2D Navier-Stokes Equations with Large Reynolds Number

Yan Guo
Division of Applied Math, Brown University
In honor of Cedric Villani

November 18, 2014
2D Navier Stokes Equations: \( \mathbf{u} = [u, \nu] \),

\[
\begin{align*}
\mathbf{u}_t + \mathbf{u} \cdot \nabla_x \mathbf{u} + \nabla \rho &= \nu \Delta \mathbf{u} \\
\nabla \cdot \mathbf{u} &= 0
\end{align*}
\]
Navier-Stokes Equation with Large Reynolds Number

- 2D Navier Stokes Equations: \( \mathbf{u} = [u, v] \),

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- \( \nu = \frac{1}{R} \ll 1 \), Reynolds Number \( R(\frac{LV}{\nu}) \), viscosity, inviscid limit is the Euler equations.

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- Instability of Poiseuille Channel flow: \( x \in \mathbb{R}, 0 \leq y \leq 2 \):

\[
u(y) = U(y) = 1 - (y - 1)^2, \quad \nu \equiv 0, \quad p = 2\nu x
\]
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- Validity of Steady Prandtl Layer Expansion ($\mathbf{u}_t \equiv 0$)
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- Validity of Steady Prandtl Layer Expansion \( (\mathbf{u}_t \equiv 0) \)

- Non-slip and Non-penetration BC
With $\nu = 0$, an example of stable shear flows for Euler:

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with non-penetration BC at $y = 0$ and $y = 2$:

$$v(x, 0) = v(x, 2) \equiv 0.$$
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Poiseuille flow is a rare example of steady NS for $\nu << 1$. 
Orr-Sommerfeld and Rayleigh Equations

- Long History: Reynolds (1883), Heisenberg (1924), Tollmien (1929), C.-C. Lin (1944), Drazin and Reid (1981), Wasaw (1948), many numerical and experimental work. This instability is believed to link to turbulence for $\nu \ll 1$.
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- Many delicate asymptotic expansions are developed, in different spatial regime in $x$. No rigorous construction of a growing modes.

\[ \psi(t, x, y) = e^{i\alpha(xct)} \phi(y), \]

\[ \varepsilon(\partial^2_y \alpha^2)\phi = (Uc)(\partial^2_y \alpha^2)\phi U_0 \phi, \]

\[ \varepsilon = \frac{i\alpha}{R}, \quad \alpha \text{ real wave number}, \quad c \text{ complex, and } \Re c \alpha > 0 \text{ is the exponential growth rate, the no-slip BC: } \phi(0) = \phi(y(0)) = \phi(2) = \phi(y(2)) = 0. \]

Is there a solution for $\varepsilon \ll 1, \alpha > 0, \Re c < 0$?

4th order ODE, non self-adjoint, no variational structure. For $\varepsilon = 0$, Rayleigh equation with stability. Bifurcation to instability for $\varepsilon \ll 1$. 

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- Assume the growing stream function $\psi(t, x, y) = e^{i\alpha(x - ct)}\phi(y)$, we obtain ODE in $y$

$$\varepsilon(\partial_y^2 - \alpha^2)^2 \phi = (U - c)(\partial_y^2 - \alpha^2)\phi - U''\phi,$$

where $\varepsilon = \frac{1}{i\alpha R}$, $\alpha$ real wave number, $c$ is complex, and $\text{Re}\{c\alpha\} > 0$ is the exponential growth rate, the no-slip BC:

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  $$\phi(0) = \phi_y(0) = \phi(2) = \phi_y(2) = 0.$$
- Is there a solution for $\varepsilon << 1$, $\alpha > 0$, $\text{Re}\ c < 0$?
- 4th order ODE, non self-adjoint, no variational structure. For $\varepsilon = 0$, Rayleigh equation with stability. Bifurcation to instability for $\varepsilon << 1$. 
Theorem (Grenier, G., Nguyen, 2014)

