Marriage matching with peer effects

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What we do

- We propose a new static empirical marriage matching function (MMF): the Log Odds MMF
- The Log Odds MMF encompasses:
  - Choo-Siow frictionless transferable utility MMF (CS)
  - The CS with frictional transfers.
  - The Dagsvik Manziel non-transferable utility MMF.
  - A subclass of the Chiappori, Salanie and Weiss MMF (CSW).
  - marriage matching with peer effects.
- Properties of this Log Odds MMF are presented.
- Existence and uniqueness proof of the marriage distribution are provided.
- Comparative statistics are derived.

The CS Benchmark

- Consider a society with I, $i = 1, .., I$, types of men and J, $j = 1, .. J$, types of women.
- Let m and f be the population vectors of men and women respectively.
- Let θ be a vector of parameters.
- A marriage matching function (MMF) is an $I \times J$ matrix valued function $\mu(m, f; \theta)$ whose typical element is μ_{ij}, the number type i men married to type j women.

The CS Benchmark(1)

- Building on the seminal papers of Becker (1973, 1974), Choo and Siow (2006; CS) developed a static frictionless transferable utility.
- Let the utility a male g of type i will get from marriage with a woman of type j, $j = 0, .., J$ be:

$$U_{ijg} = \tilde{u}_{ij} - \tau_{ij} + \epsilon_{ijg} \tag{1}$$

  - $\tilde{u}_{ij}$ is the systematic gross return to a male of type i marrying a female of type j.
  - τ_{ij} is the transfer made by the man to his wife of type j.
  - $u_{ij} = \tilde{u}_{ij} - \tau_{ij}$ a systematic net component common to all (i, j) matches
  - ϵ_{ijg} is an idiosyncratic component which is man specific.
  - ϵ_{ijg} i.i.d. random variable distributed according to the Gumbel distribution.
- Similarly, the utility which a woman k of type j who chooses to marry a type i man, $i = 0, .., I$, is:

$$V_{ijk} = \tilde{v}_{ij} + \tau_{ij} + \epsilon_{ijk}. \tag{2}$$

The CS Benchmark(2)

- Now, let extend the CS framework by assuming that we have multiple type of relationships r (ex: Cohabitation vs marriage)
- Let the utility of male g of type i who matches a female of type j in a relationship r be:

$$U^r_{ijg} = \tilde{u}^r_{ij} - \tau^r_{ij} + \epsilon^r_{ijg} \tag{3}$$

- Let a man who remains unmatched matched a type 0 woman and $\tau^r_{i0} = 0$.
- Individual g will choose according to:

$$U_{ig} = \max_{j,r}\{U_{i0g}, U^a_{i1g}, ..., U^a_{ijg}, ..., U^a_{iJg}, U^b_{i0g}, ..., U^b_{ijg}, ..., U^b_{iJg}\}.$$

- Let $(\mu^r_{ij})^d$ be the number of (r, i, j) matches demanded by i,
- $(\mu_{i0})^d$ be the number of unmatched i type men.

The CS Benchmark(3)

- Following the well known McFadden result, we have:

$$\frac{(\mu_{ij}^r)^d}{m_i} = \mathbb{P}(U_{ijg}^r - U_{ikg}^{r'} \geq 0, k = 1, ..., J; r' = a, b), \tag{4}$$

  where m_i denotes the number of men of type i.
- From above, we obtain a quasi-demand equation by type i men for (r, i, j) relationships.

$$ln \frac{(\mu_{ij}^r)^d}{(\mu_{i0})^d} = \tilde{u}_{ij}^r - \tilde{u}_{i0} - \tau_{ij}^r, \tag{5}$$

The CS Benchmark(4)

- The quasi-supply equation of type j women for (r,i,j) relationships is given by:

$$\ln \frac{(\mu_{ij}^r)^s}{(\mu_{0j})^s} = \tilde{v}_{ij}^r - \tilde{v}_{0j} + \tau_{ij}^r. \tag{6}$$

- The matching market clears when, given equilibrium transfers τ_{ij}^r, we have for all (r,i,j):

$$(\mu_{ij}^r)^d = (\mu_{ij}^r)^s = \mu_{ij}^r. \tag{7}$$

- Substituting (7) into equations (5) and (6) we get the following MMF:

$$\ln \frac{\mu_{ij}^r}{\sqrt{\mu_{i0}\mu_{0j}}} = \pi_{ij}^r \ \forall \ (r,i,j) \tag{8}$$

where $\pi_{ij}^r = \tilde{u}_{ij}^r - \tilde{u}_{i0} + \tilde{v}_{ij}^r - \tilde{v}_{0j}$. (called the the gains to marriage)

The CS Benchmark(4)

- An equilibrium of the CS matching model can be defined as a vector of single-hood $\mu \equiv (\mu_{10}, ..., \mu_{I0}, \mu_{01}, ..., \mu_{0J})'$ that verifies:
  - The CS MMF.

