Weak amenability of Fourier algebras and spectral synthesis of the antidiagonal

Nico Spronk (U. Waterloo)

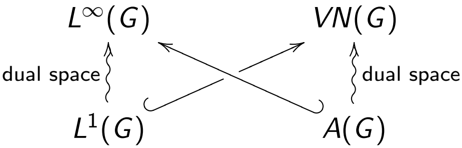
Joint work with
Hun Hee Lee (Seoul National U.)
Jean Ludwig (U. Lorraine – Metz)
Ebrahim Samei (U. Saskatchewan)

Workshop on Recent Developments in Quantum Groups, Operator Algebras and Applications
U. Ottawa
February 5, 2015

Group and Fourier algebras

G – locally compact group, m_l, m_r – left/right Haar measures
$L^1(G)$ – group algebra, convolution product
– predual of commutative $(L^\infty(G), \Gamma, m_l, m_r)$
$A(G)$ – Fourier algebra, pointwise product in $\mathcal{C}_0(G)$
– predual of co-commutative $(VN(G), \widehat{\Gamma}, \hat{m})$

Generalized Pontryagin duality diagram:



In particular, G abelian $\quad\Rightarrow\quad A(G) \cong L^1(\widehat{G})$.

Amenability

$\mathcal{A}$ – Banach algebra, $\mathcal{M}$ –Banach $\mathcal{A}$-bimodule

$$H^1(\mathcal{A},\mathcal{M}) = \frac{\{D \in \mathcal{B}(\mathcal{A},\mathcal{M}) : D(ab) = D(a)b + aD(b)\}}{\{a \mapsto ax - xa : x \in \mathcal{M}\}}$$

Definition [Johnson,'73]

$\mathcal{A}$ <u>amenable</u> if $H^1(\mathcal{A},\mathcal{M}^*) = \{0\}$, $\forall\ \mathcal{M}^*$ – dual $\mathcal{A}$-bimodule

$L^1(G)$ Banach bimodules $\leftrightsquigarrow$ bounded G-bimodules.

Theorem [Johnson,'73 &'72]

(i) $L^1(G)$ amenable $\Leftrightarrow$ G amenable.
(ii) $\mathcal{A}$ amenable $\Leftrightarrow$ $\mathcal{A}$ admits b.a.d. (averaging net)

Bounded approximate diagonal (b.a.d.): $(d_\alpha) \subset \mathcal{A}\hat{\otimes}\mathcal{A}$

$$\mathrm{mult}(d_\alpha)a \to a \quad \text{and} \quad a \otimes 1 \cdot d_\alpha - d_\alpha \cdot 1 \otimes a \to 0.$$

Weak amenability

Theorem [Singer-Wermer '55]

$\mathcal{A}$ commutative & semisimple $\Rightarrow H^1(\mathcal{A},\mathcal{A}) = \{0\}$.

Definition [Bade-Curtis-Dales '87]

$\mathcal{A}$ commutative.
$\mathcal{A}$ weakly amenable if $H^1(\mathcal{A},\mathcal{S}) = 0$, $\forall$ symmetric bimodule $\mathcal{S}$.

Proposition [Bade-Curtis-Dales '87]

$\mathcal{A}$ commutative. $\mathcal{A}$ weakly amenable $\Leftrightarrow H^1(\mathcal{A},\mathcal{A}^*) = \{0\}$.

Theorem [Johnson, '91]

$H^1(L^1(G), L^1(G)^*) = \{0\}$, i.e. $L^1(G)$ always "weakly amenable".

Weak amenability, operator (weak) amenability of $A(G)$

Theorem [Johnson, '94]

$A(\mathrm{SO}(3))$ not weakly amenable!

Motivated completely bounded versions:

Operator amenability: $H^1_{cb}(A(G), \mathcal{M}^*) = \{0\}$ $\forall$ c.b. $A(G)$-bimod.

Operator weak amenability: $H^1_{cb}(A(G), VN(G)) = \{0\}$

All $L^1(G)$ results automatically completely bounded.

Theorem [Ruan '95]

$A(G)$ operator amenable $\Leftrightarrow$ G amenable.

Theorem [S. '02, Samei '05]

$A(G)$ always operator weakly amenable

When does weak amenability fail for $A(G)$?

Theorem [Forrest-Runde '05]

(i) $A(G)$ amenable $\Leftrightarrow$ G virtually abelian.
(ii) connected component G_e abelian $\Rightarrow$ $A(G)$ w.a.

Basic Observation

Let $H \leq G$ be closed.
(i) [McMullen '72, Herz '73, *et al*] $A(G)|_H = A(H)$.
(ii) [Bade-Curtis-Dales '87] $A(H)$ not w.a. $\Rightarrow$ $A(G)$ not w.a. either.
Hence, problem of w.a. for $A(G)$ reduces to connected groups.

