Background to
‘On the subring of special cycles’

Stephen Kudla
(Toronto)

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In this talk I will provide some background for my lecture in the Fields Number Theory Seminar on June 17th.

“On the subring of special cycles on orthogonal Shimura varieties”

The main objects that will be needed are:

- Theta series and the Weil representation
- Eisenstein series
- The Siegel-Weil formula
- Shimura varieties for orthogonal groups

In this talk, I will give a casual introduction aimed at non-specialists.
Classical theta series

Classical theta series are constructed from

\[ L = \mathbb{Z}^m, \quad Q : L \rightarrow \mathbb{Z}, \text{ a } \mathbb{Z}-\text{valued quadratic form of signature } (m, 0), \]

\[ \tau = u + iv \in \mathcal{H} \]

\[ \theta(\tau, L) = \sum_{x \in L} q^{Q(x)}, \quad q = e(\tau) = e^{2\pi i \tau}. \]

\[ \theta(\tau, L) \text{ is a modular form of weight } \frac{1}{2}m. \]

It is the sum over \( L \) of the Schwartz function

\[ \varphi_\mathbb{R}(\tau, x) = q^{Q(x)} = e(Q(x)u) e^{-2\pi Q(x)v}. \]
Theta series and the Weil representation

We introduce some more machinery: \( \mathbb{A} = \text{adèles of } \mathbb{Q} \)
\( \mathbb{A}_f = \text{finite adèles of } \mathbb{Q} \)

\[
V(\mathbb{Q}) = L \otimes_{\mathbb{Z}} \mathbb{Q} \quad V(\mathbb{A}) = L \otimes_{\mathbb{Z}} \mathbb{A} \quad V(\mathbb{A}_f) = L \otimes_{\mathbb{Z}} \mathbb{A}_f
\]
\[
\hat{L} = L \otimes \hat{\mathbb{Z}} = \text{compact open subset of } V(\mathbb{A}_f),
\]
\[
S(V(\mathbb{A}_f)) = \text{Schwartz space of } V(\mathbb{A}_f)
\]
\[
= \text{locally constant functions of compact support}
\]
\[
\varphi_L = \text{characteristic function of } \hat{L}.
\]

Then
\[
\theta(\tau, L) = \sum_{x \in V(\mathbb{Q})} q^{Q(x)} \varphi_L(x)
\]
is the sum of the Schwartz function
\[
q^Q \otimes \varphi_L \in S(V(\mathbb{A})) = S(V(\mathbb{R})) \otimes S(V(\mathbb{A}_f))
\]
over the lattice \( V(\mathbb{Q}) \) in \( V(\mathbb{A}) \).
Theta series and the Weil representation

Let $G = O(V)$ and $G' = SL(2)$, algebraic groups over $\mathbb{Q}$. There is an obvious action of $G(\mathbb{A})$ on $S(V(\mathbb{A}))$ given by

$$\omega(g) \varphi(x) = \varphi(g^{-1}x).$$

Assume, for simplicity, that $m = \dim V$ is even. This is to avoid a discussion of metaplectic covers. Then there is an action of $G'(\mathbb{A})$ on $S(V(\mathbb{A}))$ given by the Weil representation. For example:

$$\omega(n(b)) \varphi(x) = \psi(bQ(x)) \varphi(x), \quad n(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix},$$

$$\omega(m(a)) \varphi(x) = \chi_V(a) \left| a \right|_{\mathbb{A}}^{\frac{1}{2}m} \varphi(xa), \quad m(a) = \begin{pmatrix} a & t \\ 0 & a^{-1} \end{pmatrix}.$$
Theta series and the Weil representation

Summation over $V(\mathbb{Q})$ gives a tempered distribution

$$\Theta : S(V(\mathbb{A})) \longrightarrow \mathbb{C}, \quad \Theta(\varphi) = \sum_{x \in V(\mathbb{Q})} \varphi(x).$$

The fundamental fact is that $\Theta$ is invariant under $\text{SL}_2(\mathbb{Q}) \times G(\mathbb{Q})$. We then obtain kernel functions, for $\varphi \in S(V(\mathbb{A}))$,

$$\theta(g, g'; \varphi) := \Theta(\omega(g, g') \varphi)$$

on $G(\mathbb{A}) \times G'(\mathbb{A})$ which are left invariant under $\text{SL}_2(\mathbb{Q}) \times G(\mathbb{Q})$. The modularity of the classical theta function is a consequence of this. Here is why.
If, for $\tau = u + iv \in \mathcal{H}$, we write

$$g'_\tau = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v^{1/2} \\ -v^{-1/2} \end{pmatrix} \in \text{SL}_2(\mathbb{R}),$$

and $\varphi_\infty(x) = e^{-2\pi Q(x)}$.

