Freeness and the Transpose

(*Matrices Just Wanna be Free*)

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Recent Developments in Quantum Groups, Operator Algebras and Applications, February 6, 2015

random variables and their distributions

- $(\mathcal{A}, \varphi)$ unital algebra with state;
- $\mathbf{C}\langle x_1, \dots, x_s\rangle$ is the unital algebra generated by the non-commuting variables $x_1, \dots, x_s$
- the *distribution* of $a_1, \dots, a_s \in (\mathcal{A}, \varphi)$ is the state $\mu : \mathbf{C}\langle x_1, \dots, x_s\rangle \to \mathbf{C}$ given by $\mu(p) = \varphi(p(a_1, \dots, a_s))$
- convergence in distribution of $\{a_1^{(N)}, \dots, a_s^{(N)}\} \subset (\mathcal{A}_N, \varphi_N)$ to $\{a_1, \dots, a_s\} \subset (\mathcal{A}, \varphi)$ means pointwise convergence of distributions: $\mu_N(p) \to \mu(p)$ for $p \in \mathbf{C}\langle x_1, \dots, x_s\rangle$.

freeness

- $\mathcal{A}_1, \mathcal{A}_2 \subseteq \mathcal{A}$ unital subalgebras are *free* if given
  - $a_1, \cdots, a_n \in \mathcal{A}$ with $\varphi(a_i) = 0$ for all i,
  - $a_i \in \mathcal{A}_{j_1}$ with $j_1 \neq \cdots \neq j_n$

  we have $\varphi(a_1 \cdots a_n) = 0$,
- a_1 and a_2 are *free* if $\text{alg}(1, a_1)$ and $\text{alg}(1, a_2)$ are free,

freeness and asymptotic freeness

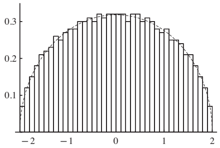
- $\{a_1^{(N)}, \ldots, a_s^{(N)}\} \subset (\mathcal{A}_N, \varphi_N)$ are *asymptotically free* if $\mu_n \to \mu$ and $x_1, \ldots, x_s$ are free with respect to μ
- if a and b are free with respect to φ then $\varphi(abab) = \varphi(a^2)\varphi(b)^2 + \varphi(a)^2\varphi(b^2) - \varphi(a)^2\varphi(b)^2$
- in general if $a_1, \ldots, a_s$ are free then all *mixed moments* $\varphi(x_{i_1} \cdots x_{i_n})$ can be written as a polynomial in the moments of individual moments $\{\varphi(a_i^k)\}_{i,k}$.
- $a_1^{(N)}, \ldots, a_s^{(N)} \in (\mathcal{A}_n, \varphi_N)$ are asymptotically free if whenever we have $b_i^{(N)} \in \mathrm{alg}(1, a_{j_i}^{(N)})$ is such that $\varphi_N(b_i^{(N)}) = 0$ and $j_1 \neq j_2 \neq \cdots \neq j_m$ we have $\varphi_N(b_1^{(N)} \cdots b_m^{(N)}) \to 0$

simple distributions: Wigner and Marchenko-Pastur

- let $f(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$ be the density of the Gauss law
- then $\log(\hat{f}(is)) = \dfrac{s^2}{2} = \displaystyle\sum_{n=1}^{\infty} k_n \frac{s^n}{n!}$ with $k_2 = 1$ and $k_n = 0$ for $n \neq 2$, so the Gauss law is characterized by having all cumulants except k_2 equal to 0
- μ a probability measure on $\mathbb{R}$, $z \in \mathbf{C}^+$, $G(z) = \int (z-t)^{-1}\, d\mu(t)$ is the Cauchy transform of μ and $R(z) = G^{\langle -1 \rangle}(z) - \frac{1}{z} = \kappa_1 + \kappa_2 z + \kappa_3 z^2 + \cdots$ is the R-transform of μ
- if $d\mu(t) = \frac{1}{2\pi}\sqrt{4-t^2}\, dt$ is the *semi-circle* law we have $\kappa_n = 0$ except for $\kappa_2 = 1$
- if $0 < c$ and $a = (1 - \sqrt{c})^2$ and $b = (1 + \sqrt{c})^2$ we let $d\mu = \frac{\sqrt{(b-t)(t-a)}}{2\pi t}\, dt$ for $c \geqslant 1$ and $d\mu = (1-c)\delta_0 + \frac{\sqrt{(b-t)(t-a)}}{2\pi t}\, dt$ for $0 < c < 1$, the Marchenko-Pastur distribution: $\kappa_n = c$ for all n

