Example 1 As a first consequence of the vorticity-stream formulation, we remark that if \( \omega^0 \in C^1_c(\mathbb{R}^2) \) is radial, then \( \omega(x, t) := \omega^0(x) \) solves (??). This follows from the Biot-Savart formula and symmetry considerations. The computation was presented in the lecture. Assume for concreteness that

\[
\text{supp}(\omega^0) \subset B_R(0) \quad \text{and} \quad \int_{B(R)} \omega^0(y) \, dy = \gamma. \tag{1}
\]

For future reference, we compute for \( |x| > R \).

\[
u(x) = K(x) \int_0^R \left( \tilde{\omega}^0(r) \int_{\partial B_r(0)} d\mathcal{H}^1(y) \right) dr = K(x) \int_{B_R(0)} \omega^0(y) \, dy \]

\[= \gamma \frac{x^\perp}{2\pi|x|^2}.
\]
The proof of the above formula uses the fact that $K$ is harmonic away from 0 and the mean value theorem for harmonic functions.

The argument we gave shows more generally:

**Lemma**

Assume that $\eta \in L^1(\mathbb{R}^N)$ is a.e. radial and that $h$ is a function (possibly vector- or matrix-valued) that is harmonic on $\text{supp}(\eta)$. Then

$$\int h(y)\eta(y) \, dy = h(0) \int \eta(y) \, dy.$$
Example 2.
When we consider concentrated vorticity, we will always have in mind

“vorticity length scale” $\varepsilon \ll 1$

We may sometimes consider a limit as $\varepsilon \to 0$.
A simple and canonical situation arises if $\omega_\varepsilon$ satisfies

$$\text{supp}(\omega_\varepsilon) \subset B_\varepsilon(0), \quad \int_{\mathbb{R}^2} \omega_\varepsilon(y) \, dy = 1, \quad \omega_\varepsilon \geq 0 \text{ everywhere.}$$

We already know that if $\omega_\varepsilon$ is radial, then it generates a stationary solution of the Euler equations, and $u_\varepsilon = K$ outside of $B_\varepsilon(0)$.

What if $\omega_\varepsilon$ is not radial?
Clearly, there is no reason to suppose that it generates a stationary solution of the Euler equations.

Nonetheless, if we suppose that, for a solution of the Euler equations the vorticity at a particular time satisfies (2), we can use the Biot-Savart law to estimate the velocity field at that time. Indeed, it is straightforward to check that there exists a constant $C$ such that

$$|u_\varepsilon(x) - K(x)| \leq \frac{C\varepsilon}{|x|^2} \quad \text{if } |x| \geq 2\varepsilon. \quad (3)$$

The proof was given in the lecture.
Example 3 Let us consider the $\varepsilon \to 0$ limit of the above example: an idealized point vortex concentrated at the origin, carrying one unit of vorticity. This may be represented by a Dirac delta-function at the origin, denoted $\delta_0$, defined by

$$\int f(x)\delta_0(dx) = f(0) \quad \text{for bounded continuous } f : \mathbb{R}^2 \to \mathbb{R}.$$ 

Applying the Biot-Savart law\(^1\) to $\delta_0$, we find that the associated velocity field is

$$u_0(x) = \mathcal{K}\delta_0 = \mathcal{K} \ast \delta_0(x) = \int K(x - y)\delta_0(dy) = K(x) = \frac{x \perp}{2\pi|x|^2}.$$ 

(4)

In other words,

"The velocity field generated by an ideal point vortex is purely rotational and scales like $\frac{1}{\text{distance}}$."

\(^1\)a particularly easy instance of the convolution of a measure and a function, an operation we may recall from basic analysis.
Also, consider an approximate identity on $\mathbb{R}^2$, that is, a sequence of functions $(\omega_\varepsilon)_{\varepsilon > 0}$ on $\mathbb{R}^2$ satisfying (2) above. Then it is straightforward to check that

$$\omega_\varepsilon \to \delta_0 \text{ weakly as measures on } \mathbb{R}^2 \text{ as } \varepsilon \to 0$$

$$u_\varepsilon := K\omega_\varepsilon \to K \text{ a.e. on } \mathbb{R}^2 \text{ as } \varepsilon \to 0$$

consistent with (4). The second assertion follows immediately from (3). The first assertion means that

$$\lim_{\varepsilon \to 0} \int f(x)\omega_\varepsilon(x) \, dx = f(0).$$

Its proof is a straightforward exercise, similar in spirit to the proof of (3).
The above argument shows that if $\omega$ is a $\delta$ function, then the associated velocity field is not $L^2_{loc}$.