$$\mu_{ij}^{r} = \mu_{i0}^{1/2}\mu_{0j}^{1/2}e^{\pi_{ij}^{r}} \qquad \text{for } r \in \{a, b\}. \tag{9}$$

  - The population constraints.

$$\sum_{j=1}^{J}\mu_{ij}^{a} + \sum_{j=1}^{J}\mu_{ij}^{b} + \mu_{i0} = m_i, \quad 1 \leq i \leq I \tag{10}$$

$$\sum_{i=1}^{I}\mu_{ij}^{a} + \sum_{i=1}^{I}\mu_{ij}^{b} + \mu_{0j} = f_j, \quad 1 \leq j \leq J \tag{11}$$

$$\mu_{0j}, \mu_{i0} > 0, \quad 1 \leq j \leq J, 1 \leq i \leq I.$$

The CS Benchmark(5)

- In others terms, an equilibrium of the CS matching model is the solution of the following quadratic system of equations:

$$\sum_{j=1}^{J} \beta_i \beta_{I+j} (e^{\pi_{ij}^a} + e^{\pi_{ij}^b}) + \beta_i^2 = m_i, \quad 1 \leq i \leq I$$

$$\sum_{i=1}^{I} \beta_i \beta_{I+j} (e^{\pi_{ij}^a} + e^{\pi_{ij}^b}) + \beta_{I+j}^2 = f_j, \quad 1 \leq j \leq J$$

$$\beta_i, \beta_{I+j} > 0, \quad 1 \leq j \leq J, 1 \leq i \leq I.$$

where $\beta_i = \sqrt{\mu_{i0}}$ and $\beta_{I+j} = \sqrt{\mu_{0j}}$

The CS Benchmark(6)

- Using a variational approach Decker et al (2013) showed that for any admissible $(m, f; \theta)$, μ exists and is unique.
- Galichon and Salanié (2013) showed also the existence and the uniqueness using an alternative approach.

Properties of the CS MMF

- The CS is just identified.
- The CS MMF fits any observed marriage distribution.
- It obeys constant returns to scale in population vectors (Graham, 2013)
- The log odds of the number of (r, i, j) relationships relative to the number of (r', i, j) relationships is independent of the sex ratio, m_i / f_j

$$ln \frac{\mu_{ij}^b}{\mu_{ij}^a} = \frac{\pi_{ij}^b - \pi_{ij}^a}{2} \tag{12}$$

  - Independence is a very strong assumption and unlikely to hold every two types of relationships.
    - Arcidiacono et al (2012, ABM) shows that it does not hold for sexual versus non-sexual boy girl relationships in high schools.
    - Mourifié and Siow (2014, in progress) shows that it does not hold for cohabitation versus marriage.
- Could we find a wider class of Matching model that can rationalize such a dependence?

Survey on the related literature

We can distinguish two main directions in the subsequent literature to the CS MMF.

1. One branch relaxes the specification of the idiosyncratic component ϵ^r_{ijg}.
   - The net systematic gains from matching, $u^r_{ij} = \tilde{u}^r_{ij} - \tau^r_{ij}$ and $v^r_{ij} = \tilde{v}^r_{ij} - \tau^r_{ij}$, from CS is retained.
   - The transfer is still used to clear the marriage market.
   - Chiappori, Salanie and Wiess (2012; CSW) allows the variance of ϵ^r_{ijg} to differ by gender and type and obtains

$$\ln \frac{\mu^r_{ij}}{(\mu_{i0})^{1-\lambda_{ij}}(\mu_{0j})^{\lambda_{ij}}} = \pi^r_{ij}(\lambda_{ij}) \ \forall\ (r,i,j) \tag{13}$$

     where $\lambda_{ij} = \frac{\sigma_{mi}}{\sigma_{m_i}+\sigma_{f_j}}$.
   - Graham (2013) provides a wide set of comparative statistics for a special case of CSW i.e. $\lambda_{ij} = \lambda$.
   - Galichon and Salanié (2012, GS) further generalizes the distributions of idiosyncratic utilities.
   - Chiappori and Salanie (2014) provides an excellent state of the art survey of the above and related models.
2. The second branch studies other behavioral specifications for the net systematic return from marriage, u^r_{ij} and v^r_{ij}

Survey on the related literature (1)

We can distinguish two main directions in the subsequent literature to the CS MMF.

1. One branch relaxes the specification of the idiosyncratic component ϵ^r_{ijg}.
2. The second branch studies other behavioral specifications for the net systematic return from marriage, u^r_{ij} and v^r_{ij}
   - Dagsvik (2000) assumes that transfers are not available to clear the marriage market i.e. $\tau^r_{ij} = 0$
   - He assumes that the idiosyncratic payoff of man g of type i marrying woman k of type j, ϵ^r_{ijgk}, is distributed i.i.d. Gumbel.
   - He uses the deferred acceptance algorithm to solve for a matching equilibrium and obtain the following non-transferable utility MMF for large marriage markets:
$$\ln \frac{\mu^r_{ij}}{\mu_{i0}\mu_{0j}} = \pi_{ij} \; \forall \; (r, i, j) \tag{14}$$
   - Manziel (2012) shows that (14) obtains under less restrictive assumptions about the distribution of ϵ^r_{ijgk}. Call (14) the DM MMF.