GUE random matrices and asymptotic freeness

- $X_N = X_N^* = \frac{1}{\sqrt{N}}(x_{ij})_{ij}$ a $N \times N$ self-adjoint random matrix with x_{ij} independent complex Gaussians with $\mathrm{E}(x_{ij}) = 0$ and $\mathrm{E}(|x_{ij}|^2) = 1$ (*modulo* self-adjointness)
- $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_N$ eigenvalues of X_N,
  $\mu_N = \frac{1}{N}(\delta_{\lambda_1} + \cdots + \delta_{\lambda_N})$ is the spectral measure of X_N,
  $\int t^k \, d\mu_N(t) = \mathrm{tr}(X_N^k)$
- X_N is the $N \times N$ GUE with limiting eigenvalue distribution given by Wigner's semi-circle law

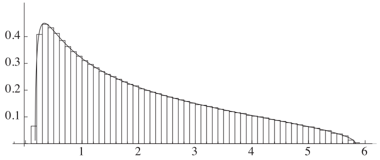


- Y_N another GUE with entries independent from those of X_N
- for large N mixed moments of X_N and Y_N are close to those of freely independent semi-circular operators (thus *asymptotically free*)

Wishart Random Matrices

- Suppose $G_1, \ldots, G_{d_1}$ are $d_2 \times p$ random matrices where $G_i = (g_{jk}^{(i)})_{jk}$ and $g_{jk}^{(i)}$ are complex Gaussian random variables with mean 0 and (complex) variance 1, i.e. $\mathrm{E}(|g_{jk}^{(i)}|^2) = 1$. Moreover suppose that the random variables $\{g_{jk}^{(i)}\}_{i,j,k}$ are independent.
- $$W = \frac{1}{d_1 d_2} \begin{pmatrix} G_1 \\ \hline \vdots \\ \hline G_{d_1} \end{pmatrix} \left(\begin{array}{c|c|c} G_1^* & \cdots & G_{d_1}^* \end{array} \right) = \frac{1}{d_1 d_2} (G_i G_j^*)_{ij}$$

  is a $d_1 d_2 \times d_1 d_2$ Wishart matrix. We write $W = d_1^{-1}(W(i,j))_{ij}$ as $d_1 \times d_1$ block matrix with each entry the $d_2 \times d_2$ matrix $d_2^{-1} G_i G_j^*$.



Partial Transposes on $M_{d_1}(\mathbf{C}) \otimes M_{d_2}(\mathbf{C})$

- G_i a $d_2 \times p$ matrix
- $W(i,j) = \frac{1}{d_2} G_i G_j^*$, a $d_2 \times d_2$ matrix,
- $W = \frac{1}{d_1}(W(i,j))_{ij}$ is a $d_1 \times d_1$ *block* matrix with entries $W(i,j)$
- $W^{\mathrm{T}} = \frac{1}{d_1}(W(j,i)^{\mathrm{T}})_{ij}$ is the "full" transpose
- $W^{\reflectbox{$\Gamma$}} = \frac{1}{d_1}(W(j,i))_{ij}$ is the "left" partial transpose
- $W^{\Gamma} = \frac{1}{d_1}(W(i,j)^{\mathrm{T}})_{ij}$ is the "right" partial transpose
- we **assume** that $\dfrac{p}{d_1 d_2} \to c$, $0 < c < \infty$
- eigenvalue distributions of W and W^{T} converge to Marchenko-Pastur with parameter c
- eigenvalues of $W^{\reflectbox{$\Gamma$}}$ and W^{Γ} converge to a shifted semi-circular with mean c and variance c (Aubrun, 2012)
- W and W^{T} are asymptotically free (M. and Popa, 2014)
- what about W^{Γ} and $W^{\reflectbox{$\Gamma$}}$?