There is thus no way of making sense of the Euler equations for such singular vorticity.

For this reason, when studying the Euler equations, if we are interested in ideal point vortex initial data

$$\omega_\varepsilon = \sum_{i=1}^{k} \gamma_i \delta_{\rho_i}$$

(a linear combination of point vortices with circulation $\gamma_i$ at points $\rho_i$) we can only make sense of this by considering initial data satisfying assumptions such as

$$\text{supp}(\omega_\varepsilon) \subset \bigcup_{i=1}^{k} B_\varepsilon(\rho_i), \quad \int_{B_\varepsilon(\rho_i)} \omega_\varepsilon \, dx = \gamma_i,$$

$$\omega \in L^\infty, \quad \gamma_i \omega_\varepsilon(x) \geq 0 \text{ in } B_\varepsilon(\rho_i)$$
questions we will consider

Ideal situation: \( \omega^o = \sum_{i=1}^{n_i} \mathbf{v}_i \delta_{p_i} \)

"quantity \( \mathbf{v}_i \in \mathbb{R} \) of vorticity concentrated at \( p_i^o \)

We may guess

\( \omega(t) = \sum_{i=1}^{n_t} \mathbf{v}_i \delta_{p_i(t)} \)
Proved Marchenko-Pastur '92

Questions

1) Given "rigid" soln of ODE \( \otimes \) \( \exists \) a corresponding soln of (E)

2) Given arbitrary soln of \( \otimes \), \( \exists \) "nearby" soln of (E)?
Remark: What are some rigid solutions of $\mathcal{F}$?

Ex 1:

$$\omega^0 = \sum_{i=1}^2 \chi_i \delta p_i$$

$$\gamma_1 = -\gamma_2 = -1 \quad p_i = -p_i^0 = (1, 0)$$

Easy to see:

$$p_1(t) = (1, ct)$$
$$p_2(t) = (-1, ct)$$

$$c = \frac{i}{\hbar \pi}$$

Ex 2:

$$p_j(t) = \hbar e^{i(\delta t + 2\pi j \omega_0)}$$

$$j = \cdots, \cdots, h$$
We now consider velocity fields generated by the kinds of 3d configurations of concentrated vorticity that we will be interested in. Let \( \Gamma \) be a closed oriented \( C^1 \) curve in \( \mathbb{R}^3 \) of arclength \( L \), parametrized by a function \( \gamma : [0, L] \rightarrow \mathbb{R}^3 \) with periodic \( C^1 \) boundary conditions

\[
\gamma(0) = \gamma(L), \quad \gamma'(0) = \gamma'(L)
\]

and such that \(|\gamma'(s)| = 1\) everywhere.

We would like to consider an (oriented, unit strength) distribution of vorticity concentrated along \( \Gamma \).

We can consider

- idealized vortex filament – zero thickness
- vortex filament of thickness \( O(\varepsilon) \ll 1 \)
3d filament, zero thickness

Toward this end, we will write $\vec{\delta}_\Gamma$ to be the vector-valued measure defined by

$$\int_{\mathbb{R}^3} \phi(x) \cdot \vec{\delta}_\Gamma(dx) := \int_0^L \phi(\gamma(s)) \cdot \gamma'(s) \, ds \quad \text{for} \, \phi \in C^0_c(\mathbb{R}^3; \mathbb{R}^3).$$

(We may also use the notation $\int_{\mathbb{R}^3} \phi \cdot d\vec{\delta}_\Gamma = \int_{\Gamma} \phi \cdot \tau$, where $\tau$ denotes the unit tangent given by the orientation.)

