

Boundary Quotients of Semigroup C^* -algebras

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- P - left cancellative semigroup
- Reduced $C_r^*(P)$ and Li's $C^*(P)$
- Li (2012) – studied $C^*(P)$ when $P \subset G$ (G group).
- What about when P does not embed in a group?
- $P \subset \mathcal{S}$, an *inverse semigroup* (always)
- $C^*(P)$ is an inverse semigroup algebra, with natural boundary quotient $\mathcal{Q}(P)$.
- Conditions on P which guarantee $\mathcal{Q}(P)$ simple, purely infinite.
- Self-similar groups

Semigroups

P countable semigroup (associative multiplication)

Left cancellative: $ps = pq \Rightarrow s = q$

Principal right ideal: $rP = \{rq \mid q \in P\}$

Elements of rP are right multiples of r

Assume $1 \in P$ (ie, P is a monoid)

Group of units: $U(P) =$ invertible elements of P

Semigroups

Study P by representing on a Hilbert space, similar to groups.

$\ell^2(P)$ – square-summable complex functions on P .

δ_x – point mass at $x \in P$. Orthonormal basis of $\ell^2(P)$.

$v_p : \ell^2(P) \rightarrow \ell^2(P)$ bounded operator $v_p(\delta_x) = \delta_{px}$ (necessarily **isometries**)

$\{v_p\}_{p \in P}$ generate the **reduced C^* -algebra of P** , $C_r^*(P)$

$v : P \rightarrow C_r^*(P)$ is called the **left regular representation**

Unlike the group case, considering **all** representations turns out to be a disaster

Li: we have to care for **ideals**.

For $X \subset P$, then $e_X : \ell^2(P) \rightarrow \ell^2(P)$ is defined by

$$(e_X \xi)(p) = \begin{cases} \xi(p) & \text{if } p \in X \\ 0 & \text{otherwise.} \end{cases}$$

Note: $v_1 = e_P$

Note that in $\mathcal{B}(\ell^2(P))$,

$$v_p e_X v_p^* = e_{pX} \quad v_p^* e_X v_p = e_{p^{-1}X}$$

If $p \in P$ and X is a right ideal, then

$$pX = \{px \mid x \in X\} \quad p^{-1}X = \{y \mid py \in X\}$$

are right ideals too.

$$pX = \{px \mid x \in X\} \quad p^{-1}X = \{y \mid py \in X\}$$

$\mathcal{J}(P)$ – smallest set of right ideals containing P , \emptyset , and closed under intersection and the above operations for all p – **constructible** ideals.

These are the ideals which are “constructible” inside $C_r^*(P)$.

- 1 $e_X e_Y = e_{X \cap Y}$
- 2 $e_P = 1$, $e_\emptyset = 0$
- 3 $v_p e_X v_p^* = e_{pX}$ and $v_p^* e_X v_p = e_{p^{-1}X}$

Definition (Li)

$C^*(P)$ is the universal C^* -algebra generated by isometries $\{v_p \mid p \in P\}$ and projections $\{e_X \mid X \in \mathcal{J}(P)\}$ satisfying the above (and $v_p v_q = v_{pq}$).

When does a semigroup embed in a group?

For $P \subset G$, we need **cancellativity** (left and right) + *something*

Examples of *something*s which work:

- commutativity (Grothendieck group)
- $rP \cap qP \neq \emptyset$ for all p, q . (Ore condition)
- **Rees conditions**:
 - principal right ideals are comparable or disjoint, and
 - each principal right ideal is contained in only a finite number of other principal right ideals
- others...

Example: Free Semigroups

X finite set, $X^0 = \{\emptyset\}$, X^n words of length n in X .

$$X^* = \bigcup_{n \geq 0} X^n$$

This is the **free semigroup** on X , under concatenation.

$X^* \subset \mathbb{F}_X$, the free group.

Not Ore (unless $|X| = 1$): if $x, y \in X$ and $x \neq y$, we have $xX^* \cap yX^* = \emptyset$

$$C^*(X^*) \cong C_r^*(X^*) \cong \mathcal{T}_{|X|}$$

\mathcal{T}_n **Toeplitz** algebra – generated by n isometries with orthogonal ranges.

