On the Almost Axisymmetric Flows with Forcing Terms

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Outline

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- A Toy Model.
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▶ Analysis of the Hamiltonian of Almost Axisymmetric Flows.

▶ A Toy Model.

▶ Challenges in the study of the Almost Axisymmetric Flows with Forcing Terms.
Time varying domain.

The time varying domain occupied by the fluid is given by

\[ \Gamma_{r_1^t} := \{(\lambda, r, z) \mid r_0 \leq r \leq r_1^t(\lambda, z), \; z \in [0, H], \; \lambda \in [0, 2\pi]\}, \]

For simplicity, we set \( r_0 = 1 \) in the sequel.
Hamiltonian

The fluid evolves with the velocity \( \mathbf{u} := \mathbf{u}(\lambda, r, z) \) expressed in cylindrical coordinates \((u, v, w)\). The temperature \( \theta \) of the fluid inside the vortex is assumed to be greater that the ambient temperature maintained constant at \( \theta_0 > 0 \).

g is the gravitational constant.

The Hamiltonian of the Almost Axisymmetric Flow is

\[ \int_{\Gamma_{r_1}} \left( \frac{u^2}{2} - g \frac{\theta}{\theta_0} \right) r dr dz d\lambda. \]

Important: The Almost Axisymmetric Flows are derived from Boussinesq’s equations with no loss of the Hamiltonian structure (George Craig).
Hamiltonian : Stable Almost axisymmetric flows

Ω : Coriolis coefficient.

\( ru + \Omega r^2 \) : angular momentum

\( \frac{g}{\theta_0} \theta \) : potential temperature.

**Stability condition:**

On each \( \lambda \)-section of the domain \( \Gamma_{r_1} \), we require that

\[
(r, z) \rightarrow [(ru^\lambda + \Omega r^2)^2, \frac{g}{\theta_0} \theta^\lambda]
\]

be invertible and gradient of a convex function.
Hamiltonian: Stable Almost axisymmetric flows

We made crucial observation that, for stable Almost axisymmetric flows for which the total mass is finite (=1), the Hamiltonian can be expressed in terms of one single measure $\sigma$:

$$\mathcal{H}[\sigma] = \int_0^{2\pi} l_0[\sigma, \lambda] + \inf_{\rho \in S} I[\sigma, \lambda](\rho) d\lambda$$

Here, $\sigma$ is a probability measure such that $\pi_1^{\#}\sigma$ is absolutely continuous with respect to $\mathcal{L}^1_{|[0,2\pi]}$.

$$l_0[\sigma, \lambda] = \int_{\mathbb{R}^2_+} \left( \frac{y_1}{2} - \Omega \sqrt{y_1} - \frac{|y|^2}{2} \right) \sigma, \lambda(dy)$$

$$I[\sigma, \lambda](\rho) := \frac{1}{2} W^2_2\left(\sigma, \frac{1}{(1 - 2x_1)^2} \chi_{D_\rho}(x)\right) + \int_{D_\rho} \left( \frac{\Omega^2}{2(1 - 2x_1)} - \frac{|x|^2}{2} \right) \frac{1}{(1 - 2x_1)^2} dx$$

Here, $S$ is the set of functions $\rho : [0, H] \rightarrow [0, 1/2)$,

$$D_\rho := \{x = (x_1, x_2) \mid x_1 \in [0, H], 0 \leq x_2 \leq \rho(x_1)\}$$
Assume $\sigma_0$ is a probability measure on $\mathbb{R}^2$ and write

$$I[\sigma_0](\rho) = \frac{1}{2} W_2^2(\sigma_0, \frac{1}{(1 - 2x_1)^2} \chi_{D\rho}(x)) + \text{good terms}$$

**Existence of a minimizer.**

Obstacle : $\{\chi_{D\rho}\}_{\rho \in S}$ is not weakly* closed in $L^\infty$. 
Analysis of the Hamiltonian

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**Existence of a minimizer.**

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However,

$$I[\sigma_0](\hat{\rho}) \leq I[\sigma_0](\rho)$$

where $\hat{\rho}$ is the increasingly monotone rearrangement of $\rho$.

Classical results in the direct methods of the calculus of variations ensures the existence of a minimizer.
Analysis of the Hamiltonian

Uniqueness of minimizers.
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**Uniqueness of minimizers.**
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Analysis of the Hamiltonian

**Uniqueness of minimizers.**
Obstacle: No convexity property for $\rho \rightarrow I[\sigma_0](\rho)$ with respect to any interpolation we can think of.

We use a Dual formulation of the minimization problem that yields existence and uniqueness.

\begin{equation}
\sup \left\{ (P, \psi) : P = \psi^*, \psi = P^* \right\} \int_{\mathbb{R}^2} \left( \frac{y_1}{2} - \Omega \sqrt{y_1} - \psi(y) \right) \sigma_0(\text{d}y) + \inf_{\rho \in \mathcal{S}} \int_0^H \Pi_P(\rho(x_2), x_2) \text{d}x_2
\end{equation}

\begin{equation}
\Pi_P(x_1, s) = \int_0^s \left( \frac{1}{2(1 - 2x_1)} - P(x_2, x_1) \right) \frac{1}{(1 - 2x_2)^2} \text{d}x_1 \quad \text{for} \quad 0 \leq x_1 < 1.
\end{equation}

(1) has a unique solution.
Analysis of the Hamiltonian.