Survey on the related literature (2)

We can distinguish two main directions in the subsequent literature to the CS MMF.

1. One branch relaxes the specification of the idiosyncratic component ϵ^r_{ijg}.
2. The second branch studies other behavioral specifications for the net systematic return from marriage, u^r_{ij} and v^r_{ij}
   - The DM MMF also fits any observed marriage distribution.
   - Unlike CS, it obeys increasing return to scale in population vectors.

- When there is more than one type of relationship, the log odds of the numbers of different types of relationships is independent of the sex ratio, for all the model surveyed i.e. The CS, CSW and DM MMF.

Log Odds MMF

- So far all the models (i.e. CS, CSW, DM MMF) can be written as follows:

$$\ln \frac{\mu_{ij}^r}{(\mu_{i0})^{\lambda_r}(\mu_{0j})^{\beta_r}} = \pi_{ij}^r \ \forall \ (r,i,j) \tag{15}$$

$$\lambda_r, \beta_r > 0$$

  with
  - $\lambda_r = \lambda_{r'} = \beta_r = \beta_{r'}$ for models with non transferable utility i.e. DM MMF
  - $\lambda_r = \lambda_{r'}; \beta_r = \beta_{r'}$ and $\lambda_r + \beta_r = 1$ for model with transferable utility i.e. CS, CSW.
- Call (25) the Log Odds MMF.
- Notice that when $\lambda_r \neq \lambda_{r'}$ or $\beta_r \neq \beta_{r'}$ we have

$$\ln \frac{\mu_{ij}^r}{\mu_{ij}^{r'}} = (\lambda^r - \lambda^{r'}) \ln(\mu_{i0}) + (\beta^r - \beta^{r'}) \ln(\mu_{0j}) + \pi_{ij}^r - \pi_{ij}^{r'} \ \forall \ (r,i,j).$$

Log Odds MMF (1)

$$\ln \frac{\mu_{ij}^{r}}{\mu_{ij}^{r'}} = (\lambda^{r} - \lambda^{r'}) \ln(\mu_{i0}) + (\beta^{r} - \beta^{r'}) \ln(\mu_{0j}) + \pi_{ij}^{r} - \pi_{ij}^{r'} \ \forall \ (r, i, j)$$

- $m_i / f_j \longrightarrow \ln \mu_{0j}; \ln \mu_{i0} \longrightarrow \ln \frac{\mu_{ij}^{r}}{\mu_{ij}^{r'}}$
- Therefore, the Log Odds MMF relaxes the strong independence property imposed by all previous surveyed MMF.
- Does an equilibrium matching distribution of the Log Odds MMF exist? is it unique?
- Could we rationalize the reduced form Log Odds MMF with some economic structural matching models?

Log Odds MMF (2)

As shown previously, an equilibrium of the CS matching model is the solution of the following system of equations:

$$\mu_{i0} + \sum_{j=1}^{J} \mu_{i0}^{\lambda_a} \mu_{0j}^{\beta_a} e^{\pi_{ij}^a} + \sum_{j=1}^{J} \mu_{i0}^{\lambda_b} \mu_{0j}^{\beta_b} e^{\pi_{ij}^b} = m_i, \text{ for } 1 \leq i \leq I, \tag{16}$$

$$\mu_{0j} + \sum_{i=1}^{I} \mu_{i0}^{\lambda_a} \mu_{0j}^{\beta_a} e^{\pi_{ij}^a} + \sum_{i=1}^{I} \mu_{i0}^{\lambda_b} \mu_{0j}^{\beta_b} e^{\pi_{ij}^b} = f_j, \text{ for } 1 \leq j \leq J. \tag{17}$$

- If λ_r and β_r are rational numbers, this system is equivalent to a polynomial system of equations.
- Could we write this system as a F.O.C of a variational problem? It seems not true in general.
- The only situation where we were able to do that is the case where $\lambda_r = \lambda_{r'}; \beta_r = \beta_{r'}$.
- As discussed previously, this case is not very interesting since it imposes "independence".

