A MARTINGALE APPROACH FOR PORTFOLIO ALLOCATION WITH STOCHASTIC VOLATILITY AND JUMPS

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Introduction

- Problem

  How to decide where to allocate their funds among the many available choices in the market. Taking into account all the relevant sources of risk.

- Risk sources
  - Volatility reveals itself to be stochastic.
  - Impact of sudden, and sometimes, high shocks to price levels.
- Estimation
  - Conditional characteristic function
  - GMM using a discretization of the empirical characteristic function, Chacko and Viceira (2003).
  - In this paper we perform a discretization of the continuous dual, then, the first four moments are restricted to match the sampling moments.

Market Model

- Securities
  - A risky asset, denoted by S = St, and a riskless asset B = Bt
  - The asset's percentage change, $\phi(J) = e^J - 1$, depends on the IID normally distributed random variable J.
  - N = N_t is a process with intensity λ_t.
  - W_1 and W_2 are independent F_t-Brownian motions correlated with $\rho W_1 + \bar{\rho} W_2$, with correlation coefficient ρ, and $\bar{\rho} = \sqrt{1-\rho^2}$.
  - Complete filtered space $(\Omega, \mathfrak{F}, (\mathfrak{F})_{0 \le t \le T}, \mathbf{P})$.

Compensated martingale of N_t

- If we note by Mt the compensated F_t-Martingale associated with N_t,
- Then
- The risk asset dynamics becomes

Market Model

- Portfolio
  - π is the number of shares in the risky asset and ϖ the number of units of the riskless asset.
  - They make up the portfolio $\Pi = (\pi, \varpi)$. In a self financing strategy $dX = \pi dS + \varpi dB$.
  - We reformulate the problem introducing the portfolio weight α_t which is the proportion of wealth invested in the risky asset.

$$d\bar{X} = \alpha \bar{X}_{-} \left\{ \left(\mu - r + \bar{\phi} \lambda \right) dt + \sqrt{V}\, dW_1 + \int_{\Re} \phi(t, z) M(dt, dz) \right\}$$

Equivalent Martingale Measure

- Equivalent martingale measure (EMM)
  - A probability measure Q equivalent to P is an EMM if the discounted asset process S is a martingale under this measure.
- It is considered probability measures Q such that the Radon - Nikodym dynamics is

$$\xi_t(\theta,\varphi) = d\left(Q^{\theta,\varphi}\Big|_{\mathfrak{F}_t}\right) / d\left(P\Big|_{\mathfrak{F}_t}\right)$$

$$d\xi = -\xi_-\left(\theta'\, dW + \int_{\Re}(1-\varphi)M(dt,dz)\right)$$

- The actual expression is the Doléans Dade exponential.

Equivalent Martingale Measure

- The current price of a risky claim differs from its expected value
- The expected value has to be corrected by the individuals risk preferences
  - Difficult to quantify the discount rates.
  - One way to solve is to adjust the probabilities of future outcomes to incorporate the risk premia for all the agents in the market and take the expectation under this new probability distribution.
- The abscence of arbitrage ensures the existence of a risk neutral measure.
- Completeness of the market ensures unicity of the EMM.
- Under Girsanov theorem we move from historical measure towards a risk neutral measure.

Equivalent Martingale Measure

- A probability measure Q equivalent to P is an equivalent martingale measure (EMM) if the discounted asset process S is a martingale under this measure.
- We consider probability measures such that the Radon Nikodym derivative is the stochastic exponential.

$$\xi_t(\theta,\varphi) \doteq \exp\left\{\int_0^t \left(\int_{\mathbb{R}} \lambda(\ln(\varphi(s,z)) + 1 - \varphi(s,z))\psi(dz) - \frac{|\theta|^2}{2}\right) ds \right. \\ \left. - \int_0^t \theta' dW_s + \int_0^t \int_{\mathbb{R}} \ln(\varphi(s,z)) M(ds,dz)\right\}$$

- and are the market prices of risk.

Equivalent Martingale Measure

- The market price of risk processes θ and φ are chosen such that the process ξS/B is a P- martingale, implying the following relationship

$$\mu - r - \theta_1 \sqrt{V} + \lambda \int_{\Re} \phi(t,z) \varphi(t,z) \psi(dz)$$

- A new processes (martingales) are defined

$$W^Q = W + \int_0^t \theta_s ds$$

$$\int_0^t \int_{\Re} M^Q(ds,dz) = \int_0^t \int_{\Re} N(ds,dz) - \lambda \int_0^t \int_{\Re} \varphi(z) \psi(dz) ds$$

- With these, the discounted wealth process happens to be a supermartingale, leading to the static budget constrain

$$E^{Q^{\theta,\varphi}}\left[\bar{\bar{X}}_t\right] \leq X_0$$

Primal and Dual Problem

- By superreplicability principle the portfolio problem (primal) is

$$F(X_0) = \sup_{X_T \in \chi} E[u(X_T)]$$

$$R.T. \quad E^{Q^{\theta,\varphi}}[\overline{\overline{X}}_t] \le X_0$$

- To solve it define the lagrangian

$$L(X_T, \eta, \mathbf{Q}) = E[u(X_T)] - \eta\left(E^{\mathbf{Q}}[\overline{X}_T] - X_0\right)$$