The associated velocity field is

$$u(x) = \mathcal{K}\vec{\delta}_\Gamma(x) = \int_{\mathbb{R}^3} \frac{(x - y)^\#}{4\pi|x - y|^3} \times \vec{\delta}_\Gamma(dy) = \int_0^L \frac{(x - \gamma(s))^\#}{4\pi|x - \gamma(s)|^3} \times \gamma'(s) \, ds.$$  \hspace{1cm} (6)

We will later study in detail the behaviour of this velocity field at points near $\Gamma$.  \hspace{1cm} \textbf{Main point for now:} $u \notin L^2_{loc}$
3d "perfect vortex filament", thickness $O(\varepsilon)$

Fix approximate identity $(\eta_\varepsilon)_\varepsilon > 0$

$$\eta_\varepsilon(x) = \frac{1}{\varepsilon^3} \eta\left(\frac{x}{\varepsilon}\right), \quad \eta \geq 0, \quad \text{radial}$$

$\text{supp } \eta \subset B_1, \quad \int \eta \, dx = 1$

Regularize $\delta_1 \rightarrow \omega^\varepsilon = \eta_\varepsilon * \delta_1$

$$\omega^\varepsilon(x) = \eta_\varepsilon * \delta_1 \quad \Rightarrow \quad \int \omega^\varepsilon \, dx = \int \eta_\varepsilon \, dx$$
\[ \Gamma \approx \tilde{\delta}_\Gamma \]

\[ \omega^e \coloneqq \int \eta^e(x-x')K \, ds' \]

\[ = 0 \quad \text{if} \quad d(x, \Gamma) > \varepsilon \]

What velocity field?

\[ u^e_I \coloneqq X \omega^e = K \ast \eta^e \ast \tilde{\delta}_\Gamma \]

\[ = \eta^e \ast (K \ast \tilde{\delta}_\Gamma) \]

radial average

harmonic away from \( \Gamma \)

\[ = K \ast \tilde{\delta}_\Gamma = \bar{u}_r \]

if \( d(x, \Gamma) > \varepsilon \)
existence and uniqueness, 2d

We state without proof the following basic result.

**Theorem (Yudovich’s Theorem)**

For any \( \omega_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2) \), there exists \( \omega \in L^\infty([0, \infty); L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)) \) that solves the vorticity-stream formulation of the Euler equations, globally in time, in the (weak) sense that

- **setting** \( u(\cdot, t) := \mathcal{K}\omega(\cdot, t) \) for every \( t \), we have \( u(\cdot, t) \in LL(\mathbb{R}^2) \cap W^{1, p}_{loc}(\mathbb{R}^2) \) for every \( p < \infty \), and \( \nabla^\perp \cdot u(\cdot, t) = \omega(\cdot, t) \).

- for every \( T > 0 \) and every bounded \( \phi \in C^1(\mathbb{R}^2 \times [0, T]) \) the identity

\[
\left. \int_{\mathbb{R}^2} \omega \phi \, dx \right|_0^T = \int_0^T \int_{\mathbb{R}^2} \frac{D\phi}{Dt} \omega \, dx \, dt
\]

holds.
Theorem (Continuation of Yudovich’s Theorem)

Moreover

- The particle trajectory map \( \alpha \mapsto X(\alpha, t) \), solving (as usual)
  \[
  \partial_t X(\alpha, t) = u(X(\alpha, t), t), \quad X(\alpha, 0) = \alpha
  \]
  is a well-defined bijection \( \mathbb{R}^2 \to \mathbb{R}^2 \) that is area-preserving and Hölder continuous.

- The vorticity transport formula
  \[
  \omega(X(\alpha, t), t) = \omega_0(\alpha)
  \]
  holds.

The notion of weak solution in Theorem 2 is strong enough that it yields expected properties of 2d Euler flows, such as

\[
\|\omega(\cdot, t)\|_{L^p(\mathbb{R}^2)} = \|\omega_0\|_{L^p(\mathbb{R}^2)} \quad \text{for all } t > 0.
\]
First, if $X$ and $Y$ are Banach spaces (for example, spaces of functions, possibly vector-valued, defined on the same domain $\mathcal{D}$) and $f \in X \cap Y$, then we will write

$$
\|f\|_{X \cap Y} := \|f\|_X + \|f\|_Y.
$$

It is standard, and very easy to check, that this defines a norm on $X \cap Y$, and that with this norm, $X \cap Y$ is a Banach space.
Second, let $X$ be a Banach space of functions $\mathcal{D} \to \mathbb{R}^k$ or some $k \geq 1$. For $1 \leq p \leq \infty$, and $T \in (0, \infty]$, we define

$$L^p([0, T); X) := \left\{ \text{measurable } f : \mathcal{D} \times [0, T) \to \mathbb{R}^k \mid \|f\|_{L^p([0, T); X)} < \infty \right\}$$

where

$$\|f\|_{L^p([0, T); X)} := \|F\|_{L^p([0, T)} \quad \text{for} \quad F(t) := \|f(\cdot, t)\|_X.$$ 