Log Odds MMF (3)

- However, let us propose an alternative strategy.
- let consider the following mapping $g : (\mathbb{R}_+^*)^{I+J} \rightarrow (\mathbb{R}_+^*)^{I+J}$

$$g_i(\mu;\theta) = \mu_{i0} + \sum_{j=1}^{J} \mu_{i0}^{\lambda_a} \mu_{0j}^{\beta_a} e^{\pi_{ij}^a} + \sum_{j=1}^{J} \mu_{i0}^{\lambda_b} \mu_{0j}^{\beta_b} e^{\pi_{ij}^b}, \text{ for } 1 \leq i \leq I, \tag{18}$$

$$g_{j+I}(\mu;\theta) = \mu_{0j} + \sum_{i=1}^{I} \mu_{i0}^{\lambda_a} \mu_{0j}^{\beta_a} e^{\pi_{ij}^a} + \sum_{i=1}^{I} \mu_{i0}^{\lambda_b} \mu_{0j}^{\beta_b} e^{\pi_{ij}^b}, \text{ for } 1 \leq j \leq J. \tag{19}$$

Log Odds MMF (4)

Hadamard's Theorem (Krantz and Park (2003, Theorem 6.2.8 p 126))

Let M_1 and M_2 be smooth, connected N-dimensional manifolds and let $f : M_1 \to M_2$ be a C^1 function if

(1) f is proper,

(2) the Jacobian of f vanishes nowhere,

(3) M_2 is simply connected,

then f is a homeomorphism.

Log Odds MMF (5)

Let see if our mapping g verifies those conditions.

- There are many ways to verify that g is proper.
- A continuous function between topological spaces is called proper if the inverse images of compact subsets are compact.
- An easy way is to use the following lemma

Lemma (Krantz and Park (2003, p 125))

Let U and V be connected open sets in $\mathbb{R}^{I+J}$, g: $\mathbb{U} \rightarrow \mathbb{V}$ is a proper mapping if and only if whenever $\{x_j\} \subseteq U$ satisfies $x_j \rightarrow \partial U$ then $g(x_j) \rightarrow \partial V$.

- Does the Jacobian of g vanish nowhere?

Log Odds MMF (6)

- Let write

$$\mu_{i0}^{\lambda_r}\mu_{0j}^{\beta_r}e^{\pi_{ij}^r} = e^{\lambda_r ln\mu_{i0}+\beta_r ln\mu_{0j}+\pi_{ij}^r}, \quad (20)$$

$$\equiv e^{\delta_{ij}^r} \quad (21)$$

- Let $J_g(\mu)$ be the Jacobian of g.
- After a simple derivation we can show that $J_g(\mu)$ takes the following form:
  $J_g(\mu) = \begin{pmatrix} (J_g)_{11}(\mu) & (J_g)_{12}(\mu) \\ (J_g)_{21}(\mu) & (J_g)_{22}(\mu) \end{pmatrix}$
  with

Log Odds MMF (7)

- $(J_g)_{11}(\mu) =$
$$\begin{pmatrix} 1+\sum_{j=1}^{J}\left[\frac{\lambda_a}{\mu_{10}}e^{\delta^a_{1j}}+\frac{\lambda_b}{\mu_{10}}e^{\delta^b_{1j}}\right] & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1+\sum_{j=1}^{J}\left[\frac{\lambda_a}{\mu_{I0}}e^{\delta^a_{Ij}}+\frac{\lambda_b}{\mu_{I0}}e^{\delta^b_{Ij}}\right] \end{pmatrix}$$

- $(J_g)_{21}(\mu) = \begin{pmatrix} \frac{\lambda_a}{\mu_{10}}e^{\delta^a_{11}}+\frac{\lambda_b}{\mu_{10}}e^{\delta^b_{11}} & \cdots & \frac{\lambda_a}{\mu_{I0}}e^{\delta^a_{I1}}+\frac{\lambda_b}{\mu_{I0}}e^{\delta^b_{I1}} \\ \vdots & \ddots & \vdots \\ \frac{\lambda_a}{\mu_{10}}e^{\delta^a_{1J}}+\frac{\lambda_b}{\mu_{10}}e^{\delta^b_{1J}} & \cdots & \frac{\lambda_a}{\mu_{I0}}e^{\delta^a_{IJ}}+\frac{\lambda_b}{\mu_{I0}}e^{\delta^b_{IJ}} \end{pmatrix},$

Log Odds MMF (8)

- $(J_g)_{12}(\mu) = \begin{pmatrix} \frac{\beta_a}{\mu_{01}} e^{\delta^a_{11}} + \frac{\beta_b}{\mu_{01}} e^{\delta^b_{11}} & \cdots & \frac{\beta_a}{\mu_{0J}} e^{\delta^a_{1J}} + \frac{\beta_b}{\mu_{0J}} e^{\delta^b_{1J}} \\ \vdots & \ddots & \vdots \\ \frac{\beta_a}{\mu_{01}} e^{\delta^a_{I1}} + \frac{\beta_b}{\mu_{01}} e^{\delta^b_{I1}} & \cdots & \frac{\beta_a}{\mu_{0J}} e^{\delta^a_{IJ}} + \frac{\beta_b}{\mu_{0J}} e^{\delta^b_{IJ}} \end{pmatrix},$
- $(J_g)_{22}(\mu) =$
$\begin{pmatrix} 1 + \sum_{i=1}^{I} \left[\frac{\beta_a}{\mu_{01}} e^{\delta^a_{i1}} + \frac{\beta_b}{\mu_{01}} e^{\delta^b_{i1}} \right] & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 + \sum_{i=1}^{I} \left[\frac{\beta_a}{\mu_{0J}} e^{\delta^a_{iJ}} + \frac{\beta_b}{\mu_{0J}} e^{\delta^b_{iJ}} \right]. \end{pmatrix}$