Semi-circle and Marchenko-Pastur Distributions

Suppose $\frac{p}{d_1 d_2} \to c$.

- limit eigenvalue distribution of W (Marchenko-Pastur)

$$\lim \mathrm{E}(\mathrm{tr}(W^n)) = \int_a^b t^n \frac{\sqrt{(b-t)(t-a)}}{2\pi t}\, dt = \sum_{\sigma \in NC(n)} c^{\#(\sigma)}$$

  $\#(\sigma)$ is the number of blocks of σ, $a = (1 - \sqrt{c})^2$, and $b = (1 + \sqrt{c})^2$

- limit eigenvalue distribution of W^Γ (semi-circle)

$$\lim \mathrm{E}(\mathrm{tr}((W^\Gamma)^n)) = \sum_{\sigma \in NC_{1,2}(n)} c^{\#(\sigma)} = \sum_{\pi \in NC(n)} \kappa_\pi$$

  $NC_{1,2}(n)$ is the set of non-crossing partitions with only blocks of size 1 and 2. (*c.f.* Fukuda and Śniady (2013) and Banica and Nechita (2013))

main theorem

- THM: The matrices $\{W, W^{\urcorner}, W^{\Gamma}, W^{\mathrm{T}}\}$ form an asymptotically free family

· let $(\epsilon, \eta) \in \{-1, 1\}^2 = \mathbb{Z}_2^2$.

· let $W^{(\epsilon,\eta)} = \begin{cases} W & \text{if } (\epsilon,\eta) = (1,1) \\ W^{\urcorner} & \text{if } (\epsilon,\eta) = (-1,1) \\ W^{\Gamma} & \text{if } (\epsilon,\eta) = (1,-1) \\ W^{\mathrm{T}} & \text{if } (\epsilon,\eta) = (-1,-1) \end{cases}$

· let $(\epsilon_1, \eta_1), \ldots, (\epsilon_n, \eta_n) \in \mathbb{Z}_2^n$

$$\mathrm{E}(\mathrm{Tr}(W^{(\epsilon_1,\eta_1)} \cdots W^{(\epsilon_n,\eta_n)}))$$

$$= \sum_{\sigma \in S_n} \left(\frac{p}{d_1 d_2}\right)^{\#(\sigma)} d_1^{f_\epsilon(\sigma)+\#(\sigma)-n} d_2^{f_\eta(\sigma)+\#(\sigma)-n}$$

where $f_\epsilon(\sigma) = \#(\epsilon\delta\gamma^{-1}\delta\gamma\delta\epsilon \vee \sigma\delta\sigma^{-1})$ (*"$\vee$" means the sup of partitions and # means the number of blocks or cycles*)

Computing Moments via Permutations, *Notation*

- $[d_1] = \{1, 2, \ldots, d_1\}$,
- given $i_1, \ldots, i_n \in [d_1]$ we think of this n-tuple as a function $i : [n] \to [d_1]$
- $\ker(i) \in \mathcal{P}(n)$ is the partition of $[n]$ such that i is constant on the blocks of $\ker(i)$ *and* assumes different values on different blocks
- if $\sigma \in S_n$ we also think of the cycles of σ as a partition and write $\sigma \leqslant \ker(i)$ to mean that i is constant on the cycles of σ
- given $\sigma \in S_n$ we extend σ to a permutation on $[\pm n] = \{-n, \ldots, -1, 1, \ldots, n\}$ by setting $\sigma(-k) = -k$ for $k > 0$
- $\gamma = (1, 2, \ldots, n)$, $\delta(k) = -k$
- given $\epsilon_1, \ldots, \epsilon_n \in \{-1, 1\}$ let $\epsilon \in S_{\pm n}$ be given by $\epsilon(k) = \epsilon_{|k|} \cdot k$
- $\delta\gamma^{-1}\delta\gamma\delta = (1, -n)(2, -1) \cdots (n, -(n-1))$