Similarly, $C([0, T); X)$ is the space of functions that are continuous from the interval $[0, T)$ into the Banach space $X$, with the norm

$$\|f\|_{C([0, T); X)} := \sup_{t \in [0, T]} \|f(\cdot, t)\|_X.$$
A function \( f \) on a domain \( \mathcal{D} \subset \mathbb{R}^N \) is said to be log-Lipschitz if

\[
\|f\|_{LL} := \sup_{0 < |x-y| < 1} \frac{|f(x) - f(y)|}{|x - y|(1 + \log \frac{1}{|x-y|})} < \infty.
\]

We will write \( LL(\mathcal{D}) \) to denote the space of all log-Lipschitz functions on \( \mathcal{D} \).

\[
LL \subseteq C^{\alpha,2} \quad \forall \alpha < 1
\]
Key difference: \[ \| \omega \| \text{ will be preserved} \]

2D
\[ \omega_t + (\omega \cdot \nu) \omega = [1] \]

3D
\[ \omega_b + (\nu \cdot \omega) \omega = (\nu \cdot \omega) \nu \]

\[ \omega_b \]
$N \geq 3$ dimensions

In 3 and higher dimensions,

- the Euler equations on all of $\mathbb{R}^N$ are known to have smooth, decaying solutions for smooth, decaying initial data.
- but only for short times
- For example, if $u_0 \in H^m(\mathbb{R}^N)$ for $m \geq \lceil \frac{N}{2} \rceil + 2$, then the Euler equations have a solution that is guaranteed to exist for times $t \in [0, T)$, where

$$\mathcal{T} \approx \frac{1}{C_m \|u_0\|_{H^m}}.$$

- For problems we want to consider, initial data $u_{0\varepsilon}$, typically satisfies

$$\|u_{0\varepsilon}\|_{H^m} \geq C \varepsilon^{-m}.$$

Indeed, the absence of any relevant well-posedness results is an indication of the difficulty of questions about concentrated vortex filaments in 3 dimensions.
Certain classes of problems in 3 or higher dimensions admit symmetry reductions that allow them to be written as PDEs for functions of 2 variables.

The most important example is problems in 3D with cylindrical symmetry. For such problems, well-posedness results are typically available. We will discuss these as needed.
conserved quantities for 3D Euler

Let $u$ be a smooth solution of the Euler equations on $\mathbb{R}^3$ with rapid decay, and let $\omega = \nabla \times u$. The following quantities are conserved:

- total velocity flux and total vorticity flux
  \[ \int_{\mathbb{R}^3} u \, dx, \quad \int_{\mathbb{R}^3} \omega \, dx \]

- kinetic energy
  \[ \frac{1}{2} \int_{\mathbb{R}^3} |u|^2 \, dx \]

- helicity
  \[ \int_{\mathbb{R}^3} u \cdot \omega \, dx \]

- fluid impulse
  \[ \int_{\mathbb{R}^3} x \times \omega \, dx \]

- moment of fluid impulse
  \[ \int_{\mathbb{R}^3} x \times x \times \omega \, dx. \]
About fluid impulse & moment of fluid impulse:

Consider \( \phi : \mathbb{R}^3 \rightarrow \mathbb{R} \)

\[
\frac{1}{2!} \int \phi \cdot \mathbf{w} = \frac{1}{4!} \int \phi \cdot (v=0) \\
= \int \nabla \times \phi \cdot \mathbf{u}_t \\
= \int (\nabla \times \phi) \cdot \left[ \nabla \cdot (\mathbf{u} \otimes \mathbf{u}) + \frac{\partial \mathbf{u}}{\partial t} \right] \\
= \int \nabla (\nabla \times \phi) : \mathbf{u} \otimes \mathbf{u} \\
= \int \left( \nabla (\nabla \times \phi) \right) : \mathbf{u} \otimes \mathbf{u} \\
= 0 \text{ if } \nabla (\nabla \times \phi) \text{ antisymmetric} \]
what if \( \omega = \frac{\delta}{\Gamma} \) and \( u = \chi \frac{\delta}{\Gamma} \)?