Log Odds MMF (9)

- Let us denote every element of $J_g(\mu)$, $J_{k,l}$ with $1 \leq k, l \leq I + J$.
- We can remark that $|J_{ll}| > \sum_{k \neq l}^{I+J} |J_{kl}|$ for $l = 1, ..., I + J$.
- So , $J_g(\mu)$ is a column diagonally dominant matrix or diagonally dominant in the sense of McKenzie (1960) far all $\mu > 0$.
- Thus, $J_g(\mu)$, for all $\mu > 0$, is a non-singular matrix. (Proof see McKenzie (1960).
- Then, our mapping g is an homeomorphism.
- Therefore, the system of equation (18) admits a unique solution.
- We can easily check that the solution is economically relevant in the sense that $0 < \mu^{eq} < (m', f')'$.

Log Odds MMF : Existence and Uniqueness

Existence and Uniqueness of the equilibrium matching

For every fixed matrix of relationship gains and coefficients $\beta_r; \lambda_r > 0$ i.e. $\theta \in \pi \times (0, \infty)^2$, the equilibrium matching of the log Odds MMF model exists and is unique.

CS with frictional transfers (CSFT), Mourifié and Siow (2014)

Let introduce an extended version of the CS model, by allowing frictions.

- Let the utility of male g of type i who matches a female of type j in a relationship r be:

$$U^r_{ijg} = \tilde{u}^r_{ij} - \alpha_r \tau^r_{ij} + \epsilon^r_{ijg} \tag{22}$$

- Similarly, the utility of a woman, is:

$$V^r_{ijk} = \tilde{v}^r_{ij} + \tau^r_{ij} + \epsilon^r_{ijk}. \tag{23}$$

- male pays $\alpha^r \tau^r_{ij}$ in utility to his partner for the relationship.
- males are assumed to be always payers and women receivers, so $\tau^r_{ij} \geq 0$.
- $\alpha_r \geq 1$, so that women value the transfer less than what it costs the men.

CSFT

Using again McFadden result, we can write the men and women quasi-demand and supply and clears the matching market using the transfer. We get the following MMF

$$\ln \frac{\mu_{ij}^{r}}{(\mu_{i0})^{1-\frac{\alpha_r}{1+\alpha_r}} (\mu_{0j})^{\frac{\alpha_r}{1+\alpha_r}}} = \frac{\pi_{ij}^{r}}{1+\alpha_r} \; \forall \, (r,i,j) \tag{24}$$

where $\pi_{ij}^{r} = \tilde{u}_{ij}^{r} - \tilde{u}_{i0} + \alpha^{r}(\tilde{v}_{ij}^{r} - \tilde{v}_{0j})$

- Let Call (24) the CSFT MM and $\lambda_r = \frac{\alpha_r}{1+\alpha_r}$.
- Then, CSFT MMF is a particular case the Log Odds MMF.
- We can remark that

$$\ln \frac{\mu_{ij}^{r'}}{\mu_{ij}^{r}} = (\lambda^{r} - \lambda^{r'}) \ln \frac{\mu_{0j}}{\mu_{i0}} + \pi_{ij}^{r} - \pi_{ij}^{r'} \; \forall \, (r,i,j)$$

- CSFT MMF is different from CS, CSW, DM MMF in the sense that $\lambda_r \neq \lambda_{r'}; \beta_r \neq \beta_{r'}$. However we still have $\lambda_r + \beta_r = 1$

CSFT (1)

- Far all the models (i.e. CS, CSW, CSFT MMF) that used the transfers to clear the market can be written as follows:

$$\ln \frac{\mu_{ij}^r}{(\mu_{i0})^{\lambda_r}(\mu_{0j})^{\beta_r}} = \pi_{ij}^r \; \forall \; (r,i,j) \tag{25}$$

$$\lambda_r, \beta_r > 0$$

  with $\lambda_r + \beta_r = 1$.

- In the DM MMF we have $\lambda_r = \beta_r = 1$.
- Let introduce an extended version of the CS model which allows presence of peer effects.
- Our model of multinomial choice with peer effects is standard and follows Blume, et. al. (2001)
- What is new is our application to two sided matching.