There exists a $0 < \varepsilon_0 << 1$, such that for all $0 < \varepsilon < \varepsilon_0$, there exists a growing mode solution with $\alpha(\varepsilon) > 0$ and $c(\varepsilon)$ such that

$$R^{-1/7} \lesssim \alpha(\varepsilon) \lesssim R^{-1/11},$$

$$\alpha(\varepsilon) \Im c(\varepsilon) \lesssim \frac{1}{\sqrt{\alpha(\varepsilon) R}} > 0.$$
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- These asymptotes are predicted by formal expansions.
- The instability occurs for large wave number $\alpha(\varepsilon)$, but if $R \sim 10^7$ (Reynolds number $10^7$), $\alpha(\varepsilon) \sim 1$. Inverse cascade?
- Our proof is constructive, no stability is known for the wave number $\alpha$ outside $[R^{-1/7}, R^{-1/11}]$.
- Our result has been extended to other general ‘shear flow’ profiles [Grenier, G., Nguyen, 2014], in particular, the Blasius boundary-layer profile $U(y) = f'(y)$ for $0 \leq y < +\infty$, where

\[
f''' + ff'' = 0
\]
Given three parameters $\varepsilon$, $c$ and $\alpha$, our goal is to represent a fundamental solution set $[\phi_1, \phi_2, \phi_3, \phi_4]$ to the Orr-Sommerfeld equation. We build starting with $\varepsilon = 0$, Rayleigh equations

$$\text{Ray}(\phi) \equiv (U - c)(\partial_y^2 - \alpha^2)\phi - U''\phi.$$ 

We further start with $\alpha = 0$, with two explicit solutions $U(y) - c$, and the singular one

$$(U(y) - c) \int_{1/2}^{y} \frac{1}{(U(y) - c)^2} dy$$

with logarithmic singularity $(y - z_c) \ln |y - z_c|$ at $z_c \sim c$ such that $U(z_c) - c = 0$. 

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Fundamental Solution Set to Rayleigh

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$$U(z_c) - c = 0.$$  

- To capture such a singularity, we define bounded function space $Y_p$

$$Y_p = \{ |\partial_y f(y)| \lesssim 1 + \ln |y - z_c|, \quad |\partial_y^l f(y)| \lesssim 1 + |y - z_c|^{1-l} \},$$

$$X_p = \{ |\partial_y^l f(y)| \lesssim |y - z_c|^{-l} \}.$$
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- We construct $\text{Ray}^{-1}$ by expansion of $\alpha \ll 1$ in $Y_p$. 

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For given $c, \alpha, \epsilon$, our goal is to construct $[\phi_i]_{i=1,2,3,4}$

$$\text{Orr}(\phi) \equiv (U - c) (\partial_y^2 - \alpha^2) \phi - U'' \phi - \epsilon (\partial_y^2 - \alpha^2)^2 \phi.$$

then to fit them to the boundary conditions. Denote

- $\text{Ray}(\phi) \equiv (U - c) (\partial_y^2 - \alpha^2) \phi - U'' \phi,$
- $\text{Diff}(\phi) \equiv -\epsilon (\partial_y^2 - \alpha^2)^2 \phi,$
- $\text{Airy}(\phi) \equiv \epsilon \partial_y^4 \phi - (U - c + 2\epsilon \alpha^2) \partial_y^2 \phi,$
- $\text{Reg}(\phi) \equiv -[\epsilon \alpha^4 + U'' + \alpha^2 (U - c)] \phi,$

$\text{Orr} = \text{Ray} + \text{Diff} = -\text{Airy} + \text{Reg}$

We start with $\phi_0$ such that $\text{Ray}(\phi_0) = 0$. Next,

- $\text{Ray}(\phi_r) = -\text{Orr}(\phi_0),$
- $\text{Orr}(\phi_0 + \phi_r) = \text{Orr}(\phi_0) + \text{Ray}(\phi_r) + \text{Diff}(\phi_r) = \text{Diff}(\phi_r),$
- $\text{Airy}(\phi_a) = \text{Diff}(\phi_r).$
Two Solutions From Rayleigh to OS

- From \([\phi_0, E_0]\) to \([\phi_1, E_1]\). Let \(E_0 = \text{Orr}(\phi_0)\), then

\[
\begin{align*}
\phi_1 &= \phi_0 + \phi_r + \phi_a, \\
E_1 &= \text{Orr}(\phi_1) = \text{Orr}(\phi_a) + \text{Orr}(\phi_r + \phi_0) \\
&= -\text{Airy}(\phi_a) + \text{Reg}(\phi_a) + \text{Diff}(\phi_r) = \text{Reg}(\phi_a) \\
&= \text{Reg} \circ \text{Airy}^{-1} \circ \text{Diff} \circ \text{Ray}^{-1}(E_0)
\end{align*}
\]

It is important to show that the operator \(\text{Reg} \circ \text{Airy}^{-1} \circ \text{Diff} \circ \text{Ray}^{-1}\) has norm < 1.
From $[\phi_0, E_0]$ to $[\phi_1, E_1]$. Let $E_0 = \text{Orr}(\phi_0)$, then