Then

$$\omega(g'_\tau)\varphi_\infty(x) = v^{1/4} q^{Q(x)} = v^{1/4} m e(Q(x)u) e^{-2\pi Q(x)v},$$

and we can write the classical theta function as

$$\theta(\tau, L) = v^{-1/4} m \Theta(\omega(g'_\tau)\varphi_\infty \otimes \varphi_L).$$
Now for $\gamma \in SL_2(\mathbb{Q})$,
\[
\Theta(\omega(g'_T)\varphi_\infty \otimes \varphi_L) = \Theta(\omega(\gamma)(\omega(g'_T)\varphi_\infty \otimes \varphi_L)) \\
= \Theta(\omega(\gamma g'_T)\varphi_\infty \otimes \omega(\gamma)\varphi_L)) \\
= \Theta(\omega(g'_{\gamma(\tau)})\omega(k)\varphi_\infty \otimes \omega(\gamma)\varphi_L))
\]
where $k = k_{\theta(\gamma,\tau)}$.

The final transformation law will come out since $\varphi_\infty$ is an eigenfunction of $SO(2)$ of weight $\frac{1}{2}m$ and $\varphi_L$ is an eigenfunction for some open compact subgroup $K \subset SL_2(\mathbb{A}_f)$.

Theta series and the Weil representation

All of this can be generalized in many ways. For example, suppose that

$$\text{sig}(V) = (p, q), \quad pq > 0.$$ 

Everything goes as before except that $q^{Q(x)}$ is not a Schwartz function!

To fix this, following Siegel, let

$$D = \{ z \subset V(\mathbb{R}) \mid \dim z = q, \langle , \rangle|_z < 0 \}$$

= space of negative $q$-planes in $V(\mathbb{R})$.

$$R(x, z) = -\langle \text{pr}_z(x), \text{pr}_z(x) \rangle$$

$$Q(x, z) = Q(x) + R(x, z).$$

So

$$Q(x, z) = \begin{cases} 
Q(x) & \text{on } z^\perp \\
-Q(x) & \text{on } z.
\end{cases}$$
Theta series and the Weil representation

Let
\[ \varphi_\infty(x, z) := e^{-2\pi Q(x, z)} = e^{-2\pi Q(x)} e^{-2\pi R(x, z)} \]
and define the theta function
\[
\theta(\tau, z; \varphi) := \nu^{-\frac{1}{4} m} \Theta(\omega(g_\tau') \varphi_\infty(\cdot, z) \otimes \varphi), \quad \varphi \in S(V(\mathbb{A}_f)).
\]
This will be a non-holomorphic modular form of weight \( \frac{1}{2}(p - q) \).

A simple calculation gives
\[
\omega(g_\tau') \varphi_\infty(x, z) = \nu^\frac{1}{4} m q^{Q(x)} e^{-2\pi R(x, z)} \nu.
\]

As a function of \( z \in D \),
\[
\theta(\tau, \gamma z; \varphi) = \theta(\tau, z; \varphi)
\]
for \( \gamma \) in the subgroup \( \Gamma = G(\mathbb{Q}) \cap K \) where \( K \subset G(\mathbb{A}_f) \) is an open compact subgroup fixing \( \varphi \).
Next there are Siegel modular theta functions. Let $G' = \text{Sp}(n)/\mathbb{Q}$. Here $Q(x) = \frac{1}{2}((x_i, x_j)) \in \text{Sym}_n(\mathbb{Q})$. Then there is a Weil representation of $G'(\mathbb{A})$ on $S(V^n(\mathbb{A}))$. For example:

\[
\omega(n(b)) \varphi(x) = \psi(\text{tr}(bQ(x))) \varphi(x), \quad n(b) = \begin{pmatrix} 1_n & b \\ b^t & 1_n \end{pmatrix}
\]

\[
\omega(m(a)) \varphi(x) = \chi_V(\det(a)) |\det(a)|^{\frac{1}{2}m} \varphi(xa), \quad m(a) = \begin{pmatrix} a \\ t_a^{-1} \end{pmatrix}
\]

The actions of $G'(\mathbb{A})$ and $G(\mathbb{A})$ on $S(V^n(\mathbb{A}))$ commute and the theta distribution

\[
\Theta(\varphi) = \sum_{x = [x_1, \ldots, x_n] \in V(\mathbb{Q})^n} \varphi(x)
\]

is invariant under $G'(\mathbb{Q}) \times G(\mathbb{Q})$. 
Theta series and the Weil representation

So again we can define theta kernels

$$\theta(g, g'; \varphi_\infty \otimes \varphi) = \Theta(\omega(g, g') \varphi_\infty \otimes \varphi),$$

which are smooth functions on $G(\mathbb{Q}) \backslash G(\mathbb{A}) \times G'(\mathbb{Q}) \backslash G'(\mathbb{A})$.