Regularity of the boundary $\partial D_\rho$

The dual problem reveals a regularity property of $\rho$ stronger than monotonicity.

More precisely, if $\text{spt}(\sigma_0) \subset (\frac{1}{L_0}, L_0) \times (0, L_0)$ $L_0 > 0$ and $P^{\sigma_0}$ solve the variational problem (1) then the study of Euler-Lagrange equation of

$$\inf_{\rho \in \mathcal{S}} \int_0^H \Pi_{P^{\sigma_0}}(\rho(x_2), x_2) dx_2$$

yields $C > 0$ such that the minimizer $\rho^{\sigma_0}$ satisfies

$$\rho^{\sigma_0}(\bar{x}_2) - \rho^{\sigma_0}(x_2) \geq C(\bar{x}_2 - x_2)$$

for all $x_2, \bar{x}_2 \in [0, H]$. Consequently, we obtain that $\partial D_{\rho^{\sigma_0}}$ is piecewise Lipschitz continuous.
A unusual Monge-Ampère equation.

Moreover, assume in addition, $\sigma_0$ is absolutely continuous with respect to the Lebesgue measure.

If $(P^{\sigma_0}, \Psi^{\sigma_0}, \rho^{\sigma_0})$ is the variational solution (1) then $P^{\sigma_0}$ is convex, $\nabla P^{\sigma_0}$ is invertible $(1 - 2x_1)^{-2} \chi_{D_\rho}(x) L^2$ a.e and

$$
\begin{cases}
(i) & \frac{1}{(1-2\partial y_2 \psi)^2} \det \nabla^2 \Psi = \sigma_0 \\
(ii) & P\left(\rho(x_2), x_2\right) = \frac{\Omega^2}{2(1-2\rho(x_2))} \text{ on } \{\rho > 0\} \quad (2) \\
(iii) & \nabla \psi \text{ maps } spt(\sigma_0) \text{ onto } D_\rho.
\end{cases}
$$
Change of variables

Let \((P_\lambda, \Psi_\lambda, \rho_\lambda)\) be the solution to the variational problem (1) corresponding to \(\sigma_\lambda\). Assume \(\sigma\) absolutely continuous with respect to Lebesgue.

Define \(u, \theta, r\) through

\[
(u_\lambda r + \Omega r^2)^2 = \partial_{x_1} P_\lambda, \quad g \frac{\theta_\lambda}{\theta_0} = \partial_{x_2} P_\lambda, \quad 2x_1 = 1 - r^{-2}. \tag{3}
\]

and

\[
\chi_{r_1} r dr dz d\lambda = (1 - 2x_1)^{-2} \chi_{\rho_\lambda} (x) dx_1 dx_2 d\lambda = \sigma dy_1 dy_2 d\lambda.
\]

Then, \((u, \theta, r_1)\) satisfy the stability condition and

\[
\mathcal{H}[\sigma] = \int_{r_1} \left( \frac{u^2}{2} - g \frac{\theta}{\theta_0} \right) r d\lambda dr dz.
\]
Forced Axisymmetric Flows: Toy Model 2D

We remove the $\lambda$ dependence on the quantities involved in the Almost axisymmetric flows with forcing terms to obtain the forced axisymmetric flows: \( \frac{D}{Dt} := \partial_t + v \partial r + w \partial z \).

\[
\begin{cases}
(r u + \Omega r^2)^2 = r^3 \partial_r [\varphi + \frac{\Omega^2}{2} r^2], & \frac{g}{\theta_0} \theta = \partial_z [\varphi + \frac{\Omega^2}{2} r^2] \quad \text{in } \Gamma_{r_1} \\
\frac{1}{r} \partial_r (rv) + \partial_z w = 0 \quad & \text{in } \Gamma_{r_1} \\
\partial_t r_1 + w \partial_z r_1 = v, \quad & \text{in } \Gamma_{r_1} \\
\frac{D}{Dt} (ru + \Omega r^2) = F, \quad \frac{D}{Dt} \left( \frac{g}{\theta_0} \theta \right) = \frac{g}{\theta_0} S \quad & \text{in } \Gamma_{r_1}
\end{cases}
\]

Here,
\[
\Gamma_{r_1} := \{(r, z) \mid r_1(t, z) \geq r \geq r_0, \ z \in [0, H]\},
\]
\[
\varphi(t, r_1(t, z), z) = 0, \quad \text{on } \partial \{r_1 > r_0\}.
\]

Neumann condition has been imposed on the rigid boundary.

Data: \( F, S \) are prescribed functions.