**Helicity**

\[
\begin{align*}
\mathbf{H} &= \int K \times \frac{\delta}{\Gamma} \\
&= \int K(\mathbf{x} - \mathbf{y}) \frac{\delta}{\Gamma}(\mathbf{d}y) \times \frac{\delta}{\Gamma}(\mathbf{d}x) \\
&= \iint \left[ \frac{\chi(s) - \chi(s')}{|\chi(s) - \chi(s')|^3} \right] \times \chi(s) \mathbf{d}s \mathbf{d}s \\
&= \text{self-linking number } \in \mathbb{T}
\end{align*}
\]

** Fluid impulse **

\[
\begin{align*}
\mathbf{I} &= \int \mathbf{x} \times \frac{\delta}{\Gamma}(\mathbf{d}x) \\
&= \int \chi(s) \times \chi'(s) \mathbf{d}s \\
&- \int \chi(s) \times \chi'(s) \mathbf{d}s
\end{align*}
\]

E.g. \( \mathbf{g}^2 \) component

\[
\begin{align*}
\mathbf{g}^2 &= \int \left[ \chi_1(s) \chi_2'(s) - \chi_2(s) \chi_1'(s) \right] \mathbf{d}s \\
&= 2 \times (\text{signed area of Proj of } \Gamma \text{ onto } x-y \text{ plane})
\end{align*}
\]

\[
\int \mathbf{y} \times (\mathbf{x}_1 \mathbf{d}x_2 - \mathbf{x}_2 \mathbf{d}x_1)
\]
Moment of fluid impulse &
related moments

\[
\int \left( \begin{array}{c} 0 \\ 0 \\ \gamma_1 + \gamma_2 \end{array} \right) \cdot \delta_r \\
\frac{d}{ds} \left( \gamma_2 \gamma_1 \right)
\]

\[
= \int \frac{\gamma_2 \gamma_1}{\sqrt{\gamma_3(s)}} \sqrt{\gamma_3(s)} \, ds
\]

cylindrical

\[
= \int_\gamma r^2 \, ds
\]

= signed area
w/ respect to r dr dz
conserved quantities for the 2D Euler equations

Let $u$ be a smooth solution of the Euler equations on $\mathbb{R}^2$, with reasonable decay, and let $\omega = \nabla \times u$. The following quantities are independent of $t$.

- total velocity flux and total vorticity flux
  $$\int_{\mathbb{R}^2} u \, dx, \quad \int_{\mathbb{R}^2} \omega \, dx.$$ 

- kinetic energy
  $$\frac{1}{2} \int_{\mathbb{R}^2} |u|^2 \, dx$$ 

- pseudo-energy
  $$\frac{1}{2} \int_{\mathbb{R}^2} \omega (G \ast \omega) \, dx, \quad G = \frac{-\log |x|}{2\pi}, \quad G \ast \equiv (-\Delta)^{-1}.$$ 

- helicity $= 0$

- fluid impulse and its moment
  $$\frac{1}{2} \int_{\mathbb{R}^2} x^{\perp} \omega \, dx,$$ 
  $$\frac{1}{2} \int_{\mathbb{R}^2} |x|^2 \omega \, dx.$$
Energy vs Pseudo-energy

Let $\Psi = \mathcal{G} \ast \omega = (-\Delta)^{-1} \omega$

**Claim 1:** $\int |u|^2 = \int \Psi \omega$ if sufficient smoothness & decay.

**Claim 2:** If $\omega$ compact supp and $\int \omega = 0$

then $\int |u|^2 = +\infty$, $\int \Psi \omega$ finite.