The following connected groups known not to have w.a. $A(G)$:

- non-abelian compact [Forrest-Samei-S. '09] (after [Plymen '94]);
- $ax + b$ (hence non-compact semi-simple Lie), and reduced Heisenberg $\mathbb{H}^r$ [Choi-Ghandehari '14];
- Heisenberg [Choi-Ghandehari '15].

Technique: use a Lie derivative to show $H^1(A(G), VN(G)) \neq \{0\}$.

Spectral and local synthesis

$A_c(G) = \{u \in A(G) : \mathrm{supp}\, u \text{ compact}\}$, $\overline{A_c(G)} = A(G)$.
$A(G)$ regular: separation of compact sets form closed sets

$E \subset G$ closed. Define ideals

$$\begin{aligned} I_G(E) &= \{u \in A(G) : u|_E = 0\} \\ J_G(E) &= \overline{\{u \in A_c(G) : u|_E = 0\}} \\ I_G^0(E) &= \{u \in A_c(G) : \mathrm{supp}\, u \cap E = \varnothing\} \\ \text{so } \overline{I_G^0(E)} &\subseteq J_G(E) \subseteq I_G(E). \end{aligned}$$

Then E is of

- spectral synthesis if $\overline{I_G^0(E)} = I_G(E)$;
- local synthesis (l.s.) if $\overline{I_G^0(E)} = J_G(E)$.

Concepts coincide if $A(G)$ admits approximate identity.
E.g. G has approximation property of Haagerup-Kraus.

The role of spectral and local synthesis

Proposition [Herz '73, Singer-Wermer '55]

$\{e\}$ spec'l synthesis $\Rightarrow \overline{I_G(\{e\})^2} = I_G(\{e\}) \Leftrightarrow H^1(A(G), \mathbb{C}) = \{0\}$.

$A(G)^\sharp$ – unitization, $m^\sharp : A(G)^\sharp \hat{\otimes} A(G)^\sharp \to A(G)^\sharp$,
$m : A(G) \hat{\otimes} A(G) \to A(G)$ multiplications

Theorem [Grønbæk '89]

$A(G)$ w.a. $\Leftrightarrow \overline{(\ker m)^2} = \overline{A(G) \otimes A(G) \cdot \ker m^\sharp}$

Theorem [Forrest-Samei-S. '05]

G SIN-group
$A(G)$ w.a. $\Leftrightarrow \check{\Delta}_G = \{(g, g^{-1}) : g \in G\}$ loc. syn. for $G \times G$

Note: In [S. '02, Samei '05] spectral synthesis of $\Delta_G = \{(g, g) : g \in G\}$ for $G \times G$ ([Herz '73]) is used to show *operator* w.a. of $A(G)$.

Our main new idea [LLSS]

Theorem

G connected Lie group.
$A(G)$ w.a. $\Rightarrow \check{\Delta}_G = \{(g, g^{-1}) : g \in G\}$ loc. syn. for $G \times G$.

Ideas:
• $[A_c(G) \times A_c(G)] \cap J_{G\times G}(\check{\Delta}_G) = J_{G\times G}(\check{\Delta}_G)$.
• Use [Grønbæk '89] and calculations to show
$\overline{J_{G\times G}(\check{\Delta}_G)^m} = J_{G\times G}(\check{\Delta}_G)$
• [Park-Samei '09] (after [Ludwig-Turowska '09]) show that
$J_{G\times G}(\check{\Delta}_G)$ is of local "weak" synthesis, whence of l.s.
Warning: result quantitative, based on dim G.

Theorem

(i) $H \leq G$ connected, $\check{\Delta}_G$ l.s. for $G \times G \Rightarrow \check{\Delta}_H$ l.s. for $H \times H$
(ii) $\Lambda \lhd G$ discrete, $\check{\Delta}_G$ l.s. for $G \times G \Leftrightarrow \check{\Delta}_{G/\Lambda}$ l.s. $G/\Lambda \times G/\Lambda$

Five (classes of) groups to check

Proposition (folklore)

Each non-abelian Lie algebra $\mathfrak{g}$ contains one of

$$\begin{aligned}
\mathfrak{su}(2) &= \langle X, Y, Z : [X, Y] = 2Z, [Y, Z] = 2X, [Z, X] = 2Y\rangle \\
\mathfrak{f} &= \langle X, Y : [X, Y] = Y\rangle \\
\mathfrak{e} &= \langle T, X_1, X_2 : [T, X_1] = X_2, [X_2, T] = X_1, [X_1, X_2] = 0\rangle \\
\mathfrak{g}_\theta &= \langle T, X_1, X_2 : [T, X_1] = X_1 - \theta X_2, [T, X_2] = \theta X_1 + X_2, \\
&\qquad [X_1, X_2] = 0\rangle, \ (\theta > 0) \\
\mathfrak{h} &= \langle X, Y, Z : [X, Y] = Z, [Y, Z] = 0 = [X, Z]\rangle
\end{aligned}$$

Hence every simply connected Lie group contains one of $\mathrm{SU}(2)$, F (affine motion), $\widetilde{E}(2)$ (Euclidean motion, simply connected cover), G_θ (Grélaud), or $\mathbb{H}$ (Heisenberg).

