Classification of spatial L^p AF algebras

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Definition of L^p-operator algebra

Definition

Let $p \in [1, \infty)$. An L^p operator algebra A is a matrix normed Banach algebra for which there exists a measure space $(X, \mathcal{B}, \mu)$ such that A is completely isometrically isomorphic to a norm closed subalgebra of $B(L^p(X, \mu))$, where $B(L^P(X, \mu))$ denotes the set of bounded linear operators on $L^p(X, \mu)$.

Given a subalgebra A of $B(L^p(X, \mu))$, for each $n \in \mathbb{N}$ we can endow $M_n(A)$ with the norm induced by the identification of $M_n(B)$ with a subalgebra of $B(A \otimes_p L^p(X, \mu))$. The collection of all these norms defines a p-operator space structure on A, as defined by M. Daws.

Example

$B(l^p\{1, 2, \ldots, n\})$ *is an* L^p*-operator algebra, denoted by* M_n^p.

What is know so far on L^poperator alegebras

N. C. Phillips has worked extensively on L^p operator algebras in recent years. He has defined

i) spatial L^p UHF algebras
ii) L^p analog $\mathcal{O}_d^p$ of the Cuntz algebra $\mathcal{O}_d$
iii) Full ad reduced crossed product of L^p operator algebras by isometric actions of second countable locally compact groups

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In a series of paper Phillips showed that many of the results we have for UHF algebras and Cuntz algebras are also valid for their L^p analogs.

a) Every spatial L^p UHF algebra has a supernatural number associated to it and two spatial L^p UHF algebras are isomorphic if and only if they have the same supernatural number.
b) Any spatial L^p UHF algebra is simple and amenable.
c) The L^p analog $\mathcal{O}_d^p$ of the Cuntz algebra $\mathcal{O}_d$ is a purely infinite, simple amenable Banach algebra.

Moreover, $K_0(\mathcal{O}_d^p) \cong \mathbb{Z}/(d-1)\mathbb{Z}$ and $K_1(\mathcal{O}_d^p) = 0$.

Some more recent work:

d) L^p analog, denoted by $F^p(G)$, of the full group C*-algebra of a locally compact group (Phillips, Gardella and Thiel). One of the results shown is that when G is discrete, amenability of $F^p(G)$ is equivalent to the amenability of G.

e) Full and reduced L^p operator algebra associated to an étale groupoid (Gardella and Lupini)

Moreover, $K_0(\mathcal{O}_d^p) \cong \mathbb{Z}/(d-1)\mathbb{Z}$ and $K_1(\mathcal{O}_d^p) = 0$.

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What about an L^p analog of AF algebras? Do we have a complete classification for them as the one given by Elliott for AF algebras?

Spatial Semisimple Finite Dimensional Algebras

Convention

Whenever $N \in \mathbb{Z}_{>0}$ *and* $A_1, A_2, \ldots, A_N$ *are Banach algebras, we make* $\bigoplus_{k=1}^{N} A_k$ *a Banach algebra by giving it the obvious algebra structure and the norm*

$$\|(a_1, a_2, \ldots, a_N)\| = \max\left(\|a_1\|, \|a_2\|, \ldots, \|a_N\|\right)$$

for $a_1 \in A_1, a_2 \in A_2, \ldots, a_N \in A_N$.

Definition

Let $p \in [1, \infty) \setminus \{2\}$. A matrix normed Banach algebra A is called a *spatial semisimple finite dimensional* L^p *operator algebra* if there exist $N \in \mathbb{Z}_{>0}$ and $d_1, d_2, \ldots, d_N \in \mathbb{Z}_{>0}$ such that A is completely isometrically isomorphic to the Banach algebra $\displaystyle\bigoplus_{i=1}^{k} M_{d_i}^p$.

We can think of A as acting on the L^p-direct sum $l^p(n_1) \oplus_p l^p(n_2) \oplus_p \cdots \oplus_p l^p(n_k) \cong l^p(n_1 + n_2 + \cdots n_k)$. So every semisimpe dinite dimensional L^p-operator algebra is an L^p operator algebra.

Proposition (Gardella and Lupini)

Let G be an étale grupoid. If A is an L^p-operator algebra, then any contractive homomorphism from $F^p(G)$ to A is automatically p-completely contractive.

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A spatial L^p AF algebra is defined as a direct limit of spatial semisimple finite dimensional L^p operator algebras with connecting maps of a certain type.

