A unified framework for complementarity in quantum information

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Recent Developments in Quantum Groups
Operator Algebras and Applications

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Open system dynamics

Schrödinger picture: Quantum states evolve under completely positive trace preserving (CPTP) maps

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$\mathcal{E}_*$ and $\mathcal{E}_*^c$ have dual properties

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If $H_S = (H_A \otimes H_B)$, then B is a correctable subsystem for

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if $\exists$ a CPTP map $\mathcal{R}_* : \mathcal{T}(H_S) \to \mathcal{T}(H_S)$ such that

$$\mathcal{R}_* \circ \mathcal{E}_* = \mathcal{F}_* \otimes \mathrm{id}_B$$

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Complementarity theorem

Theorem (Kretschmann–Kribs–Spekkens '08)

Let $H_S = (H_A \otimes H_B)$ be ***finite-dimensional*** *and*

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B *is* ε-*correctable for* $\mathcal{E}_*$ $\Leftrightarrow$ B *is* $2\sqrt{\varepsilon}$-*private for any* $\mathcal{E}_*^c$.

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Observables evolve under **normal unital completely positive** (NUCP) maps:

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iff $\exists$ a NUCP map $\mathcal{R} : \mathcal{B}(H_B) \to \mathcal{B}(H_S)$ such that

$$\mathcal{E} \circ \mathcal{R}(b) = (1 \otimes b)$$

for all $b \in \mathcal{B}(H_B)$.

Correctable subalgebras [*Bény–Kempf–Kribs '07*]

A von Neumann subalgebra $N \subseteq \mathcal{B}(H_S)$ is ε-correctable for

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Correctable subsystems $\cong$ Correctable type I factors

Duality picture

Correctable subalgebras		????
$\bigcup$		
Correctable subsystems	$\leftrightarrow$	Private subsystems

Quantum channels

Heisenberg: Observables on the output H_S evolve to observables on the input H_S.

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Definition

*Let $M \subseteq \mathcal{B}(H_S)$ be a von Neumann algebra. A **quantum channel** is a NUCP map*

$$\mathcal{E} : M \to \mathcal{B}(H_S).$$

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so that

$$\mathcal{E}(\mathcal{B}(H_S)) \subseteq (\mathcal{B}(H_A) \otimes 1) = (1 \otimes \mathcal{B}(H_B))'.$$

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Definition (C.–Kribs–Levene–Todorov '14)

A von Neumann subalgebra $N \subseteq \mathcal{B}(H_S)$ *is* *private* *for* $\mathcal{E} : M \to \mathcal{B}(H_S)$ *if*

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Given $\varepsilon > 0$, *we say that* N *is* ε-private *for* $\mathcal{E}$ *if* $\exists$ *a quantum channel* $\mathcal{F} : M \to \mathcal{B}(H_S)$ *such that*

$$\|\mathcal{E} - \mathcal{F}\|_{cb} < \varepsilon$$

and N *is* private *for* $\mathcal{F}$.

Private subalgebras

Examples: Any normal conditional expectation $E : \mathcal{B}(H) \to N'$, e.g., if $\pi : G \to \mathcal{B}(H)$ is unitary rep. a compact group,

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Private subsystems $\cong$ Private factors of type I

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Complementarity

Definition (C.–Kribs–Levene–Todorov '14)

Let $\mathcal{E} : M \to \mathcal{B}(H_S)$ be a quantum channel.

Given a Stinespring representation (π, V, H) of $\mathcal{E}$, we define the complementary channel to be the NUCP map $\mathcal{E}^c : \pi(M)' \to \mathcal{B}(H_S)$ given by

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Again, defined up to partial isometries. In this case,

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If $M = \mathcal{B}(H_S)$, then $\pi(x) = x \otimes 1_E$ and $\pi(M)' = 1 \otimes \mathcal{B}(H_E)$.

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Theorem (C.–Kribs–Levene–Todorov '14)

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Main Tools: Arveson's commutant lifting and the continuity of Stinespring rep (*Kretschmann–Schlingemann–Werner '08*):

$$\frac{\|\Phi_1 - \Phi_2\|_{cb}}{\sqrt{\|\Phi_1\|_{cb}} + \sqrt{\|\Phi_2\|_{cb}}} \leqslant \inf_{V_1, V_2} \|V_1 - V_2\|_\infty \leqslant \sqrt{\|\Phi_1 - \Phi_2\|_{cb}}$$

for CP maps $\Phi_1, \Phi_2 : A \to \mathcal{B}(H)$.

