Functional Representations of Combinatorial Sets and Applications in Optimization

Oksana Pichugina
Brock university
Sergey Yakovlev
Ukraine, Kharkov

CONFERENCE ON OPTIMIZATION, TRANSPORTATION AND EQUILIBRIUM IN ECONOMICS
Fields Institute, Toronto
September 16, 2014

Problem Statement

$$f(x) \to \min, \qquad (1)$$

$$x \in E \subset R^n, \qquad (2)$$

$$|E| = N < \infty, \qquad (3)$$

E - Euclidean combinatorial set (ECS).

Considered ECS M classes:

1) Boolean vector set (ECS of permutation with repetitions from 0,1):

$$B_n = \{x \in R^n : \ x_i \in \{0,1\}, i \in J_n\}. \qquad (4)$$

2) General permutations set from a multiset G:

$$E_{nk}(G),\ G = \{g_i\}_{i \in J_n}, |S(G)| = k,\ J_n = \{1,...,n\}. \qquad (5)$$

Different Representation of ECS

1.

$$E \subseteq R^n \text{ -} \qquad (1)$$

set representation of E (SR);

2.

$$E = \{x : f_j(x) = 0,\ j \in J_{m'};\ f_{j+m'}(x) \le 0,\ j \in J_{m''}\} \text{ -} \qquad (2)$$

functional representation of E (FR);

$m'' = 0$ - strict FR (SFR);

$m'' > 0$ - nonstrict FR (NSFR);

3. Mixture of (1)-(2)

$E = \{x \in E^* \subseteq R^n : f_j(x) = 0,\ j \in J_{m'};\ f_{j+m'}(x) \le 0,\ j \in J_{m''}\}$ set-functional representation of E (SFR).

Some Representation of B_n

1. FR: $B_n = \{x \in R^n : \; x_i^2 - x_i = 0, \; i \in J_n\}$; (1)

2. SFR: $B_n = \{x \in \Pi_n : \; \sum_{i=1}^{n} (1 - \cos(2\pi x_i)) = 0\}$, (2)

$$\Pi_n = \{x \in R^n : \; \bar{0} \le x \le \bar{1}\}. \quad (3)$$

Penalty fuctions (PF) examples:

- FR (1) - $\Phi(x,\lambda) = f(x) + \lambda \cdot \sum_{j=1}^{m'} f_j^2(x) \underset{\lambda>0}{\rightarrow} \min$

- SFR (2) -

$$\Phi(x,\lambda) = f(x) + \lambda \cdot \left(\sum_{j=1}^{m'} f_j^2(x) + \sum_{j=1}^{m''} \max\left(0, -f_{j+m'}(x)\right) \right) \underset{\lambda>0}{\rightarrow} \min$$

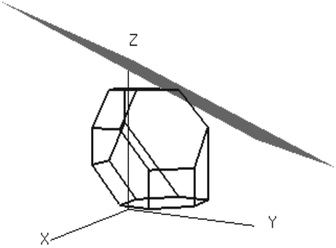
Approach 1: Combinatorial Cutting Plane Method for vertex located ECS (CCPM)

Let E be vertex located:

$$E = \text{vert}\ P, \tag{1}$$

$$P = \text{conv}\ E - \tag{2}$$

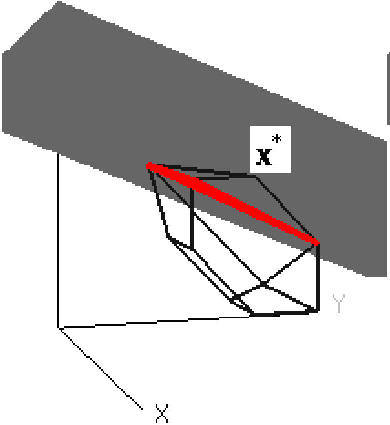
combinatorial polyhedron (CP)