Spatial Idempotents

Definition

Let $p \in [1,\infty) \setminus \{2\}$. Let $A \subset B(L^p(X,\mu))$ be a unital L^p-operator algebra, with $(X,\mathcal{B},\mu)$ a σ-finite measure space, and let $e \in A$ be an idempotent. We say that e is a *spatial idempotent* if the homomorphism $\varphi\colon \mathbb{C} \oplus \mathbb{C} \to B(L^p(X,\mu))$ given by $\varphi(\lambda_1,\lambda_2) = \lambda_1 e + \lambda_2(1-e)$ is contractive.

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Proposition

Let $p \in [1, \infty) \setminus \{2\}$. Let $(X, \mathcal{B}, \mu)$ be a σ-finite measure space, and let $e \in B(L^p(X, \mu))$. Then e is a spatial idempotent if and only if there is a measurable subset $E \subset X$ such that e is multiplication by χ_E, i.e.

$$e(f) = \chi_E \cdot f, \text{ for every } f \in L^p(X, \mu).$$

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The proof use the following structure theorem for contractive representations of $C(X)$ on an L^p space.

Proposition

Let $p \in [1, \infty) \setminus \{2\}$. *Let* X *be a compact metric space, let* $(Y, \mathcal{C}, \nu)$ *be a* σ*-finite measure space, and let* $\pi \colon C(X) \to B(L^p(Y, \nu))$ *be a contractive unital homomorphism. Let* $\mu \colon L^\infty(Y, \nu) \to B(L^p(Y, \nu))$ *be the representation of* $L^\infty(Y, \nu)$ *on* $L^p(Y, \nu)$ *given by multiplication operators. Then there exists a unital *-homomorphism* $\varphi \colon C(X) \to L^\infty(Y, \nu)$ *such that* $\pi = \mu \circ \varphi$.

Spatial Maps

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Definition

Let $p \in [1, \infty) \setminus \{2\}$. Let A be a unital σ-finitely representable L^p operator algebra, let $d \in \mathbb{Z}_{>0}$, and let $\varphi \colon M_d^p \to A$ be a homomorphism (not necessarily unital). We say that φ is *spatial* if $\varphi(1)$ is a spatial idempotent and φ is contractive.

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Lemma

Let $p \in [1, \infty) \setminus \{2\}$. Let $d, m \in \mathbb{Z}_{>0}$, and let $\psi \colon M_d^p \to M_m^p$ be a homomorphism. Then ψ is spatial iff there exists $k \in \mathbb{Z}_{>0}$ and a complex permutation matrix $s \in M_m^p$ such that $\forall\ a \in M_d^p$ we have

$$s\psi(a)s^{-1} = \mathit{diag}(a, a, \ldots, a, 0),$$

where diag$(a, a, \ldots, a, 0)$, is a block diagonal matrix in which a occurs k times and 0 is the zero element of M_{m-kd}^p.

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Definition

Let $p \in [1, \infty) \setminus \{2\}$, and let $A \cong \bigoplus_{k=1}^{N} M_{d_k}^p$ be a spatial semisimple finite dimensional L^p operator algebra. Let B be a σ-finitely representable unital L^p operator algebra, and let $\varphi \colon A \to B$ be a homomorphism. We say that φ is *spatial* if for $k = 1, 2, \ldots, N$, the restriction $\varphi|_{M_{d_k}^p}$ is spatial.

Block Diagonal maps

Definition

Let $p \in [1, \infty) \setminus \{2\}$. Let $A \cong \bigoplus_{j=1}^{M} M_{c_j}^p$, and $B \cong \bigoplus_{i=1}^{N} M_{d_i}^p$ be spatial semisimple finite dimensional L^p-operator algebras. A homomorphism $\varphi \colon A \to B$ is said to be block diagonal if

$$\varphi(a_1 \oplus a_2 \oplus \cdots \oplus a_M) = B_1 \oplus B_2 \oplus \cdots \oplus B_N.$$

where each B_j is a block diagonal matrix having the a_i's and zero matrices on the diagonal.