Boundary Quotient

Simplification: suppose that for all $p, q \in P$,

$$rP \cap qP = sP \text{ some } s \in P$$

Then $\mathcal{J}(P) = \{sP \mid s \in P\}$.

Such semigroups are called **Clifford** semigroups, or **right LCM** semigroups.

Finite $F \subset P$ is a **foundation set** if for all $r \in P$, there is $f \in F$ with $fP \cap rP \neq \emptyset$.

Definition (Brownlowe, Ramagge, Robertson, Whittaker)

The **boundary quotient** $\mathcal{Q}(P)$ is the universal C^* -algebra generated by the same elements and relations as in Li's $C^*(P)$, and also satisfying

$$\prod_{f \in F} (1 - e_{fP}) = 0 \text{ for all foundation sets } F.$$

Boundary Quotient

What the heck does “ $\prod_{f \in F} (1 - e_{fP}) = 0$ for all foundation sets F .” mean?

$D :=$ unital, commutative C^* -algebra generated by $\{e_{rP}\}_{r \in P}$

Projections in D have a “greatest lower bound”, “least upper bound”, and “complement”:

$$e \wedge f = ef \qquad e \vee f = e + f - ef \qquad \neg e = 1 - e$$

ie, they form a **Boolean algebra**. Rearranging $\prod_{f \in F} (1 - e_{fP}) = 0$ using de Morgan's laws gives

$$\bigvee_{f \in F} e_{fP} = 1.$$

Free semigroup: $\mathcal{Q}(X^*) \cong \mathcal{O}_{|X|}$.

Inverse Semigroups

Even when P does not embed into a group, it embeds into an inverse semigroup.

A semigroup S is called an **inverse semigroup** if for every element $s \in S$ there is a **unique** element s^* such that

$$ss^*s = s \quad \text{and} \quad s^*ss^* = s^*$$

Any set of **partial isometries** in a C^* -algebra closed under multiplication and adjoint is an inverse semigroup.

Many C^* -algebras of interest are generated by an **inverse semigroup of partial isometries**.

Finite dimensional, AF, Cuntz algebras, Graph algebras, Tiling algebras

For a given S , we have

- $C^*(S)$ – **universal** C^* -algebra of S (Toeplitz-type)
- $C_{\text{tight}}^*(S)$ – **tight** C^* -algebra of S (Cuntz-type)

Both come from **étale groupoids**, which can be analyzed

Inverse Semigroups

For our right LCM semigroup P ,

$$\mathcal{S} := \{v_p v_q^* \mid p, q \in P\} \cup \{0\}$$

is closed under multiplication, and so is an inverse semigroup.

$$(v_p v_q^*)(v_r v_s^*) = \begin{cases} v_{pq'} v_{sr'}^* & \text{if } qP \cap rP = kP \text{ and } qq' = rr' = k \\ 0 & \text{if } qP \cap rP = \emptyset \end{cases}$$

Theorem

- 1 (Norling) $C^*(P) \cong C^*(\mathcal{S})$
- 2 $(\mathcal{S}) \mathcal{Q}(P) \cong C_{tight}^*(\mathcal{S})$

Properties of $\mathcal{Q}(P)$

We know $\mathcal{Q}(P) \cong C_{\text{tight}}^*(\mathcal{S})$

Most of what we can say about $\mathcal{Q}(P)$ stems from knowing that $C_{\text{tight}}^*(\mathcal{S})$ comes from an étale groupoid $\mathcal{G}_{\text{tight}}$, a dynamical object.

One can formulate properties which guarantee that a groupoid algebra is simple, but they are topological and dynamical.

e.g. “ $\mathcal{G}_{\text{tight}}$ is Hausdorff,” “ $\mathcal{G}_{\text{tight}}$ is minimal,” “ $\mathcal{G}_{\text{tight}}$ is essentially free”.

We translate these statements so that they are (mostly) algebraic properties.

Properties of $\mathcal{Q}(P)$

“ $\mathcal{G}_{\text{tight}}$ is Hausdorff”

(H) For all $p, q \in P$, either

- 1 $pb \neq qb$ for all $b \in P$, or
- 2 There exists a finite $F \subset P$ with $pf = qf$ and whenever $pb = qb$ there is an $f \in F$ such that $fP \cap bP \neq \emptyset$.