These can be used to ‘theta lift’ automorphic forms from one group to the other.

For example, again there is a Siegel-Gaussian for $z \in D$,

$$\varphi_\infty(x, z) = e^{-2\pi \text{tr}(Q(x))} e^{-2\pi \text{tr}(R(x, z))},$$
and, for $\tau = u + iv \in \mathfrak{H}_n$, the function

$$\omega(g'_\tau)\varphi_\infty(x, z) = \det(v)^{1/4} m q^{Q(x)} e^{-2\pi \text{tr}(R(x, z)v)}.$$ 

Here, $x = [x_1, \ldots, x_n]$ and $R(x, z) = -((\text{pr}_z(x_i), \text{pr}_z(x_j))) \in \text{Sym}_n(\mathbb{R})_{\geq 0}$. This is an eigenfunction for $K' = U(n)$ with character $\det(k)^{1/2(p-q)}$, so the theta series

$$\theta(\tau, z; \varphi) = \det(v)^{-1/4} m \Theta(\omega(g'_\tau)\varphi_\infty(\cdot, z) \otimes \varphi), \quad \varphi \in \mathcal{S}(V^n(\mathbb{A}_f))$$

is a (non-holomorphic) Siegel modular form of weight $\ell = \frac{1}{2}(p - q)$. If $f$ is a Siegel cusp form of weight $\ell$, the Petersson inner product

$$\theta(f, \varphi)(z) := \langle \theta(\cdot, z; \varphi), f \rangle_{\text{Pet}}$$

is an automorphic function on $D$, the theta lift of $f$. 

### Theta series and the Weil representation
The Siegel-Weil formula

Of course, we could go in the other direction starting with an automorphic form on $G(\mathbb{A})$.
The most basic case is 11, the constant function on $G(\mathbb{Q}) \backslash G(\mathbb{A})$!
The Siegel-Weil formula describes the automorphic form

$$
\theta(g', 11, \varphi) = \int_{G(\mathbb{Q}) \backslash G(\mathbb{A})} \theta(g, g'; \varphi) \, dg, \quad \varphi \in S(V^n(\mathbb{A}))
$$

in terms of Eisenstein series on $G'(\mathbb{A})$.
The $\equiv$ occurs here since there can be convergence issues in defining this integral.
The Hilbert-Siegel case

Before continuing, we pass to the Hilbert-Siegel case.

\[ F = \text{a totally real number field} \]
\[ \mathbb{A} = \mathbb{A}_F = \text{adèlle ring of } F \]
\[ \psi : \mathbb{A}_F \rightarrow \mathbb{C}^1, \quad \text{a non-trivial additive character, trivial on } F \]
\[ V = \text{an } F\text{-vector space of dimension } m, \]
\[ (\ , \ ) = \text{symmetric bilinear form}, \]
\[ Q(x) = \frac{1}{2}(x, x), \quad \text{associated quadratic form}. \]
\[ \text{sig}(V_\sigma) = (p_\sigma, q_\sigma), \quad \sigma \in \text{Hom}(F, \mathbb{R}), \quad V_\sigma = V \otimes_{F, \sigma} \mathbb{R}, \]
\[ S(V^n(\mathbb{A})) = S(V^n_{\sigma_1}) \otimes \ldots \otimes S(V^n_{\sigma_d}) \otimes S(V^n(\mathbb{A}_f)) \]
\[ \theta(g, g'; \varphi) = \Theta(\omega(g, g')\varphi), \quad \varphi \in S(V^n(\mathbb{A})). \]
The Hilbert-Siegel case

We can choose more specific data as before:

\[ D = D_{\sigma_1} \times \cdots \times D_{\sigma_d}, \quad z = [z_1, \ldots, z_d] \in D \]

\[ \varphi_\infty(\tau, z) = \prod_\sigma \varphi_\sigma(\tau_\sigma, z_\sigma), \quad \tau = [\tau_{\sigma_1}, \ldots, \tau_{\sigma_d}] \in \mathfrak{H}_d^n, \]

\[ \varphi_\sigma(g'_{\tau_\sigma}, z_\sigma) = \det(v_\sigma)^{\frac{1}{4}m} q_\sigma Q(x) e^{-2\pi \text{tr}(R(x, z_\sigma) v_\sigma)} \in S(V^n_\sigma) \]

\[ \theta(\tau, z; \varphi) = N \det(v)^{-\frac{1}{4}m} \sum_{x \in V(F)^n} \varphi_\infty(g'_\tau, x, z) \varphi(x) \]