- And the problem becomes

$$\Psi(\eta, \mathbf{Q}) = \sup_{X_T \in \chi}\{L(X_T, \eta, \mathbf{Q})\}$$

- Or

$$\Psi(\eta, \mathbf{Q}) = \sup_{X_T \in \chi}\left\{E\left[u(X_T) - \eta \frac{d\mathbf{Q}}{d\mathbf{P}} \overline{X}_T\right]\right\} + \eta X_0$$

Primal and Dual Problem

- The last expression is the Legendre – Fenchel transform of u, $\tilde{u}(p) = \sup_x \{u(x) - px\}, p > 0$. Hence

$$\Psi(\eta, \mathbf{Q}) = E\left[\tilde{u}\left(\eta \beta_T \frac{d\mathbf{Q}}{d\mathbf{P}}\right)\right] + \eta X_0$$

- In this context the dual problem is defined as

$$\inf_{\eta, \mathbf{Q}} \{\Psi(\eta, \mathbf{Q})\}$$

$$\eta > 0, \mathbf{Q} \in \mathbb{Q}$$

- is continuous, solutions are equivalent: Strong duality

Solution

- Fix in the dual problem and minimize over the set of EMM.
- We chose a specific functional and the optimization runs over the parameter space for the prices of risk.
- The value function is found by means of the Hamilton-Jacobi-Bellman (HJB) for optimal control of jump diffusions
- The infinitesimal generator is

$$\mathcal{A}\Phi = -r\bar{\xi}\frac{\partial\Phi}{\partial\bar{\xi}} + \kappa(\vartheta - V)\frac{\partial\Phi}{\partial V} + \frac{|\theta|^2}{2}\bar{\xi}^2\frac{\partial^2\Phi}{\partial\bar{\xi}^2} + \frac{\sigma^2}{2}V\frac{\partial^2\Phi}{\partial V^2}$$
$$- \varrho\theta\bar{\xi}\frac{\partial^2\Phi}{\partial\bar{\xi}\partial V} + \lambda\int_{\mathbb{R}}\left(\Phi(\varphi(z)\bar{\xi}) - \Phi(\bar{\xi}) + (1-\varphi(z))\bar{\xi}\frac{\partial\Phi}{\partial\bar{\xi}}\right)\psi(dz)$$

Solution for Power Utility

- For power utility we have
  - Market price of risk, market price of volatility risk and the market price of jump risk

$$\hat{\theta}_1 = -b\rho\sigma\sqrt{V} + \gamma\hat{\alpha}\sqrt{V}$$

$$\hat{\theta}_2 = -b\bar{\rho}\sigma\sqrt{V}$$

$$\hat{\varphi} = (1+\hat{\alpha}\phi)^{-1/\gamma}$$

  - The optimal rule

$$\hat{\alpha} = \frac{1}{\gamma}\frac{\mu - r}{V} + \frac{1}{\gamma}b\rho\sigma + \frac{1}{\gamma}\frac{\lambda}{V}E\left[\phi(1+\hat{\alpha}\phi)^{-\gamma}\right]$$

  - The optimal wealth at time t, and the lagrangian multiplier

$$\hat{X}_t = \frac{X_0 B_t}{E\left[\xi_T^{\gamma}\right]}\xi_t^{\gamma-1}e^{(1-\gamma)(a'(\tau)+b(\tau)V)} \qquad \hat{\eta} = \left(\frac{X_0\beta_T^{-\gamma}}{E\left[\xi_T^{\gamma}\right]}\right)^{\frac{1}{\gamma-1}}$$

Conditional Characteristic Function

- The characteristic function $\Phi(\zeta,\tau,\Theta,Y_t)=E_t\left[e^{i\zeta Y_{t+\tau}}\right]$ is used to estimate the model parameters. Log returns are used for convenience

$$dY = d\,ln\,S = \left(\mu - \frac{V}{2}\right)dt + \sqrt{V}\,dW_1 + JdN$$

- GMM is applied using as moments $E\left[exp(i\zeta Y_n) - \Phi(\zeta,\Theta)\right] = 0$ and the restriction that the first four sampling moments should match the theoretical moments.
- Exact moments are retrieved by

$$Mean = K_1 \qquad SD = \sqrt{K_2}$$

$$Skew = \frac{K_3}{K_2^{3/2}} \qquad EK = \frac{K_4}{K_2^2}$$

where $K_n = \frac{1}{i^n}\frac{d}{d\zeta}\,ln\,\Phi(\zeta)\Big|_{\zeta=0}$

Data

- The dataset comprises the prices from January 1982 to October 20107 of
  - The Standard and Poor's Composite Index
  - Two series of the ten portfolios from Kenneth French's web page: Low book to market (Growth), and High book to market (Value).
- The timeframe spanned by the data includes
  - The crash of October 1987
  - The Gulf War I in August 1990,
  - The Mexican crisis in December 1994,
  - The Asian crisis of July 1997
  - The Russian crisis of August 1998
  - The bursting of the dot.com bubble.
- The riskless rate of interest is the simple average of the three month Treasury Bill rates, and its value is 4.7% in annualized terms.