Basic strategy

Goal: If G one of the five groups above, show that

$$\overline{I^0_{G\times G}(\check{\Delta}_G)} \subsetneq J_{G\times G}(\check{\Delta}_G).$$

Hence we find S in $VN(G)$ for which

$$S \perp I^0_{G\times G}(\check{\Delta}_G) \text{ but } S \not\perp J_{G\times G}(\check{\Delta}_G). \qquad (\heartsuit)$$

Proposition

Suppose G is a connected Lie group, and there are X in $\mathfrak{g}$ and v in $L^1(G)$ such that

$$S_{X,v} \in VN(G\times G), \quad \langle S_{X,v}, u\rangle = \int_G \partial_{(X,0)} u(g, g^{-1}) v(g)\, dg$$

for $u \in \mathcal{C}_c^\infty(G)$, then $S_{X,v}$ satisfies $(\heartsuit)$.

Remark: easier to show linear funct'l is bdd., than an operator.

Basic strategy (continued)

For each of our five basic groups pick a Lie derivative:

- any, if $\mathfrak{su}(2)$;
- $X \in \mathfrak{n}$ where $\mathfrak{g} = \mathfrak{n} \rtimes \mathfrak{a}$, if $\mathfrak{g} = \mathfrak{e}, \mathfrak{f}, \mathfrak{g}_\theta$;
- $Z \in \mathfrak{z}$ (centre), if $\mathfrak{g} = \mathfrak{h}$.

This is never a Lie derivative in a "quotient" direction.

We work in the situation with easiest Plancherel for $L^2(G)$:

- $E(2)$ (1-parameter direct interval) and $\mathbb{H}^r$ (almost atomic);
- $\mathrm{SU}(2)$, F (atomic); G_θ (1-parameter direct intergal).

We have ad-hoc choices for v in $L^1(G)$, e.g. $v = 1$ for $\mathrm{SU}(2)$.

The main result

Theorem

G connected Lie group. TFAE:
(a) G abelian; **(b)** $A(G)$ w.a.; and **(c)** $\check{\Delta}_G$ l.s. for $G \times G$

Corollary

If G is locally compact, and contains non-abelian closed, connected, Lie subgroup, then $A(G)$ not w.a.
In particular, if G is Lie, $A(G)$ w.a. $\Leftrightarrow$ G_e is abelian.

Question: Does every non-abelian connected l.c. group contain a non-abelian closed, connected, Lie subgroup?

A sufficient condition ...

[Gleason, Yamambe, Montgomery-Zippin '50s]
G connected $\Rightarrow$ G pro-Lie: $G = \varprojlim_{N \searrow \{e\}} G/N$, G/N Lie.

[Hoffman-Morris '07] G connected, pro-Lie (l.c.)

$$G^{(0)} = G,\ G^{(n)} = [G^{(n-1)}, G^{(n-1)}] \text{ and } G^{(\infty)} = \bigcap_{n=1}^{\infty} G^{(n)}.$$

G is <u>pro-solvable</u> if $G^{(\infty)} = \{e\}$.
Otherwise, $\prod_{i \in I} S_i \to G^{(\infty)} \to \prod_{i \in I} S_i / Z(S_i)$, S_i semi-simple Lie.

Proposition

G *not* pro-solvable $\Rightarrow$ G contains connected semi-simple Lie group.

Question: Does a non-abelian (l.c.) pro-solvable G always contain a closed non-abelian connected Lie H?

... which reduces us to "easy" cases

"Big" reduced Heisenberg group [Cheng-Forrest-S. '13]:

$$\overline{\mathbb{H}}^r = (\mathbb{R} \times \mathbb{R}^{\mathrm{ap}}) \rtimes \mathbb{R}, \ (y, \zeta, x)(y', \zeta', x') = (y + y', \zeta\zeta'\eta(xy'), x + x')$$

where $\eta : \mathbb{R} \to \mathbb{T}^{\mathbb{R}}$, $\eta(t) = (e^{iyt})_{y \in \mathbb{R}}$ and $\mathbb{R}^{\mathrm{ap}} = \overline{\eta(\mathbb{R})}$.

Fact: the only non-trivial closed connected Lie subgroups are $\mathbb{R} \times \{1\} \times \{0\}$ and $\{0\} \times \{1\} \times \mathbb{R}$.

Questions

(i) Is $A(\overline{\mathbb{H}}^r)$ w.a.?
(ii) If G is l.c., non-abelian pro-solvable and connected, can $A(G)$ be w.a.?

Answer to (ii) will complete the characterization of w.a. for $A(G)$.

Thank-you!

–

Merci beaucoup!