= a (non-holomorphic) Hilbert-Siegel theta series

of weight \( \left( \frac{1}{2}(p_1 - q_1), \ldots, \frac{1}{2}(p_d - q_d) \right) \).
We now return to the description of the integral

\[ \theta(g', \Pi, \varphi) \equiv \int_{G(F) \backslash G(\mathbb{A})} \theta(g, g'; \varphi) \, dg, \quad \varphi \in S(V^n(\mathbb{A})) \]

Only a very special type of Eisenstein series are involved.

\[ G' = \text{Sp}(n)/F \quad \text{P = Siegel parabolic} \]

\[ P = NM, \quad n(b) = \begin{pmatrix} 1_n & b \\ 1_n & \end{pmatrix}, \quad m(a) = \begin{pmatrix} a & \\ & t a^{-1} \end{pmatrix}, \]

\[ l(s, \chi) = \text{degenerate principal series}, \]

\[ \Phi(n(b)m(a)g', s) = \chi(\det(a)) |\det(a)|^{s+\rho_n} \Phi(g', s), \quad \rho_n = \frac{1}{2}(n + 1). \]

Note that \( \Phi(g', s) \) is left \( P(F) \)-invariant.
Siegel-Eisenstein series

Given a function $\Phi(g', s) \in I(s, \chi)$, we can form the Eisenstein series

$$E(g', s, \Phi) = \sum_{\gamma \in P(F) \backslash G'(F)} \Phi(\gamma g, s).$$

- This is absolutely convergent for $\text{Re}(s) > \rho_n = \frac{1}{2}(n + 1)$.
- It has a meromorphic analytic continuation to the whole $s$-plane and a functional equation relating $s$ and $-s$.
- When $s_0$ is not a pole, we get an intertwining operator

$$E(s_0) : I(s_0, \chi) \longrightarrow \mathcal{A}(G') = \text{space of automorphic forms on } G'(F) \backslash G'(\mathbb{A}).$$

- It is useful to note that

$$I(s, \chi) = \bigotimes_v I_v(s, \chi_v), \quad \text{restricted } \bigotimes_v.$$
To relate this to the theta series machinery, define the Rallis map:

$$\lambda_V : S(V(\mathbb{A})^n) \rightarrow I_n(s_0, \chi),$$

$$\varphi \mapsto \lambda_V(\varphi)(g') = \omega(g')\varphi(0), \quad s_0 = \frac{1}{2} \dim V - \rho_n.$$

$$\Phi(g', s; \varphi) = \omega(g')\varphi(0) \cdot |a(g')|^{s-s_0}$$

= standard section of $I(s, \chi)$ attached to $\varphi$,

$$E(g', s, \lambda_V(\varphi)) = \sum_{\gamma \in P(F) \backslash G'(F)} \Phi(\gamma g', s; \varphi)$$

= the Eisenstein series, $\Re(s) > \rho_n = \frac{1}{2}(n + 1)$.

We refer to $s_0 = \frac{1}{2}m - \frac{1}{2}(n + 1)$ as the Siegel-Weil point.
The Siegel-Weil formula

Rallis and I proved the following result.

**Theorem:** (Siegel-Weil formula, KR, Crelle 1988)
Assume that $V$ is anisotropic. Recall $G = O(V)$.
(1) Then, for any $\varphi \in S(V^n(\mathbb{A}))$, the Eisenstein series $E(g', s, \lambda_V(\varphi))$ is holomorphic at $s = s_0$.
(2)
$$E(g', s_0, \lambda_V(\varphi)) = \int_{G(F)\backslash G(\mathbb{A})} \theta(g', g; \varphi) \, dg,$$
where $\text{vol}(G(F)\backslash G(\mathbb{A}), dg) = 1$.
Taking the Siegel-Gaussian as the archimedean component of $\varphi$, $g' = g'_\tau$, and multiplying by $N \det(\nu)^{-\frac{1}{4}m}$, yields a more classical identity:
$$E(\tau, s_0, \lambda_V(\varphi)) = \int_{O(V)(F)\backslash O(V)(\mathbb{A})} \theta(\tau, g; \varphi) \, dg.$$
Orthogonal Shimura varieties