Choo-Siow with peer effect (CSPE)

- Let the utility of male g of type i who matches a female of type j in a relationship r be:

$$U^r_{ijg} = \tilde{u}^r_{ij} + \phi^r \ln \mu^r_{ij} - \tau^r_{ij} + \epsilon^r_{ijg}, \text{ where} \tag{26}$$

  - $\tilde{u}^r_{ij} + \phi^r \ln \mu^r_{ij}$: Systematic gross return to a male of type i matching to a female of type j in relationship r.
  - ϕ^r: Coefficient of peer effect for relationship r, $1 \geq \phi^r \geq 0$.
  - μ^r_{ij}: Equilibrium number of (r, i, j) relationships.
  - $\tilde{u}_{i0} + \phi^0 \ln \mu^0_{i0}$ is the systematic payoff that type i men get from remaining unmatched, $1 \geq \phi^0 \geq 0$.
  - We allow the peer effect to differ by relationship.
  - For example, unmarried individuals spend more time with their unmarried friends than married individuals with their married friends.
  - There is no apriori reason to rank ϕ^0 versus ϕ^r.

CSPE

- Similarly, the utility of a woman, is: male g of type i who matches a female of type j in a relationship r be:

$$V^r_{ijk} = \tilde{v}^r_{ij} + \Phi^r \ln \mu^r_{ij} + \tau^r_{ij} + \epsilon^r_{ijk}, \tag{27}$$

Using again McFadden result, we can write the men and women quasi-demand and supply and clears the matching market using the transfers. We get the following MMF

$$\ln \mu^r_{ij} = \frac{1-\phi^0}{2-\phi^r-\Phi^r} \ln \mu_{i0} + \frac{1-\Phi^0}{2-\phi^r-\Phi^r} \ln \mu_{0j} + \frac{\pi^r_{ij}}{2-\phi^r-\Phi^r} \tag{28}$$

where $\pi^r_{ij} = \tilde{u}^r_{ij} - \tilde{u}_{i0} + \tilde{v}^r_{ij} - \tilde{v}_{0j}$.

- Call (28), the CSPE MMF

CSPE (1)

- With the CSPE MMF we no longer have necessarily $\lambda_r + \beta_r = 1$.
- When there is no peer effect or all the peer effect coefficients are the same,

$$\phi^0 = \Phi^0 = \phi^r = \Phi^r$$

  we recover the CS MMF
- When

$$\frac{1-\phi^0}{2-\phi^r-\Phi^r} = \frac{1-\Phi^0}{2-\phi^r-\Phi^r} = 1$$

  we recover the DM MMF

Log Odds MMF: Properties

- Let us summarize the different models and some of their properties.

Models and restrictions on λ^r and β^r			
Model	λ^r	β^r	Restrictions
Log Odds MMF	λ^r	β^r	$\lambda^r \geq 0, \beta^r \geq 0$
CS	$\frac{1}{2}$	$\frac{1}{2}$	$\lambda^r = \beta^r = \frac{1}{2}$
DM	1	1	$\lambda^r = \beta^r = 1$
CSW	λ^r	1-λ^r	$\lambda^r = \lambda^{r'} > 0$
CSFT	$\frac{\alpha^r}{1+\alpha^r}$	$\frac{1}{1+\alpha^r}$	$\lambda^r + \beta^r = 1, 1 > \lambda^r > 0$
CSPE	$\frac{1-\phi^0}{2-\phi^r-\Phi^r}$	$\frac{1-\Phi^0}{2-\phi^r-\Phi^r}$	$\lambda^r, \beta^r \geq 0, \frac{\lambda^r}{\lambda^{r'}} = \frac{\beta^r}{\beta^{r'}}$

Log Odds MMF : Comparative statics

- Unlike the Variational approach this methodology does not provide us intuitions about some comparative statics.
- Following Graham (2013), let us propose a fixed point representation of the equilibrium of the log Odds MMF.
- The log Odds MMF ca be written as follows: for all (i,j) pairs:

$$\frac{\mu_{ij}^r}{\mu_{i0}} = exp\left[\pi_{ij}^r + (\lambda_r - 1)ln\mu_{i0} + \beta_r\mu_{0j}\right] \equiv \eta_{ij}^r \qquad \text{for } r \in \{a,b\}, \tag{29}$$

$$\frac{\mu_{ij}^r}{\mu_{0j}} = exp\left[\pi_{ij}^r + \lambda_r ln\mu_{i0} + (\beta_r - 1)\mu_{0j}\right] \equiv \zeta_{ij}^r \qquad \text{for } r \in \{a,b\}. \tag{30}$$

Log Odds MMF : Comparative statics (1)

- Manipulating the population constraints (10), (11) we have the following:

$$\mu_{i0} = \frac{m_i}{1 + \sum_{j=1}^{J} \left[\eta_{ij}^a + \eta_{ij}^b \right]} \equiv B_{i0}, \quad 1 \leq i \leq I \tag{31}$$

$$\mu_{0j} = \frac{f_j}{1 + \sum_{i=1}^{I} \left[\zeta_{ij}^a + \zeta_{ij}^b \right]} \equiv B_{0j}, \quad 1 \leq j \leq J. \tag{32}$$