Characterization of spatial homomorphism

Lemma

Let $p \in [1,\infty) \setminus \{2\}$, *let* $A \cong \bigoplus_{j=1}^{L} M_{c_k}^p$ *and* $B \cong \bigoplus_{k=1}^{N} M_{d_k}^p$ *be spatial semisimple finite dimensional* L^p*-operator algebras, and* $\varphi\colon A \to B$ *be a homomorphism. Then* φ *is spatial if and only if there exist permutation matrices*

$$s_1 \in M_{d_1}^p,\ s_2 \in M_{d_2}^p,\ \ldots,\ s_N \in M_{d_N}^p$$

such that, if $s = (s_1, s_2, \ldots, s_N) \in B$, *the homomorphism* $a \mapsto s\varphi(a)s^{-1}$ *is block diagonal.*

Definition

Let $p \in [1, \infty) \setminus \{2\}$. A *spatial L^p AF direct system* is a direct system $\left((A_m)_{m \geq 1}, (\varphi_{n,m})_{m \leq n}\right)$ satisfying the following:

(1) For every $n \geq 1$, the algebra A_n is a spatial semisimple finite dimensional L^p operator algebra.

(2) For all $m \leq n$, the map $\varphi_{m,n}$ is a spatial homomorphism.

A Banach algebra A is a *spatial L^p AF algebra* if it is sometrically isomorphic to the direct limit of a spatial L^p AF direct system.

L^p AF algebras

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Proposition

Let $p \in [1, \infty) \setminus \{2\}$, and let A be a spatial L^p AF algebra. Then A is a separable nondegenerately representable L^p operator algebra.

Moreover, there exists a unique p-operator space structure on A since A can be realized as a groupoid L^p operator algebra (Gardella-Lupini).

Equivalence of idempotents

Definition

Let A be a Banach algebra. Let e, f be idempotents in A. Denote by $\bar{A}$ the unitalization of A.

(1) e is algebraic equivalent to f, denoted by $e \sim f$, if there exist $x, y \in A$ such that $xy = e$ and $yx = f$.

(2) e is similar to f, denoted by $e \sim_s f$ if there exists an invertible element z in $\bar{A}$ such that $zez^{-1} = f$

(3) e is homotopic equivalent to f, denoted by $e \sim_h f$, if there exists a norm continuous path of idempotents in A from e to f.

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Algebraic equivalence, similarity and homotopic equivalence coincide on the set of idempotents of $M_\infty(A)$.

Scaled preorder K_0-group

Definition

Let A be a Banach algebra with an approximate identity of idempotents. We define $K_0(A)_+ = \text{Idem}(M_\infty(A))$, where $\text{Idem}(M_\infty(A))$ denotes the set of similarity classes of idempotents in $M_\infty(A)$. Then $K_0(A)_+$ is an Abelian semigroup with respect to
$$[e] + [f] = \left[\begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix} \right]$$

Let $K_0(A)$ be the Grothendieck group of $K_0(A)_+$, and set

$$\Sigma(A) = \{[e] \in K_0(A)_+ \,|\, e \text{ is an idempotent in } A\}.$$

We refer to the triplet $(K_0(A), K_0(A)_+, \Sigma(A))$ as the scaled preordered K_0-group of A.

Proposition

Let $((A_m)_{m\geq 0}, (\varphi_{m,n})_{m\leq n})$ *be a contractive direct system of Banach algebras, and suppose each* A_m *has an approximate identity of idempotents. Denote by* A *the direct limit of the direct system. Then,*

$$(K_0(A), K_0(A)_+, \Sigma(A)) = \varinjlim (K_0(A_n), K_0(A_n)_+, \Sigma(A_n)).$$

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$$(K_0(A), K_0(A)_+, \Sigma(A)) = \varinjlim (K_0(A_n), K_0(A_n)_+, \Sigma(A_n)).$$

A Riesz group (G, G_+) is an unperforated partially ordered group satisfying the following condition: for every $a_1, a_2, b_1, b_2 \in G$ satisfying $a_i \leq b_j$ for $1 \leq i, j \leq 2$ there exists an element $z \in G$ such that $a_i \leq z \leq b_j$ for $i, j = 1, 2$. Let Σ be a scale for (G, G_+), i.e. a hereditary, directed, generating subset of G_+.

Riesz groups and K_0 groups

If A is a spatial L^p AF algebra then the scaled ordered K_0 group $(K_0(A), K_0(A)_+, \Sigma(A))$ is a Riesz group.

