1. The notion of escape function

In order to characterize the spectrum of the wave operator $H$, we would like to connect spectral properties of $H$ and dynamical properties of its principal symbol $h$.

For a differential operator of order $m$

$$D = \sum_{|j| \leq m} a_j(x) \frac{\partial^j}{\partial x^j}$$

the principal symbol is

$$d = \sum_{|j| = m} a_j(x) \left(i \frac{\partial}{\partial \xi}\right)^j$$

freeze the coefficients and take the Fourier transform.
This definition can be extended to pseudo differential operators, looking at the action of $H$ on fast oscillating functions:

$$\langle H(\alpha e^{iS}), \alpha e^{iS} \rangle = \int |a(x)|^2 e^{i\theta} h(x, \xi(x)) + o(\tau^{-1})$$

The key property is that the principal symbol, even though it does not determine $H$ completely, contains almost all the information, up to smoother error terms.

- order $(\mathcal{H} - \text{Op}(h)) \leq \text{order } H$
- order $[\mathcal{H}, \mathcal{H}_2] \leq \text{order } H + \text{order } H_2 - 1$

One can therefore obtain expansions at any order.

**Example 1**

Typical operators with continuous spectrum are multiplications by smooth functions $\gamma = \gamma(x)$ (the spectrum is then the range of $\gamma$).

The Hamiltonian dynamics associated to the principal symbol $\tilde{h}(x, p) = \gamma(x)$ is given by

$$\frac{dx}{dt} = \frac{\partial h}{\partial p} = 0, \quad \frac{dp}{dt} = -\frac{\partial h}{\partial x} = -\gamma'(x_0).$$

In particular, for all trajectories, the function $|\rho(t)|$ goes to infinity as $t \to +\infty$. 
Example 2

In the case of operators with discrete spectrum, the eigenfunctions have finite energy.

Under a suitable factorization assumption on $H = u, v$, (satisfied by 2D internal waves), eigenfunctions have to be constant along the trajectories of the symbol $h$.

Both properties are compatible iff the dynamics has compact invariant sets, which rules out the existence of a function going to infinity as $t \to +\infty$.

This is of course not a proof but a strong indication that the fact that $H$ has continuous spectrum is related to the existence of an escape function $y$ of the form

$$\frac{d}{dt} \psi (x(t), p(t)) \geq \kappa > 0$$

or equivalently

$$\{ h, \psi \} \geq \kappa > 0$$

Remark: This property can be rephrased in terms of operators $i [H, A] \ni \kappa \text{Id} + K$ for some compact operator $K$.  

- Example 2
2. The conjugate operator method

Let $A$ be a self-adjoint (unbounded) operator. We define $H^3 = \{ u \in L^2 \mid (1 + A^2)^{3/2} u \in L^2 \}$. $H$ is $n$-smooth with respect to $A$ if $B_1 = i[H, A]$, $B_2 = [B_{k-1}, A]$ are bounded up to $k = n$.

$A$ is conjugate to $H$ if

$$X(H) B_1 X(H) \geq \alpha \overline{X(H)} + K$$

for $X, \overline{X} \in C^\infty$ with $\overline{X} \overline{X} = \overline{X}$.

Theorem (Mourre): for any closed interval $I \subset \text{supp} \overline{X}$

(i) $H$ has a finite set $\sigma_p(H)$ of eigenvalues in $I$

(ii) the resolvent $(H - \lambda)^{-1}$ defined for $\text{Im} \lambda \neq 0$ admits boundary values at $\lambda \in I \setminus \sigma_p(H)$ in the space $O_s = L(H^s, H^{-s})$ for $s > \frac{1}{2}$

(iii) $(H - \lambda \pm i \delta)^{-1} \in C^\infty_{\delta} (O_s)$ for $s > \frac{1}{2}$

(iv) $(H - \lambda \pm i \delta)^{-1} \in C^\infty_{\delta} (O_s)$ for $s > n - \frac{1}{2}$
Sketch of proof

- If the discrete spectrum is not finite
  \( \exists (\phi_n) \) orthonormal \( H \phi_n = \omega_n \phi_n \)
  \( \langle \phi_n, K \phi_n \rangle \to 0 \) by compactness of \( K \)
  \( \langle \phi_n, K \phi_n \rangle \leq -\kappa \Vert \phi_n \Vert^2 \)

\( \text{contradiction} \)

- Define the approximation \( G_\varepsilon = (H - z - i\varepsilon P B_\varepsilon P)^{-1} \)
  and \( F_\varepsilon = A^{-1} G_\varepsilon A^{-1} \)

\[
\||P G_\varepsilon A^{-1}||^2 = \frac{1}{\varepsilon^2} \|A^{-1}(G_\varepsilon - G_\varepsilon) A^{-1}\| \leq \frac{1}{\varepsilon^2} \| F_\varepsilon \|
\]