**Proof of Claim 1**

$I$ by $P$:

$$\int_{B_R} |u|^2 = \int_{B_R} \mathcal{G}^2 \Psi^2$$

$$= -\int_{B_R} (\mathcal{G}^2 \Psi^2) \Psi + \int_{\partial B_R} \Psi (\mathcal{G}^2 \Psi^2)$$

$$= \int_{B_R} \omega \Psi + \int_{\partial B_R} \Psi (\omega, \omega)$$

**Proof of Claim 2**

If $\omega \in C^0$ with compact supp, say in $B_R(c)$

then

1) $u(x) = \frac{x^4}{2\pi^2lx^2} \int_{B_R} w(x) \to 0$ as $x \to 0$

2) $\Psi$ odd $\implies \int w \Psi < 0$. 
Example:

Say \( \nu \) radially, supp \( \nu \in B_e \),

\[
\int \nu \, dx = 1 \quad \text{for simplicity.}
\]

Bounds for \( \int \psi \nu \, dx \)?

We know \( u(x) = \frac{x^1}{2 \pi x^2} \) if \( \|x\| > e \)

Similarly, \( \nu(x) = G_{x} \omega = \frac{-1g(\|x\|)}{2\pi} \) if \( \|x\| > e \)

So by earlier computation, for \( R > e \),

\[
\int_{R^2} \nu \cdot \psi \, dx = \int_{\overline{B_R}} \nu \cdot \psi \, dx
\]

\[
= \int_{B_R \setminus B_{2e}} \nu_1 \cdot \psi_1 \, dx + \int_{B_{2e}} \nu_2 \cdot \psi_2 \, dx
\]

\[
= \int_{B_R \setminus B_{2e}} \frac{1}{2\pi} \, dx + \int_{B_{2e}} \frac{-1g(\|x\|)}{2\pi} \cdot \frac{1}{R} \, dx
\]

\[
= \frac{1}{2\pi} \int_{\epsilon}^{R} \frac{1}{r} \, dr + \left(-\log R \right) \frac{1}{2\pi}
\]

\[
= \left( \frac{1}{2\pi} \log \frac{1}{\epsilon} \right) \frac{1}{2\pi}
\]
Also, since \( \omega \text{ not nodal} \),

because:

1) For \( s > \epsilon, \)

\[
1 = \int_{B_s} \omega = \int_{\partial B_s} \mathbf{v} \cdot \mathbf{n} = \int_{\partial B_s} \mathbf{n} \cdot 2
\]

and

\[
\left| \int_{\partial B_s} \mathbf{n} \cdot 2 \right| \leq \left( \int_{\partial B_s} |\mathbf{n}|^2 \int_{B_s} 1 \right)^{1/2}
\]

So

\[
\int_{\partial B_s} |\mathbf{n}|^2 \geq \frac{1}{2 \pi s^2}
\]

2) \( u(x) = \frac{\chi}{2 \pi |x|^3} + O(\frac{s}{|x|^2}) \),

\[
\psi(x) = -\frac{\pi}{2 \pi} \log \frac{|x|}{\epsilon} + O(\frac{1}{|x|^3})
\]

**Conclusion:**

\[ \text{Supp} \omega \subseteq B_{s \epsilon}, \]

\[ \int_{\partial B_s} \mathbf{n} \cdot 2 \]

\[ \Rightarrow \int_{B_{s \epsilon}} \omega \psi \geq \frac{s^2}{2 \pi} \log \frac{1}{s \epsilon}. \]
Quick Intro to Gross-Pitaevskii:

\[ \psi - \Delta \psi - \frac{1}{\epsilon^2} (1-i^2 \gamma^2) \psi = 0 \]

\[ \psi : D \times [0,1] \to \mathbb{C} \]

\[ D \; \text{open} \subset \mathbb{R}^N \]

\[ \psi |_{\partial D} = 0 \quad \text{on} \quad \partial D \quad (\text{if } \neq \emptyset) \]

**Notation:** For \( v, w \in \mathbb{C} \)

\[ (v, w) = \Re (v \bar{w}) \]

\( \psi \) = wave fun for QM gas.