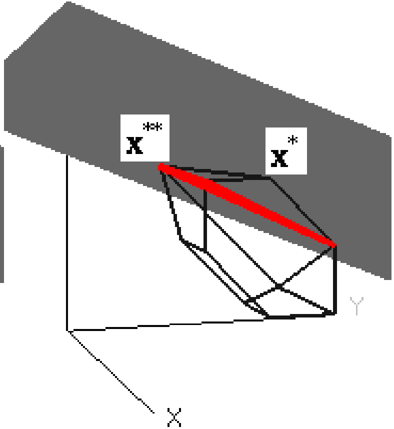
$$cx \underset{E}{\rightarrow} \min$$

$$Ax \le b, A = (a_i)_{i \in J_m}.$$

CCPM (Part 2)



Pic.1



Pic.2

Representations of ECS inscribed into a sphere

$$E \subset S_r(a), \qquad (1)$$

$$S_r(a) = \{x \in R^n : \left(x - a\right)^2 = r^2\}. \qquad (2)$$

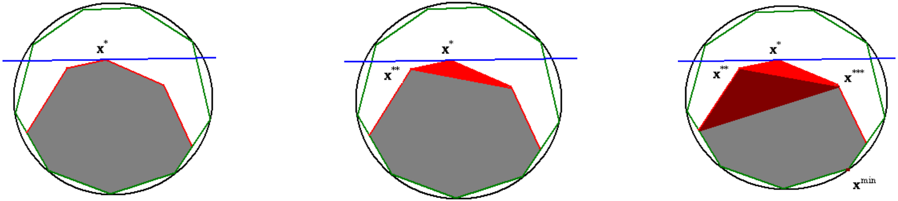
$$E = P \cap S_r\left(a\right) \qquad (3)$$

$$P = \{x : A'x \le b'\} \qquad (4)$$

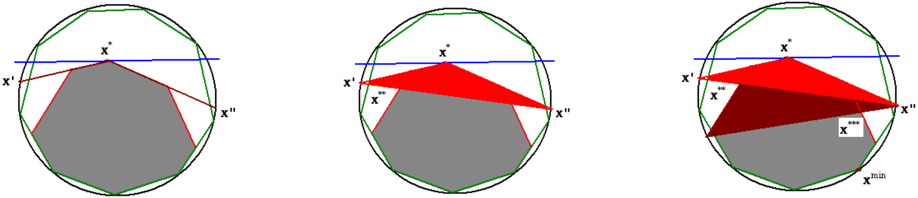
- (3)- SR of E;
- (2)-(3), (3)-(4)- SFR of E;
- (2)-(4)- FR of E.

Approach 2: Modification of CCPM for ECS inscribed into a sphere

Approach 1



Approach 2



Approach3: Polyhedral Spherical Method (PSM) for ECS inscribed into a sphere

Representation 1- $$f(x) \to \min_{x \in E} \quad (1)$$

Representation 2- $$f(x) \to \min_{x \in P \cap S_r(a) \subset R^n} \quad (2)$$

Relaxation 1

$$f(x) \to \min_{x \in P \subset R^n} \quad (3)$$

Relaxation 2

$$f(x) \to \min_{x \in S_r(a) \subset R^n} \quad (4)$$

Remark. We can think that $f(x)$ is convex, because it is known that any optimization problem on the vertex located set can be converted to the optimization of its convex extension in R^n.

PSM (Part 2 – Additional problems)

Subproblem 1.

$$cx \underset{E}{\rightarrow} \min \quad \left(cx \underset{P}{\rightarrow} \min \right) \tag{1}$$

Subproblem2.

$$f(x) \underset{P}{\rightarrow} \min \tag{2}$$

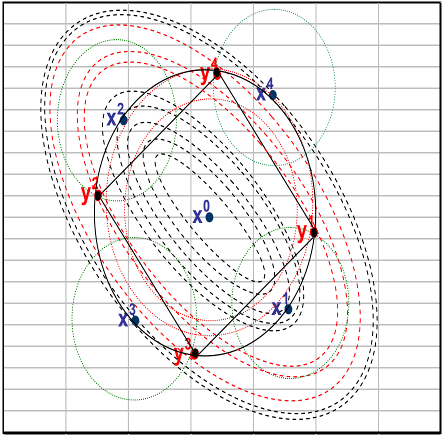
Subproblem 3.