\( \| (I - P) G_\varepsilon A^{-1} \| \leq C \)

- \( \frac{dF_\varepsilon}{d\varepsilon} = A^{-1} G_\varepsilon iPB_\varepsilon P G_\varepsilon A^{-1} \)

\[
= A^{-1} [G_\varepsilon, A] A^{-1} + \varepsilon A^{-1} G_\varepsilon [A, iPB_\varepsilon P] G_\varepsilon A^{-1} + A^{-1} G_\varepsilon (P - I) [A, H] P G_\varepsilon A^{-1} + A^{-1} G_\varepsilon [A, H^2] (P - I) G_\varepsilon A^{-1}
\]

main term

\[
\| \frac{dF_\varepsilon}{d\varepsilon} \| \leq C \left( 1 + \frac{\|F_\varepsilon\|^{1/2}}{\varepsilon^{1/2}} + \|F_\varepsilon\| \right)
\]

- Integrating this differential inequality, we obtain uniform bounds as \( \varepsilon \to 0 \), so that \( G_\varepsilon \)
  has boundary values in \( O_3 = L(H^4, H^4) \)
• To obtain the Hölder continuity, we combine

\[
\begin{align*}
    \| F_\varepsilon(z) - F_\varepsilon(0) \| & \leq C \varepsilon \\
    \| F_\varepsilon(z) - F_\varepsilon(z') \| & \leq \frac{dF_\varepsilon}{dz} \| z-z' \| \leq \frac{C}{\varepsilon} |z-z'| 
\end{align*}
\]

Remarks: to get the fractional Sobolev regularity, one has to replace $A^{-1}$ by $A^{-s}(i+\varepsilon A)^{-s-1}$.

• To obtain additional regularity w.r.t. $\lambda$ a better approximation is needed.

3 Construction of the conjugate operator

We will consider a general (scalar) equation of the form $\frac{d}{dt} u - iH u = f \exp(i\omega t)$ on $T^2$

(H0) $H$ is a pseudodifferential operator with smooth principal symbol homogeneous of degree 0 and vanishing subprincipal symbol.

(H1) $\Sigma_{\omega_0} = k^{-1}(\omega_0)$ is non degenerate \( \pi : Z = \partial \Sigma_{\omega_0} \rightarrow X \) is a finite covering of degree
\(X_h\) induces a field of oriented directions on \(Z\). We therefore have a foliation \(F\) on \(Z\) (non-singular by (H1)). The leaves are the orbits of the Hamiltonian dynamics.

(H2) \(F\) is finite simple; i.e. \(F\) has a finite number of compact leaves, which are hyperbolic. And all other leaves accumulate only along these compact leaves at \(\infty\).

Remark: this model reproduces the main features of the internal wave operator, but requires more regularity.

---

**Normal form**

Let \(B_f\) be the basin of attraction (resp. repulsion) of the cycle \(f\). Denote by \(S\) a local Poincaré section transverse to \(f\) and by \(\Pi : S \to S\) the Poincaré return map.

1. By Sternberg’s theorem, \(\Pi(y) = \mu y\) for a good choice of \(\mu\).
2. Choose a vector field tangent to the foliation such that the return time is \(2\Pi\). Denote by \(x\) the coordinate along the flow.
3. The foliation \(F_0\) is given by \(dy + \delta y\, dx = 0\), it differs from \(F\) only by a reparameterization.
4. The normal form is extended on \(B_f\) by scattering.
Construction of the escape function

We define first a local escape function \( \psi_y \) on \( B_r \) let \( y \in \mathbb{C} \) such that \( y = 1 \) in \([-1, 1] \) and \( y \psi'(y) \leq 0 \). Define \( \psi_y = \lambda \psi(y) \)
\[ \{ \Phi^2 h_o, \psi_y \} = \Phi^2 \{ h_o, \psi_y \} + h_o \{ \phi, \psi_y \} = \lambda^2 \Phi^2 (y - y \psi') > \alpha_o \text{ on } \Gamma_{rk} \]

From (H2), we know that \( \Sigma_{w_o} \subset \bigcup \Gamma_{rk} \).
Extracting a finite covering, and adding the local escape functions, we end up with a global escape function.
**Construction of the conjugate operator**

We first extend \( y \) as a smooth function homogeneous of degree 1 on \( T^*X \) which satisfies \( \{ h, y \} \geq \alpha > 0 \) in some conical neighborhood of \( \Sigma_{w_0} \).

Then we choose a self-adjoint operator \( A \) of principal symbol \( y \). By the sharp Garding inequality \( \langle (H)[H,A]X(H) \rangle \geq \|X(H)\|_k \) we can apply Howe's theory and define the resolvents \( (H - \lambda \geq 0)^{-1} \) for \( \lambda \) close to \( w_0 \).