Gradient interfaces with and without disorder

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Outline

1. Physics motivation
   - Example 1: Elasticity
   - Recap-Gaussian Measure
   - Example 2: Effective interface models

2. The model
   - Dimension $d = 1$
   - Generalization to dimension $d \geq 2$

3. Questions

4. Known results
   - Results: Strictly Convex Potentials
   - Techniques: Strictly Convex Potentials
   - Results: Non-convex potentials
   - Interfaces with disorder

5. Open questions: non-convex potentials
Microscopic model ↔ emerging macroscopic structures.

Macroscopic phases → microscopic interfaces

Approach: Microscopic modelling of the interface itself.
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- Crystals are macroscopic objects, with ordered arrangements of atoms or molecules in microscopic scale.
- Mechanical model of a crystal: little balls connected by springs, where heat causes the jiggling.

Configuration: snapshot of the atoms’ positions at a given time.
In thermal equilibrium, the jiggings explore samples of a probability measure on the configurations. This is the **Gibbs measure**:

$$\text{Prob}(\text{Configuration}) \propto \exp(-\beta \text{ Energy of Configuration}),$$

where $\beta = 1/\text{temperature} > 0$.

- Moving every atom in the same direction the same amount does not change the energy, and hence the probability, of the configuration (**shift-invariance**).

- If Hook’s law holds, the elastic energy between two atoms with displacements $x, y$ is given by $c(x - y)^2$ (the force $F$ needed to extend or compress a spring by some distance $|x - y|$ is proportional to that distance).

- Then the measure on the atoms’ configurations is **multi-dimensional Gaussian**.
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1D Gaussian random variables

Recall: A standard 1D Gaussian random variable $X$ has distribution given by the density

$$P(X \in [x, x + dx]) = \frac{\exp(-x^2/2)}{\sqrt{2\pi}} dx.$$
Gaussian random variables in $\mathbb{R}^n$

- If $\langle x, y \rangle$ is an inner product in $\mathbb{R}^n$, then

$$ \frac{1}{(2\pi)^{n/2}} \exp \left( \frac{\langle x, x \rangle}{2} \right) $$

is the density of an associated multidimensional Gaussian.

- This is the same as taking

$$ \sum_{j=1}^{n} z_{j} e_{j} $$

where $\{e_{j}\}$ is an orthonormal basis and $\{z_{j}\}$ are independent 1D Gaussians.
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The interface for the Ising model - simplest description of ferromagnetism

The spontaneous magnetization on cooling down the substance below a critical temperature, the so-called Curie temperature.

The Ising model on a domain \( \Omega \subset \mathbb{Z}^d \) with free boundary condition, at inverse temperature \( \beta = 1/T > 0 \) and external field \( h \in \mathbb{R} \), is given by the following Gibbs measure on spin configurations \( (\sigma_x)_{x \in \Omega} \in \{\pm 1\}^\Omega \)

\[
P_{\Omega,h,\beta}(\sigma) := \frac{1}{Z_{\Omega,h,\beta}} \exp \left( \beta \sum_{x,y \in \Omega, |x-y|=1} \sigma_x \sigma_y + h \sum_{x \in \Omega} \sigma_x \right) \mathbb{P}(\sigma),
\]

where \( \mathbb{P} \) is the uniform distribution on \( \{\pm 1\}^\Omega \).
- Assume $d = 2$ and $\Omega = [0, N] \times [0, N]$.
- Spin configuration $\sigma = \{\sigma_x\}_{x \in \{0, \ldots, N\} \times \{0, \ldots, N\}}$, spins $\sigma_x \in \{-1, 1\}$
- Goal: Modelling and analysis of the interface phase boundary
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The model

Dimension $d = 1$

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The model

Dimension \( d = 1 \)

- Interface — transition region that separates different phases

- \( \Lambda_n := \{-n, -n + 1, \ldots, n - 1, n\} \), \( \partial \Lambda_n = \{-n - 1, n + 1\} \)

- Height Variables (configurations) \( \phi_i \in \mathbb{R}, i \in \Lambda_n \)