- Let $B(\mu; m, f, \theta) \equiv (B_{10}(.), ..., B_{I0}(.), B_{01}(.), ..., B_{0J}(.))'$.
- For a fixed θ we have shown that the $(I+J)$ vector μ of unmatched is a solution to $(I+J)$ vector of implicit functions

$$\mu - B(\mu; m, f, \theta) = 0. \tag{33}$$

- Now, let $J(\mu) = I_{I+J} - \bigtriangledown_{\mu} B(\mu; m, f, \theta)$ with $\bigtriangledown_{\mu} B(\mu; m, f, \theta) = \frac{\partial B(\mu; m, f, \theta)}{\partial \mu'}$ be the $(I+J) \times (I+J)$ Jacobian matrix associated with (33).

Log Odds MMF : Comparative statics (2)

- To derive the different comparative statics, we generalize the Graham (2013) approach and then show the following results.

1. $J(\mu^{eq}) = U(\mu^{eq})H(\mu^{eq})U(\mu^{eq})^{-1}$ where $U(\mu^{eq})$ is a diagonal matrix and
   - $H(\mu^{eq})$ a matrix with non-negative elements when $\lambda_r \leq 1$ and $\beta_r \leq 1$
   - $H(\mu^{eq})$ is a row stochastic matrix (a matrix with non-negative elements where the rows sum to one) whenever $\lambda_r + \beta_r = 1$
2. $H(\mu^{eq})$ is row diagonally dominant if
   1. $\max(\beta_b - \lambda_b, \beta_a - \lambda_a) < \min_{i \in I} \left(\frac{1-\rho_i^m}{\rho_i^m} \right)$;
   2. $\min(\beta_b - \lambda_b, \beta_a - \lambda_a) > -\max_{j \in J} \left(\frac{1-\rho_j^f}{\rho_j^f} \right)$;

   where ρ_i^m is the rate of matched men of type i and ρ_j^f is the rate of matched women of type j.
3. Therefore, we can show that H^{-1} has the following sign structure

$$H^{-1}(\mu^{eq}) = \begin{pmatrix} + & \vdots & - \\ \cdots & & \cdots \\ - & \vdots & + \end{pmatrix}.$$

Log Odds MMF : Comparative statics (3)

- the sign structure of H^{-1} is obtained because:
  1. The Schur complements of the $H(\mu)$ upper $I \times I$ and lower $J \times J$ diagonal blocks are $SH_{11} = H_{22} - H_{21}(H_{11})^{-1}H_{12}$ and $SH_{22} = H_{11} - H_{12}(H_{22})^{-1}H_{21}$.
  2. H row diagonally dominant $\Rightarrow$ the two Schur complement are also diagonally dominant. See Theorem 1 of Carlson and Markham (1979 p 249).
  3. SH_{11} and SH_{22} are also Z-matrices (i.e., members of the class of real matrices with nonpositive off-diagonal elements).
  4. Then, SH_{11} and SH_{22} are M-matrices $\implies SH_{11}^{-1} \geqq 0$ and $SH_{22}^{-1} \geqq 0$. see Theorem 4.3 of Fiedler and Ptak (1962).
- By applying the implicit function theorem to the equation (33) we have:
  1. $\frac{\partial \mu}{\partial m_i} = J(\mu)^{-1} \frac{\partial B}{\partial m_i}$ for $1 \leq i \leq I$
  2. $\frac{\partial \mu}{\partial f_j} = J(\mu)^{-1} \frac{\partial B}{\partial f_j}$ for all $1 \leq j \leq J$

Log Odds MMF : Comparative statics (4)

Comparative Statics (1)

Let μ be the equilibrium matching distribution of the log Odds MMF model. If the coefficients β_r and λ_r respect the restrictions

1. $0 < \beta_r; \lambda_r \leq 1$ for $r \in \{a, b\}$;
2. $\max(\beta_b - \lambda_b, \beta_a - \lambda_a) < \min_{i \in I} \left(\frac{1-\rho_i^m}{\rho_i^m} \right)$;
3. $\min(\beta_b - \lambda_b, \beta_a - \lambda_a) > -\max_{j \in J} \left(\frac{1-\rho_j^f}{\rho_j^f} \right)$;