Computing Moments via Permutations, II

- $\delta\gamma^{-1}\delta\gamma\delta = (1,-n)(2,-1)\cdots(n,-(n-1))$
- if $A_k = (a^{(k)}_{ij})_{ij}$, a $N \times N$ matrix, then

$$\mathrm{Tr}(A_1 \cdots A_n) = \sum_{i_1,\dots,i_n=1}^{N} a^{(1)}_{i_1 i_2} a^{(2)}_{i_2 i_3} \cdots a^{(n)}_{i_n i_1} = \sum_{\substack{i_{\pm 1},\dots,i_{\pm n} \\ \delta\gamma^{-1}\delta\gamma\delta \leqslant \ker(i)}} a^{(1)}_{i_1 i_{-1}} a^{(2)}_{i_2 i_{-2}} \cdots a^{(n)}_{i_n i_{-n}}$$

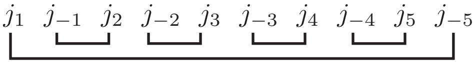
$$\begin{aligned}
&\mathrm{Tr}\big(W^{(\epsilon_1,\eta_1)} \cdots W^{(\epsilon_n,\eta_n)}\big) \\
&\quad = d_1^{-n} \sum_{i_1,\dots,i_n} \mathrm{Tr}\Big(\big(W^{(\epsilon_1,\eta_1)}\big)_{i_1 i_2} \cdots \big(W^{(\epsilon_n,\eta_n)}\big)_{i_n i_1}\Big) \\
&\quad = d_1^{-n} \sum_{i_{\pm 1},\dots,i_{\pm n}} \mathrm{Tr}\Big(\big(W^{(\epsilon_1,\eta_1)}\big)_{i_1 i_{-1}} \cdots \big(W^{(\epsilon_n,\eta_n)}\big)_{i_n i_{-n}}\Big) \\
&\quad = d_1^{-n} \sum_{j_{\pm 1},\dots,j_{\pm n}} \mathrm{Tr}\Big(W(j_1,j_{-1})^{(\eta_1)} \cdots W(j_n,j_{-n})^{(\eta_n)}\Big)
\end{aligned}$$

where $\delta\gamma^{-1}\delta\gamma\delta \leqslant \ker(i)$, $\epsilon\delta\gamma^{-1}\delta\gamma\delta\epsilon \leqslant \ker(j)$ and $j = i \circ \epsilon$

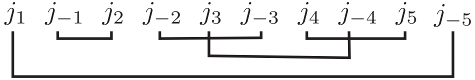
Example of Twisting

$n = 5$, $\epsilon = (1, 1, -1, -1, 1)$

$\delta\gamma^{-1}\delta\gamma = (1, -5)(2, -1)(3, -2)(4, -3)(5, -4)$



$\epsilon\delta\gamma^{-1}\delta\gamma\epsilon = (1, -5)(2, -1)(3, -4)(-3, -2)(4, 5)$



Computing Moments via Permutations, III

$$\mathrm{Tr}\big(W^{(\epsilon_1,\eta_1)}\cdots W^{(\epsilon_n,\eta_n)}\big) = d_1^{-n}\sum_{j_{\pm 1},\dots,j_{\pm n}} \mathrm{Tr}\Big(W(j_1,j_{-1})^{(\eta_1)}\cdots W(j_n,j_{-n})^{(\eta_n)}\Big)$$

with $\epsilon\delta\gamma^{-1}\delta\gamma\delta\epsilon \leqslant \ker(j)$. Let $s = r\circ\eta$ then for $\delta\gamma^{-1}\delta\gamma\delta \leqslant \ker(r)$