Unknown: \( u, v, w, \varphi, \theta \) and \( r_1 \)
In view of the change of variable discussed above, existence of a variational solution to the MA equation, formal computations yield

Toy Model $\iff$ 

$$
\begin{cases}
\partial_t \sigma_t + \text{div}(\sigma_t V_t[\sigma_t]) = 0 \\
\sigma|_{t=0} = \bar{\sigma}_0
\end{cases}
$$
In view of the change of variable discussed above, existence of a variational solution to the MA equation, formal computations yield

\[
\text{Toy Model} \iff \begin{cases} 
\partial_t \sigma_t + \text{div}(\sigma_t V_t[\sigma_t]) = 0 \\
\sigma|_{t=0} = \bar{\sigma}_0
\end{cases}
\]

**Task we completed:**

Identify the operator \(\sigma \mapsto V_t[\sigma]\).
Forced axisymmetric flows: Velocity field

Regular initial data:

\[ V_t[\sigma](y) = \mathbb{L}_t(\nabla \psi^\sigma(y); y) \]

where

\[ \mathbb{L}_t(x; y) = \left( 2\sqrt{y_1}F_t((1 - 2x_1)^{-\frac{1}{2}}, x_2), \frac{g}{\theta_0}S_t((1 - 2x_1)^{-\frac{1}{2}}, x_2) \right). \]

and

\[ \psi^\sigma \] is a solution in the variational problem (1).

General initial data:

Use the Riesz representation theorem to uniquely define \( V_t[\sigma] \) by

\[ \int_{\mathbb{R}^2} \langle V_t[\sigma], G \rangle d\sigma = \int_{D_{\rho^\sigma}} e(x_1)\langle \mathbb{L}_t(x, \nabla P^\sigma), G(\nabla P^\sigma) \rangle dx_1 dx_2 \]

\( \forall G \in C_c(\mathbb{R}^2, \mathbb{R}^2) \) and \((P^\sigma, \rho^\sigma)\) solves the variational problem (1).
Existence of solutions for the Forced axisymmetric flows.

- Appropriate conditions of the forcing terms.
- Continuity property in $\sigma \rightarrow V_t[\sigma]$ (and $\sigma \rightarrow \sigma V_t[\sigma]$).

$\Rightarrow$ Global solution in time.
Almost Axisymmetric Flow with Forcing Terms

Back to the full physical model

These equations are given by (here, $\frac{D}{Dt} := \partial_t + \frac{u}{r} \partial_{\lambda} + v \partial r + w \partial z$)

\[
\begin{align*}
    r \left( \frac{Du}{Dt} + \frac{uv}{r} + \frac{1}{r} \partial_{\lambda} \varphi + 2\Omega v \right) &= F, \\
    \frac{u^2}{r} + 2\Omega u &= \partial_r \varphi, \\
    \frac{D\theta}{Dt} &= S, \\
    \frac{1}{r} \partial_r (rv) + \frac{1}{r} \partial_{\lambda} u + \partial_z w &= 0, \\
    \partial_z \varphi - g \frac{\theta}{\theta_0} &= 0, \\
    \partial_t r_1 + \frac{u}{r_1} \partial_{\lambda} r_1 + w \partial_z r_1 &= \nu \text{ on } \{r = r_1\}
\end{align*}
\]

in the region

\[\Gamma_{r_1} := \{(\lambda, r, z) \mid r_1(\lambda, z) \geq r \geq r_0, \; z \in [0, H], \; \lambda \in [0, 2\pi]\},\]

subject to the boundary condition

\[\varphi(t, \lambda, r_1(t, \lambda, z), z) = 0, \; \text{on} \; \partial\{r_1 > r_0\}.\]  \hspace{1cm} (7)

Neumann condition has been imposed on the rigid boundary.
Almost axisymmetric Flow with Forcing Terms : Dual space 3D

The equations above can be recast as a transport equation:

$$\partial_t \sigma_t + \text{div}(\sigma_t X_t[\sigma_t]) = 0; \quad \sigma|_{t=0} = \sigma_0 \ll L^3 \quad (8)$$

Here

$$X_t[\sigma](y) = L_t(\nabla \psi^\sigma(y), y)$$

$$\psi^\sigma(\lambda, \cdot)$$ solves the Monge Ampère equations (2)

and

$$L_t(x, y) =$$

$$\left( \frac{\sqrt{y_1}}{r_0} - \Omega - 2x_1\sqrt{y_1}, 2\sqrt{y_1}F_t(\lambda, e^{\frac{1}{4}}(x_1), x_2) + 2x_1\sqrt{y_1}, \frac{g}{\theta_0} S_t(\lambda, e^{\frac{1}{4}}(x_1), x_2) \right)$$

with $x = (\lambda, x_1, x_2), y = (\lambda, y_1, y_2)$ and $e(x_1) = (1 - 2x_1)^{-2}$. 
Challenges in the continuity equation

- Defining well the velocity $X_t[\sigma]$.

- Existence and Regularity of
  \[ \nabla \psi = \begin{pmatrix} \frac{\partial \psi}{\partial \lambda}, \frac{\partial \psi}{\partial \gamma}, \frac{\partial \psi}{\partial z} \end{pmatrix} \]

- Regularity in a Monge–Ampere equation with respect to a parameter:
  \[ \frac{1}{(1 - 2\partial_{y_1} \psi^\lambda)^2} \det \nabla_{y_1, y_2}^2 \psi^\lambda = \sigma^\lambda \]
Thank you for your attention!