Let

\[ F = \text{a totally real number field}, \quad d = |F : \mathbb{Q}|, \]
\[ V = \text{inner product space with} \]
\[ \text{sig}(V) = ((m, 2), \ldots, (m, 2), (m + 2, 0), \ldots (m + 2, 0)) \]
\[ D = D_{\sigma_1} \times \cdots \times D_{\sigma_{d_+}}, \quad d_+ > 0. \]
\[ D_{\sigma} = \{ z \in \text{Gr}^o_2(V_\sigma) \mid (, )_z < 0 \} \]
\[ \simeq \{ w \in \mathbb{P}(V \otimes_{F, \sigma} \mathbb{C}) \mid (w, w) = 0, (w, \bar{w}) < 0 \} \]
\[ = \text{open subset of a quadric in } \mathbb{P}^{m+1}(\mathbb{C}) \]
\[ \dim_{\mathbb{C}} D = md_+. \]
\[ G = \text{GSpin}(V) \to \text{SO}(V) \quad G(\mathbb{R}) \text{ acts on } D \]
\[ \text{Sh}(G, D)_K = G(F) \backslash (D \times G(\mathbb{A}_f)/K) = \text{Shimura variety} \]
\[ \simeq \bigsqcup_j \Gamma_j \backslash D, \quad K = \text{open compact in } G(\mathbb{A}_f). \]
Orthogonal Shimura varieties

Here are some examples.

- \( m = 1, \ d_+ = 1, \ SO(V)(\mathbb{R}) = SO(1, 2) \times SO(3)^{d-1}, \)
  \( D_{\sigma_1} \simeq \mathcal{H}, \) the upper half plane,
  \[
  Sh(G, D)_K = \begin{cases} 
  \text{modular curve, if } F = \mathbb{Q}. \\
  \text{Shimura curve, if } d > 1, \ F \neq \mathbb{Q} 
  \end{cases}
  \]

- \( m = 1, \ d_+ = d > 1, \ SO(V)(\mathbb{R}) = SO(1, 2)^d, \ D \simeq \mathcal{H}^d, \)
  \[
  Sh(G, D)_K = \text{Hilbert modular variety for } F \\
  \text{e.g. } SL_2(O_F)\mathcal{H}^d.
  \]

- \( m = 1, \ d_+ < d, \ SO(V)(\mathbb{R}) = SO(1, 2)^{d_+} \times SO(3)^{d-d_+}, \ D \simeq \mathcal{H}^{d_+}, \)
  \[
  Sh(G, D)_K = \text{intermediate between the first two examples.}
  \]
Orthogonal Shimura varieties

- $m = 2, d_+ = d = 1$, $\text{SO}(V)(\mathbb{R}) = \text{SO}(2, 2)$, $D \cong \mathfrak{H} \times \mathfrak{H}$.
  There is a real quadratic field $E/\mathbb{Q}$ associated to $V$ and $\text{Sh}(G, D)$ is a
  Hilbert modular surface for $E$.

- $m = 2, d_+ \leq d, d > 1$, $\text{SO}(V)(\mathbb{R}) = \text{SO}(2, 2)^{d_+} \times \text{SO}(4)^{d-d_+}$,
  $D \cong (\mathfrak{H} \times \mathfrak{H})^{d_+}$.
  There is a totally real quadratic extension $E/F$ associated to $V$ and
  $\text{Sh}(G, D)$ is a generalized relative Hilbert modular surface for $E/F$.

- $m = 3, d_+ = d = 1$, $\text{SO}(V)(\mathbb{R}) = \text{SO}(3, 2)$, $D \cong \mathfrak{H}_2$, the Siegel space of
  genus 2, and $\text{Sh}(G, D)$ is a (twisted) Siegel modular variety.

- $m = 3, d_+ \leq d, d > 1$, $\text{SO}(V)(\mathbb{R}) = \text{SO}(3, 2)^{d_+} \times \text{SO}(5)^{d-d_+}$,
  $D \cong (\mathfrak{H}_2)^{d_+}$, and
  \[ \text{Sh}(G, D) = \bigsqcup_j \Gamma_j \backslash \mathfrak{H}_2^{d_+} \]
  is a generalized Hilbert-Siegel modular variety, of dimension $3d_+$. 
Thus the orthogonal Shimura varieties provide a large class of (quasi-)projective varieties of interest from the point of view of both geometry and number theory.

The theta functions $\theta(\tau, z; \varphi)$ provide a useful tool in their study. An example will be the topic of the forthcoming Fields Number Theory Seminar.