- Type-specific elasticities of single-hood.
  The following inequalities hold in the neighbourhood of μ^{eq}:

$$\frac{m_i}{\mu_{k0}} \frac{\partial \mu_{k0}}{\partial m_i} \geq \begin{cases} \frac{1}{m_i^*} \frac{m_k}{m_k^*} \sum_{j=1}^{J} \frac{[\lambda_a \mu_{kj}^a + \lambda_b \mu_{kj}^b][\beta_a \mu_{kj}^a + \beta_b \mu_{kj}^b]}{f_j^*} > 0 & \text{if } k \neq i \\ \frac{m_i}{m_i^*} [1 + \frac{1}{m_i^*} \sum_{j=1}^{J} \frac{[\lambda_a \mu_{ij}^a + \lambda_b \mu_{ij}^b][\beta_a \mu_{ij}^a + \beta_b \mu_{ij}^b]}{f_j^*}] > 1 & \text{if } k = i, \end{cases}$$

$1 \leq k \leq I$.

$$\frac{f_j}{\mu_{0k}} \frac{\partial \mu_{0k}}{\partial f_j} \geq \begin{cases} \frac{1}{f_j^*} \frac{f_k}{f_k^*} \sum_{i=1}^{I} \frac{[\lambda_a \mu_{ik}^a + \lambda_b \mu_{ik}^b][\beta_a \mu_{ik}^a + \beta_b \mu_{ik}^b]}{m_i^*} > 0 & \text{if } k \neq j \\ \frac{f_j}{f_j^*} [1 + \frac{1}{f_j^*} \sum_{i=1}^{I} \frac{[\lambda_a \mu_{ij}^a + \lambda_b \mu_{ij}^b][\beta_a \mu_{ij}^a + \beta_b \mu_{ij}^b]}{m_i^*}] > 1 & \text{if } k = j, \end{cases} \quad 1 \leq k \leq J,$$

Log Odds MMF : Comparative statics (5)

Comparative Statics (2)

- Type-specific elasticities of single-hood.
  The following inequalities hold in the neighbourhood of μ^{eq}:

$$\frac{m_i}{\mu_{0j}}\frac{\partial\mu_{0j}}{\partial m_i} \leq -\frac{[\lambda_a\mu_{ij}^a + \lambda_b\mu_{ij}^b]}{m_i^* f_j^*} m_i < 0, \text{ for } 1 \leq i \leq I \text{ and } 1 \leq j \leq J,$$

$$\frac{f_j}{\mu_{i0}}\frac{\partial\mu_{i0}}{\partial f_j} \leq -\frac{[\beta_a\mu_{ij}^a + \beta_b\mu_{ij}^b]}{m_i^* f_j^*} f_j < 0, \text{ for } 1 \leq i \leq I \text{ and } 1 \leq j \leq J,$$

  where

$$m_i^* \equiv m_i - \sum_{j=1}^{J}[(1-\lambda_a)\mu_{ij}^a + (1-\lambda_b)\mu_{ij}^b], \text{ for } 1 \leq i \leq I,$$

$$f_j^* \equiv f_j - \sum_{i=1}^{I}[(1-\beta_a)\mu_{ij}^a + (1-\beta_b)\mu_{ij}^b], \text{ for } 1 \leq j \leq J.$$

Log Odds MMF : Comparative statics (6)

- We can derive comparative statistics for $\ln \frac{\mu^r_{kl}}{\mu^{r'}_{kl}}$.
- For $\lambda_r > \lambda_{r'}$ and $\beta_r > \beta_{r'}$ we have

$$\frac{1}{\partial m_i} \ln \left(\frac{\mu^r_{kl}}{\mu^{r'}_{kl}} \right) > 0$$

$$\frac{1}{\partial f_j} \ln \left(\frac{\mu^r_{kl}}{\mu^{r'}_{kl}} \right) > 0$$

Log Odds MMF : Return to scale

- One important question is to know under which conditions on λ_r and β_r the Log Odds MMF admit constant (increasing or decreasing) return to scale?
- In other terms, holding the type distributions of men and women fixed, does increasing market size has an effect on the probability of matching?
- We can show

$$\sum_{i=1}^{I} U(\mu)^{-1} \frac{\partial \mu}{\partial m_i} m_i + \sum_{j=1}^{J} U(\mu)^{-1} \frac{\partial \mu}{\partial f_j} f_j = \sum_{i=1}^{I} [H(\mu)^{-1}]_{\cdot i} + \sum_{j=1}^{J} [H(\mu)^{-1}]_{\cdot (I+j)}.$$

- Whenever $\lambda_r + \beta_r = 1$, $H(\mu)$ is a row stochastic matrix $\Rightarrow$ the row of the $H(\mu)^{-1}$ matrix sum to one $\Rightarrow$ CRS.
- Our intuition is whenever $\lambda_r + \beta_r > 1$ ($\lambda_r + \beta_r < 1$) we have increasing (decreasing) return to scale.
- However, we have not be able to show that yet? any suggestions ?

ConclUion

- We propose a new static empirical marriage matching function (MMF): the Log Odds MMF
- The Log Odds MMF encompasses CS, CSW, CSFT, CSPE, DM MMF
- Properties of this Log Odds MMF are presented.
- Existence and uniqueness proof of the marriage distribution are provided.
- Comparative statistics are derived.