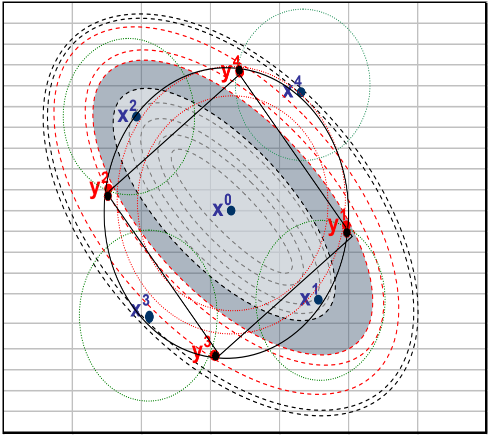
$$f(x) \underset{S_r(a)}{\rightarrow} \min \tag{3}$$

Subroblem4. Projection on ECS

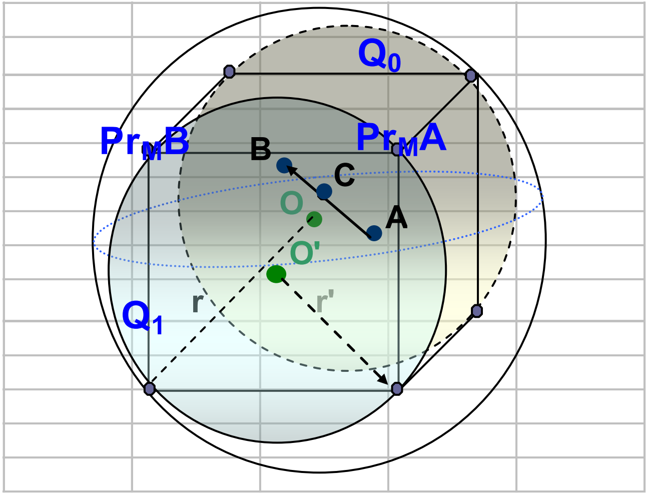
$$x' = Pr_E\, y \tag{4}$$

PSM (Part 3 – Quadratic optimization in R^2)





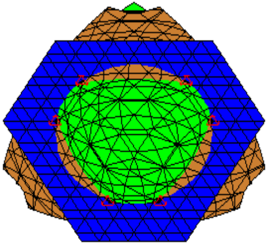
PSM (Part 4 –Optimization over B_3)

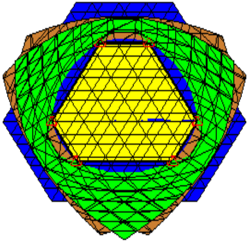


Functional Representation of $E_{nk}(G)$, n=3

$$E = M, M = \left\{ x : \sum_{i=1}^{n} x_i^j = \sum_{i=1}^{n} g_i^j, j \in J_n \right\} \qquad (1)$$

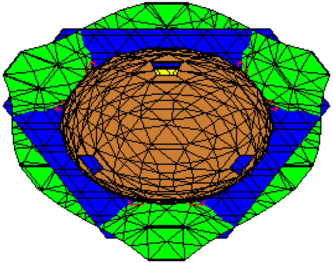
Example 1. k=n=3, $E = E_3(G), G \geq 0$, convex FR (1)

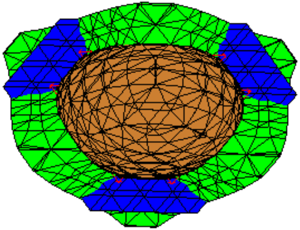




Functional Representation of $E_{nk}(G)$, $n=3$ (Part 2)

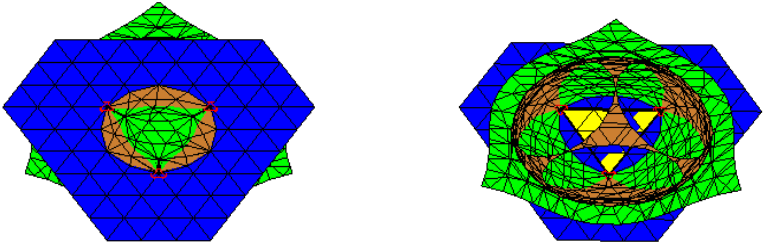
Example 2. $k=n=3$, $E=E_3(G), G \geq 0$, nonconvex FR with r_{min}