Estimation

- We use the characteristic function of the continuously compounded returns.
- An "unconstraint" optimization is performed with wide intervals on the variables.
- A "constraint" optimization is performed calibrating the long run variance to GARCH estimates.
- GARCH estimates similar to sampling variance
- Long run variance of constraint similar to sampling variance.
- Growth volatility is higher than for Value volatility
- Value mean (0.14%) is twice the Growth mean (0.07%)

Results

- Value skewnees (negative) and excess kurtosis (leptokurtic) are the highest in absolute value.
- Jump frequency of one per week for S&P500, once each 20 days for Growth series, and less frequently, once each1.3 months, for Value.
- Variance jump component Value – Growth (45.39%)
  - Magnitude component similar for Growth and Value
  - Intensity of Value is half the Growth's intensity (49.34%)
- Variance stochastic volatility component (1.97)
  - Contribution given by the long run variance. The ratio Growth – Value is 1.97

Conclusions

- Portfolio weight is inversely proportional to the risk aversion and reduces to the standard myopic rule when
  - The correlation between the asset prices and volatility is zero, and the frequency of extreme event vanishes.
- The market price of risk was found to be composed by the Brownian market price of risk plus a jump contribution.
- The market price of volatility risk is approximately proportional to the market price of risk for a very low risk adverse investor. It is zero if
  - The innovations in the returns are perfectly correlated with the instantaneous volatility

Conclusions

- Impact of the jump and diffusion part on moments
  - The mean is dominated by the diffusion mean compensated by a half of the long run variance. It also includes a Poisson term, however, its contribution is minimal.
  - The variance is made up of a jump and a stochastic volatility component in the same order of magnitude, It depends on model volatility parameters.
  - Skewness and excess kurtosis depend almost exclusively on the jump contribution. Both depends on Poisson intensity. Skewness is mainly due to jump mean, which also determines the sign, And kurtosis strongly depends on jump volatility.

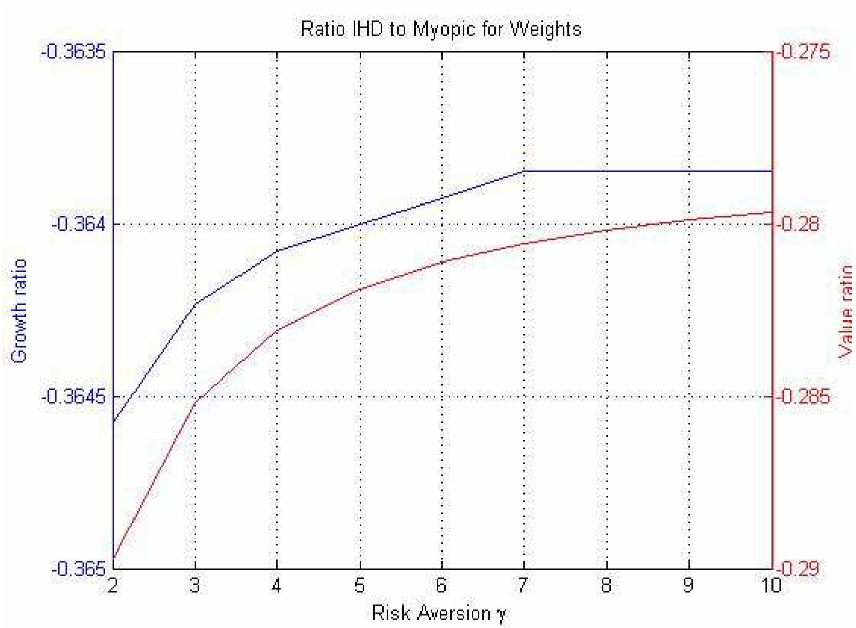
Conclusions

- The largest variance of Growth series, compared with that of Value series, is mainly due to
  - The low frequency of occurrence of unexpected large events in Value series,
  - The larger long term volatility of the Growth series.
- Regarding the jump mean magnitude, i.e., its absolute value, smaller shocks happen more frequently that larger shocks.

Conclusions

- The portfolio weights are found to be low compared with those of a standard diffusion model because the investor perceives more risk coming from jumps and stochastic volatility. Participation in Value is greater than in Growth.
- We calculate a myopic (excess return over variance times gamma) demand and is greater than the total portfolio weight. We define the intertemporal hedging demand.
  - The negative jump mean causes a lower allocation of wealth in the risky asset.
  - Given that the expectations of asset performance worsen, the intertemporal hedging demand is negative reducing the participation of the myopic component.
  - The reduction diminishes as γ increases for Growth and Value series. It is constant and larger for the S&P500.

Conclusions



A negative IHD is equivalent to going short in the asset and as a result the myopic participation is reduced.

- As expectations of asset performance worsen, investors hedge against adverse changes in investment opportunities.
  - The S&P500 series performs worse. The myopic demand is reduced by 61.42%.
  - Growth's myopic component is reduced by a lower proportion than that of Value series. It decreases as risk aversion increases.

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