$$\begin{aligned}
&\mathrm{Tr}\big(W(j_1,j_{-1})^{(\eta_1)}\cdots W(j_n,j_{-n})^{(\eta_n)}\big)\\
&= \sum_{r_{\pm 1},\dots,r_{\pm n}} \big(W(j_1,j_{-1})^{(\eta_1)}\big)_{r_1 r_{-1}}\cdots\big(W(j_n,j_{-n})^{(\eta_n)}\big)_{r_n r_{-n}}\\
&= \sum_{s_{\pm 1},\dots,s_{\pm n}} \big(W(j_1,j_{-1})\big)_{s_1 s_{-1}}\cdots\big(W(j_n,j_{-n})\big)_{s_n s_{-n}}\\
&= d_2^{-n}\sum_{s_{\pm 1},\dots,s_{\pm n}} \big(G_{j_1}G_{j_{-1}}^*\big)_{s_1 s_{-1}}\cdots\big(G_{j_n}G_{j_{-n}}^*\big)_{s_n s_{-n}}\\
&= d_2^{-n}\sum_{s_{\pm 1},\dots,s_{\pm n}}\sum_{t_1,\dots,t_n} g_{s_1 t_1}^{(j_1)}\overline{g_{s_{-1}t_1}^{(j_{-1})}}\cdots g_{s_n t_n}^{(j_n)}\overline{g_{s_{-n}t_n}^{(j_{-n})}}
\end{aligned}$$

Gaussian entries

$\mathrm{E}(\mathrm{Tr}(W^{(\epsilon_1,\eta_1)} \cdots W^{(\epsilon_1,\eta_1)}))$

$$= (d_1 d_2)^{-n} \sum_{j_{\pm 1},\dots,j_{\pm n}} \sum_{s_{\pm 1},\dots,s_{\pm n}} \sum_{t_1,\dots,t_n} \mathrm{E}(g^{(j_1)}_{s_1 t_1} \overline{g^{(j_{-1})}_{s_{-1} t_1}} \cdots g^{(j_n)}_{s_n t_n} \overline{g^{(j_{-n})}_{s_{-n} t_n}})$$

$$= (d_1 d_2)^{-n} \sum_{j_{\pm 1},\dots,j_{\pm n}} \sum_{s_{\pm 1},\dots,s_{\pm n}} \sum_{t_1,\dots,t_n} \mathrm{E}(g^{(j_1)}_{s_1 t_1} \cdots g^{(j_n)}_{s_n t_n} \overline{g^{(j_{-1})}_{s_{-1} t_1}} \cdots \overline{g^{(j_{-n})}_{s_{-n} t_n}})$$

[subject to the condition that $\epsilon\delta\gamma^{-1}\delta\gamma\delta\epsilon \leqslant \ker(j)$ and $\eta\delta\gamma^{-1}\delta\gamma\delta\eta \leqslant \ker(s)$]

$$= (d_1 d_2)^{-n} \sum_{j_{\pm 1},\dots,j_{\pm n}} \sum_{s_{\pm 1},\dots,s_{\pm n}} \sum_{t_1,\dots,t_n} \mathrm{E}(g_{\alpha(1)} \cdots g_{\alpha(n)} \overline{g_{\beta(1)}} \cdots \overline{g_{\beta(n)}})$$

where $g_{\alpha(k)} = g^{(j_k)}_{s_k t_k}$ and $g_{\beta(k)} = g^{(j_{-k})}_{s_{-k} t_k}$. Using $\mathrm{E}(g_{\alpha(1)} \cdots g_{\alpha(n)} \overline{g_{\beta(1)}} \cdots \overline{g_{\beta(n)}}) = |\{\sigma \in S_n \mid \beta = \alpha \circ \sigma\}|$