- Boundary condition 0, such that

\[
\phi_i = 0, \quad \text{when } i \in \partial \Lambda_n.
\]

- The energy \( H(\phi) := \sum_{i=-n}^{n+1} V(\phi_i - \phi_{i-1}) \), with \( V(s) = s^2 \) for Hooke’s law.
The finite volume Gibbs measure

\[ \nu_{\Lambda_n}^0(\phi_{-n}, \ldots, \phi_1, \ldots, \phi_n) = \frac{1}{Z_{\Lambda_n}^0} \exp(-\beta H(\phi)) d\phi_{\Lambda_n} = \]

\[ \frac{1}{Z_{\Lambda_n}^0} \exp(-\beta \sum_{i=-n}^{n+1} (\phi_i - \phi_{i-1})^2) \prod_{i=-n}^n d\phi_i, \]

where \( \beta = 1/T > 0 \), \( \phi_{-n-1} = \phi_{n+1} = 0 \) and

\[ Z_{\Lambda_n}^0 := \int_{\mathbb{R}^{2n+1}} \exp(-\beta \sum_{i=-n}^{n+1} (\phi_i - \phi_{i-1})^2) \prod_{i=-n}^n d\phi_i, \]

is a multidimensional centered Gaussian measure.

We can replace the 0-boundary condition in \( \nu_{\Lambda_n}^0 \) by a \( \psi \)-boundary condition in \( \nu_{\Lambda_n}^\psi \) with \( \phi_{-n-1} := \psi_{-n-1}, \phi_{n+1} := \psi_{n+1} \).
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Replace the discrete interval \([-n, -n + 1, \ldots, 1, 2, \ldots, n]\) by a discrete box

\[ \Lambda_n := \{-n, -n + 1, \ldots, 1, \ldots, n - 1, n\}^d, \]

with boundary

\[ \partial \Lambda_n := \{i \in \mathbb{Z}^d \setminus \Lambda_n : \exists j \in \Lambda_n \text{ with } |i - j| = 1\}. \]

The energy \( H(\phi) := \sum_{i,j \in \Lambda_n \cup \partial \Lambda_n} V(\phi_i - \phi_j) \), where \( V(s) = s^2 \) and \( \phi_i = 0 \) for \( i \in \partial \Lambda_n \).

The corresponding finite volume Gibbs measure on \( \mathbb{R}^{\Lambda_n} \) is given by

\[ \nu^0_{\Lambda_n}(\phi) := \frac{1}{Z_{\Lambda_n}} \exp(-\beta H(\phi)) \prod_{i \in \Lambda_n} d\phi_i. \]

It is a Gaussian measure, called the Gaussian Free Field (GFF).
For GFF

- If \( x, y \in \Lambda_n \)
  \[
  \text{cov}_{\nu^0_{\Lambda_n}} (\phi_x, \phi_y) = G_{\Lambda_n}(x, y),
  \]
  where \( G_{\Lambda_n}(x, y) \) is the Green’s function, that is, the expected number of visits to \( y \) of a simple random walk started from \( x \) killed when it exits \( \Lambda_n \).

- GFF appears in many physical systems; two-dimensional GFF has close connections to Schramm-Loewner Evolution (SLE).

- Random, fractal curve in \( \Omega \subseteq \mathbb{C} \) simply connected.

- Introduced by Oded Schramm as a candidate for the scaling limit of loop erased random walk (and the interfaces in critical percolation).

- Contour lines of the GFF converge to SLE (Schramm-Sheffield 2009).
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The model