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Proposition

Let $p \in [1, \infty)$. Let (G, G_+, Σ) be a countable Riesz group with scale Σ. Then there exists a spatial L^p AF direct system in which the maps are block diagonal and whose direct limit A satisfies $\big(K_0(A),\ K_0(A)_+, \Sigma(A)\big) \cong (G, G_+, \Sigma)$.

Theorem (The L^p Elliott Theorem)

Let $p \in [1, \infty) \setminus \{2\}$. Let A and B be spatial L^p AF algebras. Suppose that there is an isomorphism $f \colon K_0(A) \to K_0(B)$ such that

$$f(K_0(A)_+) = K_0(B)_+ \text{ and } f(\Sigma(A)) = \Sigma(B).$$

Then there exists an isomorphism $F \colon A \to B$ such that $F_* = f$.

Idea of Proof: First we show that a spatial L^p-AF algebra A is isometrically isomorphic to the direct limit of a spatial L^p AF direct system in which all the maps are block diagonal and injective. An intertwining argument, similar to the one used in the classic Elliott's theorem, can then be used to complete the proof.

Incompressibility

Definition

Let A be a Banach algebra. A is said to be *incompressible* if whenever B is another Banach algebra and $\varphi\colon A \to B$ is a contractive homomorphism, then the induced homomorphism $\overline{\varphi}\colon A/\ker(\varphi) \to B$ is isometric.

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Theorem (Dales)

Every C^*-algebra is incompressible.

p-incompressibility

Definition

Let $p \in [1, \infty) \setminus \{2\}$, and let A be a σ-finitely representable L^p operator algebra. Then A is *p-incompressible* if whenever $(Y, \mathcal{C}, \nu)$ is a σ-finite measure space and $\varphi \colon A \to B(L^p(Y, \nu))$ is a contractive homomorphism then the induced homomorphism $\overline{\varphi} \colon A / \ker(\varphi) \to B(L^p(Y, \nu))$ is isometric.

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Lemma

Let $p \in [1, \infty) \setminus \{2\}$. Then every spatial semisimple finite dimensional L^p operator algebra is p-incompressible.

System of ideals

Definition

Let $p \in [1, \infty) \setminus \{2\}$. Let $\big((A_m)_{m \in \geq 1}, \, (\varphi_{n,m})_{m \leq n}\big)$ be an L^p AF direct system where all connecting maps are injective. A *system of ideals in* $\big((A_m)_{m \in \mathbb{Z}_{\geq 0}}, \, (\varphi_{n,m})_{m \leq n}\big)$ is a family $(J_m)_{m \geq 1}$ such that J_m is an ideal in A_m for all $m \in \mathbb{Z}_{\geq 0}$ and $\varphi_{n,m}^{-1}(J_n) = J_m$ for all $m, n \geq 1$ with $m \leq n$.

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Lemma

Let $p \in [1, \infty) \setminus \{2\}$. Let$\big((A_m)_{m \geq 1}, (\varphi_{n,m})_{m \leq n}\big)$ be an L^p AF direct system with injective maps, and let $(J_m)_{m \geq 1}$ be a system of ideals in $\big((A_m)_{m \geq 1}, (\varphi_{n,m})_{m \leq n}\big)$. Set $A = \varinjlim A_m$, and for $m \geq 1$ let $\varphi_m \colon A_m \to A$ be the map associated to the direct system. Then $J = \overline{\bigcup_{n \geq 1} \varphi_m(J_m)}$ is a closed ideal in A. Moreover, if the direct system is spatial, then $\varphi_m^{-1}(J) = J_m$ for all $m \geq 1$.

Definition

Let $p \in [1, \infty) \setminus \{2\}$. Let $\left((A_m)_{m \geq 1}, (\varphi_{n,m})_{m \leq n}\right)$ be an L^p AF direct system with injective maps, and let $A = \varinjlim_n A_n$. An ideal $J \subset A$ of the form in the previous lemma is called a *direct limit ideal*.

In general, it is false that the quotient of an L^p operator algebra by a closed ideal is also an operator algebra (counterexample given by Gardella and Thiel, 2014).

Proposition

Let $p \in [1, \infty) \setminus \{2\}$. Let A be a spatial L^p AF algebra, and let $J \subset A$ be a direct limit ideal. Then A/J is a spatial L^p AF algebra.

Theorem

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- Work in progress: determine what can be said about the structure of ideals in a spatial L^p-AF algebra.
- Is a spatial L^p-AF algebra incompressible?