Thus

$E(\mathrm{Tr}(W^{(\epsilon_1,\eta_1)} \dots W^{(\epsilon_1,\eta_1)}))$

$$= (d_1 d_2)^{-n} \sum_{j_{\pm 1},\dots,j_{\pm n}} \sum_{s_{\pm 1},\dots,s_{\pm n}} \sum_{t_1,\dots,t_n} |\{\sigma \in S_n \mid \text{"various conditions"}\}|$$

where "various conditions" means

- $\epsilon\delta\gamma^{-1}\delta\gamma\delta\epsilon \leqslant \ker(j)$
- $\eta\delta\gamma^{-1}\delta\gamma\delta\eta \leqslant \ker(s)$
- $j_{-k} = j_{\sigma(k)}$ which is equivalent to $\sigma\delta\sigma^{-1} \leqslant \ker(j)$
- $s_{-k} = s_{\sigma(k)}$ which is equivalent to $\sigma\delta\sigma^{-1} \leqslant \ker(s)$
- $t_k = t_{\sigma(k)}$ which is equivalent to $\sigma \leqslant \ker(t)$

$$E(\mathrm{tr}(W^{(\epsilon_1,\eta_1)} \dots W^{(\epsilon_n,\eta_n)}))$$
$$= \sum_{\sigma \in S_n} \left(\frac{p}{d_1 d_2}\right)^{\#(\sigma)} d_1^{f_\epsilon(\sigma)+\#(\sigma)-(n+1)} d_2^{f_\eta(\sigma)+\#(\sigma)-(n+1)}.$$

where $f_\epsilon(\sigma) = \#(\epsilon\delta\gamma^{-1}\delta\gamma\delta\epsilon \vee \sigma\delta\sigma^{-1})$ ("$\vee$" *means the sup of partitions*)

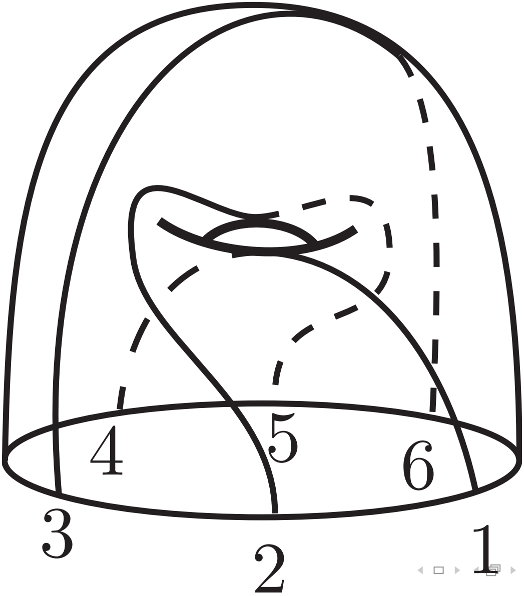
finding the highest order terms

- general fact: if p and q are pairings then $\#(p \vee q) = \frac{1}{2}\#(pq)$. In fact we can write the permutation pq as a product of cycles $c_1 c_1' \cdots c_k c_k'$ where $c_i' = q c_i^{-1} q$ and the blocks of $p \vee q$ are $c_i \cup c_i'$
- $\#(\epsilon\delta\gamma^{-1}\delta\gamma\delta\epsilon \vee \sigma\delta\sigma^{-1}) = \frac{1}{2}\#(\delta\gamma^{-1}\delta\gamma \cdot \epsilon\delta\sigma\delta\sigma^{-1}\epsilon)$
- if $\pi, \sigma \in S_n$ and $\langle \pi, \sigma \rangle$ (the subgroup generated by π and σ) has only one orbit then there is an integer g (the "genus") such that

$$\#(\pi) + \#(\pi^{-1}\sigma) + \#(\sigma) = n + 2(1 - g)$$

  and $g = 0$ only when π is planar or non-crossing with respect to σ.
- $\delta\gamma^{-1}\delta\gamma$ has two cycles so $\langle \delta\gamma^{-1}\delta\gamma, \epsilon\delta\sigma\delta\sigma^{-1}\epsilon \rangle$ can have either 1 or 2 orbits
- if $\langle \delta\gamma^{-1}\delta\gamma, \epsilon\delta\sigma\delta\sigma^{-1}\epsilon \rangle$ has one orbit then $\#(\epsilon\delta\gamma^{-1}\delta\gamma\delta\epsilon \vee \sigma\delta\sigma^{-1}) + \#(\sigma) \leqslant n$