Generalization to dimension $d \geq 2$

General potential $V$, general boundary condition $\psi$, general $\Lambda$

- $V: \mathbb{R} \to \mathbb{R}$, $V \in C^2(\mathbb{R})$ with $V(s) \geq As^2 + B, A > 0, B \in \mathbb{R}$ for large $s$.
- The finite volume Gibbs measure on $\mathbb{R}^{\Lambda}$

$$\nu_{\psi}^{\Lambda}(\phi) := \frac{1}{Z_{\psi}^{\Lambda}} \exp(-\beta \sum_{i,j \in \Lambda \cup \partial \Lambda, |i-j|=1} V(\phi_i - \phi_j)) \prod_{i \in \Lambda} d\phi_i,$$

where $\phi_i = \psi_i$ for $i \in \partial \Lambda$.
- **Tilt** $u = (u_1, \ldots, u_d) \in \mathbb{R}^d$ and tilted boundary condition $\psi_i^u = i \cdot u, i \in \partial \Lambda$.
- Finite volume surface tension (free energy) $\sigma_{\Lambda}(u)$: macroscopic energy of a surface with tilt $u \in \mathbb{R}^d$.

$$\sigma_{\Lambda}(u) := \frac{1}{|\Lambda|} \log Z_{\psi}^{\Lambda}.$$

- Gradients $\nabla \phi$: $\nabla \phi_b = \phi_i - \phi_j$ for $b = (i,j), |i - j| = 1$
Questions (for general potentials $V$):

- **Existence and (strict) convexity** of infinite volume (i.e., infinite dimensional) surface tension

  $$\sigma(u) = \lim_{\Lambda \uparrow \mathbb{Z}^d} \sigma_\Lambda(u).$$

- **Existence** of shift-invariant infinite dimensional Gibbs measure

  $$\nu := \lim_{\Lambda \uparrow \mathbb{Z}^d} \nu^\psi_\Lambda$$

- **Uniqueness** of shift-invariant Gibbs measure under additional assumptions on the measure.

- **Quantitative results for $\nu$: decay of covariances** with respect to $\phi$, central limit theorem (CLT) results, log-Sobolev inequalities, large deviations (LDP) results.
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Known results

Results: Strictly Convex Potentials

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Known results for potentials $V$ with

$$0 < C_1 \leq V'' \leq C_2 :$$

- Existence and strict convexity of the surface tension $\sigma$ for $d \geq 1$ and $\sigma \in C^1(\mathbb{R}^d)$.
- Gibbs measures $\nu$ do not exist for $d = 1, 2$.
- We can consider the distribution of the $\nabla \phi$-field under the Gibbs measure $\nu$. We call this measure the $\nabla \phi$-Gibbs measure $\mu$.
- $\nabla \phi$-Gibbs measures $\mu$ exist for $d \geq 1$.
- (Funaki-Spohn (CMP-2007)) For every $u = (u_1, \ldots, u_d) \in \mathbb{R}^d$ there exists a unique shift-invariant ergodic $\nabla \phi$-Gibbs measure $\mu$ with $E_{\mu}[\phi_{e_k} - \phi_0] = u_k$, for all $k = 1, \ldots, d$.
- CLT results, LDP results

Bolthausen, Brydges, Deuschel, Funaki, Giacomin, Ioffe, Naddaf, Olla, Peres, Sheffield, Spencer, Spohn, Velenik, Yau, Zeitouni
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Known results

Techniques: Strictly Convex Potentials

For

\[ 0 < C_1 \leq V'' \leq C_2 : \]

- **Brascamp-Lieb Inequality (Brascamp-Lieb JFA 1976/Caffarelli-CMP 2000):** for all \( x \in \Lambda \) and for all \( i \in \Lambda \)

  \[ \text{var} \, \nu^\psi_i (\phi_i) \leq \text{var} \, \tilde{\nu}^\psi_i (\phi_i), \]

  \( \tilde{\nu}^\psi_i \) is the Gaussian Free Field with potential \( \tilde{V}(s) = C_1 s^2 \).

- **Random Walk Representation (Deuschel-Giacomin-Ioffe 2000):**
  Representation of Covariance Matrix in terms of the Green function of a particular random walk.