**Relevant quantities**

1) energy density

\[ e_\epsilon(\psi) = \frac{1}{2} \| \psi \|^2 + \frac{(1+i^2 \gamma^2 \epsilon^2)}{4 \epsilon^2} \]

2) mass density:

\[ 14^2 \]

3) momentum density = supercurrent

\[ j(\psi) = (i \psi, \bar{\psi}) \]

vector \( \psi \)

\[ j(\psi)^k = (i \psi, \bar{\psi}) \]

4) vorticity

\[ \omega(\psi) = \begin{cases} 
\frac{i}{2} \nabla \times j(\psi) & N = 3 \\
\frac{i}{2} \nabla \cdot j(\psi) & N = 2
\end{cases} \]
Fact: $N = 3$, $\psi$ smooth enough $\Rightarrow$

$$\omega(\psi) = \psi' \times \omega \psi_2$$

when $\psi = \psi' + i \psi_2$.

**Proof:**

$$\epsilon_{123} \partial_x \psi'^m = \epsilon_{123} \left[ (i \partial_x \psi \psi_n) \rightarrow 0 + \psi_1 \epsilon_{1m} \psi_n \right]$$

Now consider $k = 3$ say.

$$(i \partial_x \psi, i \psi_2) - (i \partial_x \psi, i \psi_2)$$

$$= (- \partial_1 \psi + \partial_1 \psi, \partial_2 \psi_2 + i \partial_2 \psi_2)$$

$$= (\partial_1 \psi_2 + \partial_1 \psi_1, \partial_1 \psi_1 + i \partial_1 \psi_2)$$

$$= 2(\partial_1 \psi_2 - \partial_1 \psi_1, \partial_2 \psi_1)$$

**Note:** when non-degenerate,

$$\omega(\psi) = \psi_1 \times \psi_2$$

is parallel to level curve of $\psi$.

Vortex lines are integral curves of $\omega(\psi)$

are level curves of $\psi$.

**Remark:**

No Biot-Savart
No Vorticity Transport
No (easy) pseudo-energy
Nonetheless, many analogies to Euler.

**About energy:**

1. Finite energy on $\mathbb{R}^2$:

   \[ \Psi(x) \sim \text{even} \to \Psi(x) \sim e^{i(\theta + \varphi)} \]

   \[ \text{de} x \quad \Psi \text{ single valued} \]

   \[ \text{easy to see: } \| \Psi \| = 0 \Rightarrow \int_{\mathbb{R}^2} \epsilon_{\epsilon}(x) = + \infty. \]

   So we usually consider bounded $\Psi$ in $\mathbb{R}^2$.

2. We will see:

   - Concentrated
     - Vortex regime
     \[ E_{\epsilon}(\Psi) = \int \epsilon_{\epsilon}(x) \approx \frac{1}{2} \log \frac{1}{\epsilon} \]

     \[ (N \geq 2) \]

     \[ \| 1 \| - 1 \|_2 \leq \epsilon \sqrt{\log \frac{1}{\epsilon}} \]

3. For both of (GP)

   \[ E_{\epsilon}(\Psi(x)) = \text{constant} \]
About mass

Conservation of mass is

\[ \frac{d}{dt} \frac{1}{2} \left| \mathbf{v} \right|^2 = \text{div} \mathbf{j}(\mathbf{v}). \]

Consequence:

\[ \text{div} \mathbf{j}(\mathbf{v}) = \frac{1}{2} \frac{d}{dt} \left( \left| \mathbf{v} \right|^2 - 1 \right) \]

\[ \approx 0 \left( \log \frac{1}{\varepsilon} \right) \quad \text{in} \quad H^1(d\mathbf{x}, d\mathbf{y}) \]

in point vortex concentrated regime

\[ \Rightarrow \text{incompressible (in weak sense)} \quad \text{in} \quad \varepsilon \to 0 \quad \text{limit.} \]
About momentum:

multiply \((A^p)\) by \(\psi\), rearrange... \\
\[ \Rightarrow \partial_t j(y) = 2 \nabla \cdot (\nabla \psi \otimes \psi) + \nabla (--) \]

Take \(w=1\) to find

\[ \partial_t w(y) = \nabla \times \nabla \cdot (\nabla \psi \otimes \psi) \]

compare Euler

\[ \partial_t w = \nabla \times \nabla \left( u \otimes u \right) \]

Analogy

\[ \omega(y) \sim w(x) \]
\[ j(y) \sim u \]