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Corollary (C.–Kribs–Levene–Todorov '14)

Let $\mathcal{E} : M \to \mathcal{B}(H_S)$ be a quantum channel. Then a von Neumann subalgebra $N \subseteq \mathcal{B}(H_S)$ is correctable for $\mathcal{E}$ iff $\exists$ a normal faithful $*$-homomorphism $\pi : N \to M$ such that

$$y\mathcal{E}(x) = \mathcal{E}(\pi(y)x), \quad \text{and} \quad \mathcal{E}(x)y = \mathcal{E}(x\pi(y))$$

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*In particular, the recovery operation $\mathcal{R}$ may always to taken to be a $*$-**homomorphism** (Bény–Kempf–Kribs '09).*

Examples: Gaussian Channels

Weyl representation

Let $\mathbb{R}^{2n}$ represent the phase space of a system of n-bosonic modes.

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View vectors as $z = (x_1, y_1, x_2, y_2, \cdots, x_n, y_n)$, where $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ are the canonical coordinates of the n-modes.

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Define $U, V : \mathbb{R}^n \to \mathcal{B}(L_2(\mathbb{R}^n))$ by

$$V_x\psi(s) = e^{i\langle x,s\rangle}\psi(s) \quad \text{and} \quad U_y\psi(s) = \psi(s+y).$$

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These satisfy the Weyl (CCR):

$$U_y V_x = e^{i\langle x,y\rangle} V_x U_y.$$

Weyl representation

Composing, we obtain $W : \mathbb{R}^{2n} \to \mathcal{B}(L_2(\mathbb{R}^n))$ given by

$$W(z) = e^{\frac{i}{2}\langle x,y\rangle} V_x U_y.$$

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$$W(z) = e^{\frac{i}{2}\langle x,y\rangle} V_x U_y .$$

This satisfies Weyl–Segal form of the CCR:

$$W(z + z') = e^{\frac{i}{2}\Delta(z,z')} W(z) W(z'),$$

where $\Delta(z, z') = \sum_{i=1}^{n} (x_i y_i' - x_i' y_i)$ is the **symplectic form** on $\mathbb{R}^{2n}$, represented by the matrix

$$\Delta = \oplus_{i=1}^{n} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} .$$

Symplectic transformations

Let $Sp(2n, \mathbb{R}) = \{ T \in GL(2n, \mathbb{R}) \mid \Delta(Tz, Tz') = \Delta(z, z') \}$.

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Stone-von Neumann: $\forall\ T \in Sp(2n, \mathbb{R})$ there exists a unitary $U_T \in \mathcal{B}(L_2(\mathbb{R}^n))$ such that

$$W(Tz) = U_T^* W(z) U_T, \quad z \in \mathbb{R}^{2n}.$$

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Metaplectic representation of $Sp(2n, \mathbb{R})$.

Characteristic Functions

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A state $\rho \in \mathcal{T}(L_2(\mathbb{R}^n))$ is **Gaussian** if φ_ρ is of the form

$$\varphi_\rho(z) = \exp\left(i\langle m, z\rangle - \frac{1}{2}\alpha(z,z)\right)$$

where $m \in \mathbb{R}^{2n}$ and α is a symmetric bilinear form.

Linear bosonic channels

$\mathcal{E} : \mathcal{B}(L_2(\mathbb{R}^n)) \to \mathcal{B}(L_2(\mathbb{R}^n))$ is **linear bosonic** if

$$\mathcal{E}_*(\rho) = (\mathrm{id} \otimes \mathrm{tr}_E)(U_T(\rho \otimes \rho_E)U_T^*), \quad \rho \in \mathcal{T}(L_2(\mathbb{R}^n)),$$

where $\rho_E \in \mathcal{T}(L_2(\mathbb{R}^l))$ and $U_T \in \mathcal{B}(L_2(\mathbb{R}^{(n+l)}))$ representing a **symplectic matrix** $T \in Sp(2(m+l), \mathbb{R})$ of the form

$$T = \begin{pmatrix} K & L \\ K_E & L_E \end{pmatrix}$$

where $K : \mathbb{R}^n \to \mathbb{R}^n$, $L : \mathbb{R}^l \to \mathbb{R}^n$, $K_E : \mathbb{R}^n \to \mathbb{R}^l$ and $L_E : \mathbb{R}^l \to \mathbb{R}^l$.