  - **GFF:** If \( x, y \in \Lambda \)

    \[ \text{cov} \, \nu^0_\Lambda (\phi_x, \phi_y) = G_\Lambda (x, y). \]

- **General** \( 0 < C_1 \leq V'' \leq C_2 : \)

  \[ 0 \leq \text{cov} \, \nu^\psi_\Lambda (\phi_x, \phi_y) \leq \frac{C}{||x-y||^{d-2}}, \quad |\text{cov} \, \mu^\rho_\Lambda (\nabla i \phi_x, \nabla j \phi_y)| \leq \frac{C}{||x-y||^{d-2+\delta}}. \]
The dynamic: **SDE** satisfied by \((\phi_x)_{x \in \mathbb{Z}^d}\)

\[
d\phi_x(t) = -\frac{\partial H}{\partial \phi_x}(\phi(t))\,dt + \sqrt{2}dW_x(t), \quad x \in \mathbb{Z}^d,
\]

where \(W_t := \{W_x(t), x \in \mathbb{Z}^d\}\) is a family of independent 1-dim Brownian Motions.
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Why look at the case with non-convex potential $V$?

- Probabilistic motivation: **Universality** class
- Physics motivation: For lattice spring models a realistic potential has to be **non-convex** to account for the phenomena of fracturing of a crystal under stress.
- **The Cauchy-Born rule**: When a crystal is subjected to a small linear displacement of its boundary, the atoms will follow this displacement.
- **Friesecke-Theil**: for the 2-dimensional mass-spring model, Cauchy-Born holds for a certain class of non-convex potentials. Generalization to $d$-dimensional mass-spring model by **Conti, Dolzmann, Kirchheim and Müller**.
Results for non-convex potentials

- For the potential

\[ e^{-V(s)} = p e^{-k_1 \frac{s^2}{2}} + (1-p) e^{-k_2 \frac{s^2}{2}}, \beta = 1, k_1 << k_2, p = \left( \frac{k_1}{k_2} \right)^{1/4} \]

- Biskup-Kotecký (PTRF-2007): Existence of several $\nabla \phi$-Gibbs measures with expected tilt $E_\mu [\phi_{e_k} - \phi_0] = 0$, but with different variances.
Cotar-Deuschel-Müller (CMP-2009)/ Cotar-Deuschel (AIHP-2012):
Let
\[ V = V_0 + g, \quad C_1 \leq V''_0 \leq C_2, \quad g'' < 0. \]
If
\[ C_0 \leq g'' < 0 \quad \text{and} \quad \sqrt{\beta}||g''||_{L^1(\mathbb{R})} \quad \text{small}(C_1, C_2) \]
uniqueness for shift-invariant \( \nabla \phi \)-Gibbs measures \( \mu \) such that
\[ E_\mu [\phi_{e_k} - \phi_0] = u_k \quad \text{for} \quad k = 1, 2, \ldots, d. \]
Our results includes the Biskup-Kotecký model, but for different range of choices of \( p, k_1 \) and \( k_2 \).

Adams-Kotecký-Müller (preprint): Strict convexity of the surface tension for very small tilt \( u \) and very large \( \beta \).
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Adding disorder (for example, making potentials random variables) tends to destroy non-uniqueness.

Consider for simplicity the disordered model

\[ e^{-V_b(\eta_b)} := pe^{-k_1(\eta_b)^2+\omega_b} + (1-p)e^{-k_2(\eta_b)^2-\omega_b}, \ (w_b)_b \text{ i.i.d. Bernoulli.} \]

**Adaptation** of the Aizenman-Wehr (CMP-1990) argument: gives uniqueness of gradient Gibbs in \( d = 2 \)

**Conjecture**

- **uniqueness** for low enough \( d \leq d_c \);
- **uniqueness/non-uniqueness phase transition** for high enough \( d > d_c \geq 2 \).

**Techniques:** Poincarre inequalities (Gloria/Otto), log-Sobolev inequalities (Milman 2012).
- Log-Sobolev inequality for moderate/low temperature.
- Relaxation of the Brascamp-Lieb inequality.
- Example of potential where the surface tension is non-strictly-convex.
- Conjecture: Surface tension (plus maybe some additional assumption) \( \Rightarrow \) uniqueness of the shift-invariant Gibbs measure.
- Conjecture: Surface tension is in \( C^2(\mathbb{R}^d) \) (both for strictly convex and for non-convex potentials).
THANK YOU!