genus of a pairing



$$\mathrm{E}(\mathrm{tr}(W^{(\epsilon_1,\eta_1)}\cdots W^{(\epsilon_n,\eta_n)}))$$
$$= \sum_{\sigma\in S_n} \Big(\frac{p}{d_1 d_2}\Big)^{\#(\sigma)} d_1^{f_\epsilon(\sigma)+\#(\sigma)-(n+1)} d_2^{f_\eta(\sigma)+\#(\sigma)-(n+1)}.$$

- σ will not contribute to the limit unless $\langle \delta\gamma^{-1}\delta\gamma, \epsilon\delta\sigma\delta\sigma^{-1}\epsilon\rangle$ has two orbits, i.e. ϵ is constant on the cycles of σ (write $\epsilon\delta\sigma\delta\sigma^{-1}\epsilon = \delta\epsilon\sigma\epsilon\delta(\epsilon\sigma\epsilon)^{-1}$)
- if ϵ is constant on the cycles of σ there is $\sigma_\epsilon \in S_n$ such that $\epsilon\delta\sigma\delta\sigma^{-1}\epsilon = \delta\sigma_\epsilon\delta\sigma_\epsilon^{-1}$ (if $\sigma = c_1 c_2 \cdots c_k$ then $\sigma_\epsilon = c_1^{\lambda_1}\cdots c_k^{\lambda_k}$ where λ_i is the sign of ϵ on c_i)
- then $\frac{1}{2}\#(\delta\gamma^{-1}\delta\gamma \cdot \epsilon\delta\sigma\delta\sigma^{-1}\epsilon) = \#(\gamma\sigma_\epsilon^{-1})$
- $\#(\sigma) + f_\epsilon(\sigma) = \#(\sigma_\epsilon) + \#(\gamma\sigma_\epsilon^{-1}) \leqslant n+1$ with equality only if σ_ϵ is non-crossing
- $\#(\sigma) + f_\eta(\sigma) = \#(\sigma_\eta) + \#(\gamma\sigma_\eta^{-1}) \leqslant n+1$ with equality only if σ_η is non-crossing

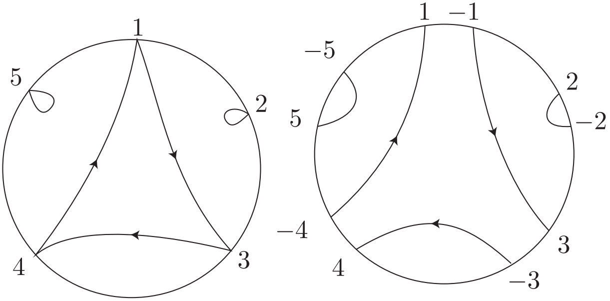
$$\mathrm{E}(\mathrm{tr}(W^{(\epsilon_1,\eta_1)}\cdots W^{(\epsilon_n,\eta_n)})) = \sum_{\sigma}\left(\frac{p}{d_1d_2}\right)^{\#(\sigma)} + O\left(\frac{1}{d_1d_2}\right).$$

where the sum runs over $\sigma \in S_n$ such that

- ϵ and η are constant on the cycles of σ and
- both σ_ϵ and σ_η are both non-crossing.
- if $\epsilon \neq \eta$ on a cycle of σ then this cycle must be either a fixed point or a pair; $\sigma_\epsilon = \sigma_\eta$ and so $f_\epsilon(\sigma) = f_\eta(\sigma)$
- σ can only connect $W^{(1,1)}$ to another $W^{(1,1)}$, a $W^{(-1,1)}$ to another $W^{(-1,1)}$, a $W^{(1,-1)}$ to another $W^{(1,-1)}$, and a $W^{(-1,-1)}$ to another $W^{(-1,-1)}$

σ and σ^{-1}

$\sigma = (1,3,4)(2)(5)$



$\sigma = (1,4,3)(2)(5)$