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If we start with Rayleigh solution $\text{Reg}(\phi_0) = 0$, then we show that

$$
norm{\text{Airy}^{-1} \circ \text{Diff}(f)}_{X_2} \lesssim \norm{f}_{Y_4} \delta^2 (1 + |\ln \delta|) (1 + |z_c / \delta|)^{3/2}, \quad \delta \sim \varepsilon^{1/3}.
$$
Two Solutions From Airy to OS

Via a sequence of translation and scaling (Langer), near $U(z_c) - c = 0$ or $y \sim z_c$,

$$
\text{Airy}(\phi) \equiv \epsilon \partial_y^4 \phi - (U - c + 2\epsilon \alpha^2) \partial_y^2 \phi \sim \partial_z^4 \phi - z \partial_z^2 \phi = 0
$$

for localized data (critical layer)

$$
\frac{y - z_c}{\epsilon^{1/3}} \sim \xi.
$$

Take $\text{Airy}(\phi_0) = 0$, then $\phi_0$ is highly localized near $z_c$.

$E_0 = \text{Orr}(\phi_0) = \text{Reg}(\phi_0)$ is highly localized near $z_c$. Then construct

$$
\phi_1 \equiv \phi_0 - \text{Airy}^{-1}(E_0), \quad \text{then}
$$

$$
E_1 \equiv \text{Orr}(\phi_1) = \text{Orr}(\phi_0) - \text{Orr} \circ \text{Airy}^{-1}(E_0) = \text{Reg} \circ \text{Airy}^{-1}(E_0).
$$

We have the crucial estimate for ‘localized data’ $E_1$ such that

$$
|\text{Airy}^{-1}(E_0)| \leq \delta |E_1|.
$$

• Repeat the process in $X_2$, then estimate $\phi, \partial_y \phi$ at $y = 0$ with $z_c^{-1}$. 
Consider the 2D steady Navier-Stokes equations for \(0 \leq X \leq L, 0 \leq Y \leq \infty\)

\[
UU_X + VU_Y + P_X = \varepsilon U_{XX} + \varepsilon U_{YY} \\
UV_X + VV_X + P_Y = \varepsilon V_{XX} + \varepsilon V_{YY}
\]

with \(U_X + V_Y = 0\) and non-slip boundary condition at \(Y = 0\)

\[
U(X, 0) = u_b > 0 \text{ (moving)}, \quad V(X, 0) = 0.
\]

We would like to understand when \(\varepsilon \to 0\), the relation to the inviscid Euler Equations. We assume that the outer Euler flow is

\[
[u_e^0(Y), 0]
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where

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0 < u_e^0(0) = u_e \neq u_b.
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Steady Prandtl Layer Expansion

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1905, L. Prandtl resolves this mismatch by constructing a boundary solutions of width \(\sqrt{\nu}\). We reformulate his theory by letting \(x = X, y = \frac{Y}{\sqrt{\nu}}\).
New 2D Navier-Stokes system

\[ \begin{align*}
UU_x + VU_y + P_x &= U_{yy} + \nu U_{xx} \\
VV_x + VV_y + \frac{P_y}{\nu} &= V_{yy} + \nu V_{xx}
\end{align*} \]

Prandtl hypothesizes as \( \nu \ll 1 \)

\[
[U, V] \text{ (NS)} \sim [u_e^0(\sqrt{\nu}y), 0](\text{Euler}) + [u(x, y), \nu(x, y)]\text{(Prandtl Layer)}
\]

where Prandtl Layer equations is a parabolic equation \( u_x + \nu_y = 0 \),

\[
[u_e(0) + u]u_x + \nu v_y + p_x = u_{yy}
\]
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- Prandtl Theory is regarded as one of the most important achievements in modern fluid mechanics, which connects the Euler theory and Navier-Stokes’ theory. Prandtl’s equation is a parabolic equation which is much easier to solve numerically. Prandtl’s analysis has broken new ground for asymptotic analysis.
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The validity and error estimate of the steady Prandtl expansion is an important open question.
We expand further the Prandtl’s expansion to get the remainder

\[
\begin{align*}
U &= u_e^0(\sqrt{\nu}y) + u_p^0(x, y) + \sqrt{\nu}u_e^1(x, \sqrt{\nu}y) + \sqrt{\nu}u_p^1(x, y) + \varepsilon^{\gamma + \frac{1}{2}}u^\varepsilon(x, y), \\
V &= v_p^0(x, y) + \sqrt{\nu}v_e^1(x, \sqrt{\nu}y) + \sqrt{\nu}v_p^1(x, y) + \varepsilon^{\gamma + \frac{1}{2}}v^\varepsilon(x, y), \\
P &= \sqrt{\nu}p_e^1(x, \sqrt{\nu}y) + \nu p_p^1(x, y) + \nu^{\gamma + \frac{1}{2}}p^\varepsilon(x, y).
\end{align*}
\]

where \([u_e^i, v_e^i, p_e^i] \ i = 1, 2\), are unknown with constructed ‘approximate profiles’. They can be constructed under rather general conditions.