Gaussian channels

$$\mathcal{E}_*(\rho) = (\mathrm{id} \otimes \mathrm{tr}_E)(U_T(\rho \otimes \rho_E)U_T^*)$$

Since $W_{n+l}(Tz) = U_T^* W_{n+l}(z) U_T$, and

$$T = \begin{pmatrix} K & L \\ K_E & L_E \end{pmatrix}$$

we get

$$\mathcal{E}(W_n(z)) = \hat{f}(z) W_n(Kz),$$

where

$$\hat{f}(z) = \varphi_{\rho_E}(K_E z), \quad z \in \mathbb{R}^{2n}.$$

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If $\hat{f} = \varphi_{\rho'_E}$ for a Gaussian state, then $\mathcal{E}$ is a **Gaussian channel**.

Gaussian channels

Example: If $f(z) = \frac{1}{(2\pi\alpha)^{n/2}} e^{-\frac{\|z\|^2}{2\alpha}}$, then

$$\mathcal{E}(x) = \int_{\mathbb{R}^{2n}} W_n(z)^* x W_n(z) f(z) dz$$

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IDEA: Use complementarity to produce explicit private subalgebras for $\mathcal{E}$.

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$$\mathcal{E}(W(z)) = \hat{f}(z)W(Kz) = \hat{f}(z)V_x \ \in L_\infty(\mathbb{R})'.$$

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$$\mathcal{E}_*(\rho) = (\mathrm{id} \otimes \mathrm{tr}_E)(U_T(\rho \otimes \rho_E)U_T^*)$$

with

$$T = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

Gaussian channels

Then $\mathcal{E}^c : \mathcal{B}(L_2(\mathbb{R}))\bar{\otimes}\mathcal{B}(L_2(\mathbb{R})) \to \mathcal{B}(L_2(\mathbb{R}))$ is

$$\mathcal{E}^c(W(z) \otimes W(z')) = \mathrm{tr}(|\psi\rangle\langle\psi| W((0,y)) \otimes W(z')) W(-z),$$

where $(\mathrm{id} \otimes \mathrm{tr})(|\psi\rangle\langle\psi|) = \rho_E$.

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$$\mathcal{R}(W(x,0)) = W((-x,0)) \otimes 1$$

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Symplectic duality

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will do. In general, if $A, B : \mathbb{R}^n \to \mathbb{R}^n$ satisfy

$$\Delta = A^t \Delta A + B^t \Delta B$$

then

$$\begin{pmatrix} A & * \\ B & * \end{pmatrix}$$

can be completed to $Sp(2n, \mathbb{R})$.

Symplectic duality

When $[A, B] = 0$, canonical choice:

$$\begin{pmatrix} A & -B^{\Delta} \\ B & A^{\Delta} \end{pmatrix} \in Sp(2n, \mathbb{R}),$$

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Complementarity & symplectic duality?

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Given **linear bosonic** $\mathcal{E} : \mathcal{B}(L_2(\mathbb{R}^n)) \to \mathcal{B}(L_2(\mathbb{R}^n))$

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with $K : \mathbb{R}^n \to \mathbb{R}^n$, if $S = K(\mathbb{R}^n) < \mathbb{R}^n$, then

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Since $[W_n(z), W_n(z')] = 0 \Leftrightarrow \Delta_n(z, z') = 0$, we have

$$W_n(S^\Delta) \subseteq W_n(S)'$$

where

$$S^\Delta := \{z \in \mathbb{R}^n \mid \Delta_n(z, w) = 0 \quad \forall w \in S\}$$

is the **symplectic complement** of S.

Symplectic duality

Thus,

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and $W_n(S^{\Delta})''$ is private for $\mathcal{E}$.

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Thank You!