module as $A$, but the filtrations are shifted: $W_m(A(n)_\mathbb{Q}) = W_{m+2n} A_\mathbb{Q}$ and $F^p(A(n)_\mathbb{C}) = F^{p+n}(A_\mathbb{C})$.

The internal Hom $\text{Hom}(A, B)$ is a mixed Hodge structure defined as follows: the underlying $\mathbb{Z}$-module is $\text{Hom}_\mathbb{Z}(A_\mathbb{Z}, B_\mathbb{Z})$; the filtrations are defined by

$$W_n \text{Hom}_\mathbb{Q}(A_\mathbb{Q}, B_\mathbb{Q}) = \{ f \in \text{Hom}_\mathbb{Q}(A_\mathbb{Q}, B_\mathbb{Q}) : f(W_r A_\mathbb{Q}) \subset W_{r+n} B_\mathbb{Q} \text{ for all } r \}$$
and

\[ F^p \text{Hom}_\mathbb{C}(A_\mathbb{C}, B_\mathbb{C}) = \{ f \in \text{Hom}_\mathbb{C}(A_\mathbb{C}, B_\mathbb{C}) : f(F^r A_\mathbb{C}) \subset F^{r+p} B_\mathbb{C} \text{ for all } r \} \].

We denote \( \text{Hom}(A, \mathbb{Z}(0)) \) by \( A^\vee \), the dual of \( A \). (This is consistent with the earlier definition of \( A^\vee \) for Hodge structures of a given weight.)

The usual canonical isomorphisms involving Homs, tensors, duals, etc. in categories of modules also hold in the category of mixed Hodge structures (with Hom being \( \text{Hom} \) now).
Extensions of mixed Hodge structures

What makes mixed Hodge structures more complicated is existence of nontrivial extensions.

**Review of Ext groups:** Let $A$ and $B$ be objects in an abelian category. By a (Yoneda) extension of $A$ by $B$ we mean a short exact sequence

$$0 \rightarrow B \rightarrow E \rightarrow A \rightarrow 0. \quad (1)$$

If there is a commutative diagram

$$\begin{array}{cccccc}
0 & \rightarrow & B & \rightarrow & E & \rightarrow & A & \rightarrow & 0 \\
\| & & \downarrow & & \| & & \downarrow & & \\
0 & \rightarrow & B & \rightarrow & E' & \rightarrow & A & \rightarrow & 0
\end{array}$$

with exact rows, we say the two extensions given in the two rows are equivalent. We denote the set of equivalence classes of extensions of $A$ by $B$ by $\text{Ext}(A, B)$ (or $\text{Ext}^1(A, B)$, because similarly one can define extensions of higher lengths).
There is a natural operation called Baer summation which makes $\text{Ext}(A, B)$ an abelian group. To discuss this we should first discuss pullbacks and pushouts of extensions.

A morphism $f : A' \rightarrow A$ induces a map $\text{Ext}(A, B) \rightarrow \text{Ext}(A', B)$ called pullback (along $f$). Indeed, the pullback of the extension class of Eq. (1) is the class of the fibred product $E \times_A A'$, with the obvious maps from $B$ and to $A'$. (For us, $E \times_A A'$ is just \{(e, a') \in E \times A' : \pi(e) = f(a')\}, where $\pi : E \rightarrow A$ is the surjective arrow in Eq. (1).)

Dually, a morphism $g : B \rightarrow B'$ induces a map $\text{Ext}(A, B) \rightarrow \text{Ext}(A, B')$ called pushout (along $g$). The pushout of the class of the extension Eq. (1) is the class of the fibred coproduct $B' \sqcup_B E$ with the obvious maps from $B'$ and to $A$. (The fibred coproduct $B' \sqcup_B E$ is the quotient of $B' \oplus E$ by the image of $(f, -\iota)$, where $\iota : B \rightarrow E$ is the injective arrow in Eq. (1).)
Now the sum of two extensions Eq. (1) and an extension with $E'$ in the middle is obtained by first taking the direct sum of the two extensions

$$0 \longrightarrow B \oplus B \longrightarrow E \oplus E' \longrightarrow A \oplus A \longrightarrow 0,$$

then pulling it back along the diagonal map $A \longrightarrow A \oplus A$ (i.e. $a \mapsto (a, a)$) and pushing it out along the codiagonal $B \oplus B \longrightarrow B$ (i.e. $(b_1, b_2) \mapsto b_1 + b_2$). This is well-defined and makes $\text{Ext}(A, B)$ an abelian group. The identity is the trivial extension

$$0 \longrightarrow B \longrightarrow B \oplus A \longrightarrow A \longrightarrow 0$$

(with the natural inclusion and projection maps).
Now we return to the category of integral mixed Hodge structures. We assume that $A$ and $B$ have free $\mathbb{Z}$-modules. Also assume, for convenience, that the highest weight of $B$ is less than the lowest weight of $A$ (the highest (resp. lowest) weight is the smallest weight where the weight filtration stabilizes (resp. is nonzero)). There is canonical isomorphism

$$\text{Ext}(A, B) \cong \frac{\text{Hom}_\mathbb{C}(A_\mathbb{C}, B_\mathbb{C})}{F^0\text{Hom}_\mathbb{C}(A_\mathbb{C}, B_\mathbb{C}) + \text{Hom}_\mathbb{Z}(A_\mathbb{Z}, B_\mathbb{Z})}$$

due to J. Carlson. The isomorphism assigns to the class of an extension

$$0 \rightarrow B \xrightarrow{\iota} E \xrightarrow{\pi} A \rightarrow 0$$

the class of the map $r \circ s$ in the quotient on the right hand side, where $s$ is a section (= right inverse) of the map $\pi : E_\mathbb{C} \rightarrow A_\mathbb{C}$ compatible with the Hodge filtration and $r$ is a retraction (= left inverse) of the map $\iota : B_\mathbb{Z} \rightarrow E_\mathbb{Z}$. (A different choice of $s$ (resp. $r$) results in a difference in $F^0\text{Hom}_\mathbb{C}(A_\mathbb{C}, B_\mathbb{C})$ (resp. $\text{Hom}_\mathbb{Z}(A_\mathbb{Z}, B_\mathbb{Z})$).)

There is a similar result in the category of rational Hodge structures.

Remark: The statement can be modified so that we won’t need any condition on the weights of $A$ and $B$. 
Some references

Here are some references for the interested reader to learn more about the objects defined in these slides:

- For Hodge theory (including Hodge structure on cohomology of smooth projective varieties, cycles class maps and Abel-Jacobi maps), see Voisin’s book *Hodge Theory and Complex Algebraic Geometry* (two volumes).

- For the theory of mixed Hodge structures and the mixed Hodge structure on cohomology of an arbitrary complex variety, see Deligne’s Hodge Theory I-III papers (also Voisin’s book).
For algebraic cycles, Fulton’s book is the standard source. For a crash course, I suggest Murre’s notes of his lectures in ICTP summer school and conference on "Hodge Theory and Related Topics" 2010. The notes also discuss a lot of other things that we included in these slides, such as adequate relations, class maps and Abel-Jacobi maps. They even include a discussion of Griffiths’ theorem on algebraic versus homological equivalence and Mumford’s theorem on the kernel of the Abel-Jacobi map.

For extensions of mixed Hodge structures, the original reference is Carlson’s paper (Extensions of mixed Hodge structures, Journees de Géometrie Algébrique d’Angers, Juillet 1979/Algebraic Geometry, Angers, 1979, pp. 107-127, Sijthoff & Noordhoff, Alphen aan den Rijn, Md., 1980).