\[
\begin{align*}
\Delta u_e^1 - u_{e y y}^1 &= 0, \ (\text{1st Euler}) \\
[u_e(0) + u]u_{p x}^1 + v_p^1 v_{p y}^0 + v_p^0 v_{p y}^1 + p_x &= \text{given. (1st Prandtl)}
\end{align*}
\]
Theorem (G., Nguyen, 2014)

Assume

\[
[u^\varepsilon, v^\varepsilon]_{y=0} = 0, \quad [u^\varepsilon, v^\varepsilon]_{x=0} = 0, \\
p^\varepsilon - 2\nu u^\varepsilon_x = 0, \quad u^\varepsilon_y + \nu v^\varepsilon_x = 0 \text{ at } x = L.
\]

Let \( \nabla_\varepsilon = [\sqrt{\nu}\partial_x, \partial_y] \). Assume

\[
\min_{0 \leq y \leq \infty} \{ u^0_e (\sqrt{\nu} y) + u^0_p (y) \} > 0
\]

then there exists \( L > 0 \), such that for some \( \gamma > 0 \)

\[
\| \nabla_\nu u^\varepsilon \|_2 + \sqrt{\nu} \| \nabla_\nu v^\varepsilon \|_2 + \nu^{\frac{\gamma}{2}} \| u^\varepsilon \|_\infty + \nu^{\frac{1}{2} + \frac{\gamma}{2}} \| v^\varepsilon \|_\infty \lesssim 1.
\]

In particular, in terms of \( X \) and \( Y \),

\[
\| U(\cdot) - u^0_e(\cdot) - u^0_p(\cdot, \frac{\cdot}{\sqrt{\nu}}) \|_\infty \lesssim \sqrt{\nu} \\
\| V(\cdot) - \sqrt{\nu} v^0_p(\cdot, \frac{\cdot}{\sqrt{\nu}}) - \sqrt{\nu} v^1_e(\cdot) \|_\infty \lesssim \nu^{\frac{1}{2} + \gamma}.
\]
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- $L \downarrow 0$ as $\min_{0 \leq y \leq \infty} \left\{ u_e^0(\sqrt{\nu}y) + u_p^0(y) \right\} \to 0$, so the classical motionless boundary is open, $L \nearrow$ for $|\nabla u_e^0| + |\nabla u_p^0| \searrow$. 
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It is well-known that energy estimates leads to

$$\| \nabla u^\epsilon \|^2_2 + \nu \| \nabla v^\epsilon \|^2_2 \lesssim \int v^\epsilon \partial_y u_s u^\epsilon + ... \lesssim L \| \nabla v^\epsilon \|^2_2 + \text{good},$$

for which $\| \nabla v^\epsilon \|^2_2$ is out of control. The following new estimate is found:

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$L \downarrow 0$ as \( \min_{0 \leq y \leq \infty} \{ u_{e}^{0}(\sqrt{\nu} y) + u_{p}^{0}(y) \} \rightarrow 0 \), so the classical motionless boundary is open, $L \nearrow$ for $|\nabla u_{e}^{0}| + |\nabla u_{p}^{0}| \searrow$.

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It is well-known that energy estimates lead to

\[
\| \nabla_{\nu} u^{\varepsilon} \|_{2}^{2} + \nu \| \nabla_{\nu} \nu^{\varepsilon} \|_{2}^{2} \lesssim \int \nu^{\varepsilon} \partial_{y} u_{s} u^{\varepsilon} + \ldots \lesssim L \| \nabla_{\nu} \nu^{\varepsilon} \|_{2}^{2} + \text{good},
\]

for which $\| \nabla_{\nu} \nu^{\varepsilon} \|_{2}^{2}$ is out of control. The following new estimate is found:

\[
\| \nabla_{\nu} \nu^{\varepsilon} \|_{2}^{2} \lesssim \| \nabla_{\nu} u^{\varepsilon} \|_{2}^{2} + \text{good}.
\]

The conditions at $x = 0$ and $x = L$ are conveniently chosen.