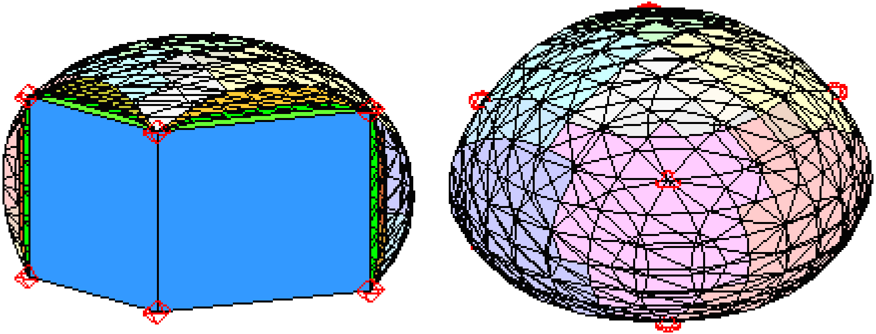
Functional Representation of $E_{nk}(G)$, $n=3$ (Part 3)

Example 3. $k=2$, $E=E_{32}(G), G \geq 0$, nonconvex FR with r_{min}



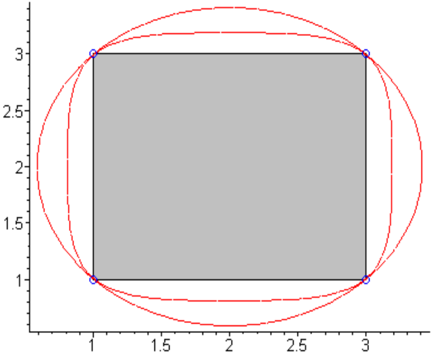
Functional Representation of B_n

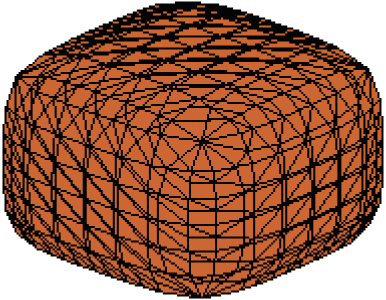
$$E = M, M = \left\{ x : \sum_{i=1}^{n} (x_i - \frac{1}{2})^{2j} = \frac{n}{2^{2j}}, \ j \in J_n \right\}$$



Irredundant Functional Representation of B_n

$$E = M, M = \left\{ x : \sum_{i=1}^{n} (x_i - \frac{1}{2})^2 = \frac{n}{4}, \ \sum_{i=1}^{n} (x_i - \frac{1}{2})^4 = \frac{n}{16} \right\}$$

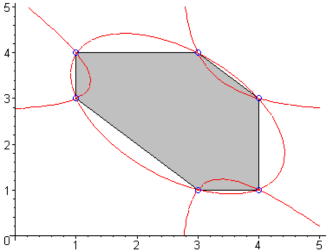


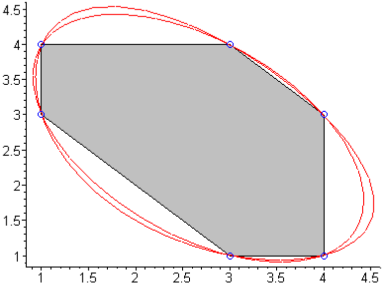


Pic. 1 $n = 2$ Pic. 2 $n = 3$ Inscribed suface in $S_r(a)$

Intersectional and Tangent FR

1. Intersectional FR $m' = n, m \geq n$





2. Tangent FR: $E = S' \cap S_r(a)$.

$$S' = \{x \in R^n : h(x) = 0\}$$

$$\Phi(x,\lambda) = f(x) + \lambda \cdot \left(\left((x-a)^2 - r^2 \right)^2 + h^2(x) \right) \underset{\lambda > 0}{\longrightarrow} \min \cdot$$

Thank you for your attention!