P satisfies condition (H) if the counterexamples to right cancellativity have a “finite cover”.

P right cancellative $\Rightarrow P$ satisfies (H)

“ $\mathcal{G}_{\text{tight}}$ is minimal”

It turns out that it is always minimal.

Properties of $Q(P)$

" $\mathcal{G}_{\text{tight}}$ is essentially free"

$$P_0 = \{q \in P \mid qP \cap rP \neq 0 \text{ for all } r \in P\}$$

This is the **core** of P .

$U(P) \subset P_0$, and if P is Ore, $P_0 = P \rightarrow \text{cOre}$

(EP) For all $p, q \in P_0$ and for every $k \in P$ such that

$$qkaP \cap pkaP \neq \emptyset$$

for all $a \in P$, there exists a foundation set F such that $qkf = pkf$ for all $f \in F$.

Properties of $Q(P)$

Theorem (S)

Let P be a right LCM semigroup which satisfies (H). Then $Q(P)$ is simple if and only if

- 1 P satisfies (EP), and
- 2 $Q(P) (\cong C^*(\mathcal{G}_{\text{tight}})) \cong C_r^*(\mathcal{G}_{\text{tight}})$

So we see [amenability](#) plays a rôle here.

Theorem (S)

Let P be a right LCM semigroup which satisfies (H) and such that $Q(P)$ is simple. Then $Q(P)$ is purely infinite if and only if $Q(P) \not\cong \mathbb{C}$.

Example: Self-similar groups

Suppose we have an action of G on X^* and a **restriction** $G \times X \rightarrow G$

$$(g, x) \mapsto g|_x.$$

such that the action on X^* can be defined **recursively**

$$g(x\alpha) = (gx)(g|_x \alpha)$$

The pair (G, X) is called a **self-similar action**.

Restriction extends to words

$$g|_{\alpha_1\alpha_2\cdots\alpha_n} := g|_{\alpha_1} |_{\alpha_2} \cdots |_{\alpha_n}$$

$$g(\alpha\beta) = (g\alpha)(g|_\alpha \beta)$$

Example: The Odometer

$$G = \mathbb{Z} = \langle z \rangle$$

$$X = \{0, 1\}$$

Then the action of \mathbb{Z} on X^* is determined by

$$z0 = 1 \quad z|_0 = e$$

$$z1 = 0 \quad z|_1 = z$$

A word α in X^* corresponds to an integer in binary (written backwards), and z adds 1 to α , ignoring carryover.

$$z(001) = 101 \quad z|_{001} = e$$

$$z^2(011) = 000 \quad z^2|_{011} = z$$

Example: Self-similar groups

Given (G, X) , the product $X^* \times G$ with the operation

$$(\alpha, g)(\beta, h) = (\alpha(g\beta), g|_{\beta} h)$$

is a left cancellative semigroup – **Zappa-Szép product** $X^* \bowtie G$

Lawson-Wallis – all semigroups with **Rees conditions** arise like this, and so they embed in a group \Leftrightarrow cancellative.

Many interesting examples are not cancellative!

Example: Self-similar groups

A word α is **strongly fixed** by $g \in G$ if $g\alpha = \alpha$ and $g|_\alpha = 1_G$. A strongly fixed word is **minimal** if no prefix is strongly fixed.

Proposition

$X^* \rtimes G$ satisfies (H) iff for all $g \neq 1_G$, there are only a finite number of minimal strongly fixed words for g .

$X^* \rtimes G$ is cancelative iff for all $g \neq 1$, g has no strongly fixed words.

The core of $X^* \rtimes G$ is $\{(\emptyset, g) \mid g \in G\} \cong G$.

Proposition

$X^* \rtimes G$ satisfies (EP) if the action of G on X^* is faithful.

If G is amenable, then the amenability condition is satisfied.

Example: The Odometer

The odometer has no strongly fixed words, so $X^* \rtimes \mathbb{Z}$ is cancelative (and satisfies (H)).

$X^* \rtimes \mathbb{Z}$ embeds into $BS(1, 2)$.

$\mathcal{Q}(X^* \rtimes \mathbb{Z})$ is simple, purely infinite, and in fact isomorphic to \mathcal{Q}_2 (Brownlowe-Ramagge-Robertson-Whittaker)

If we modify the odometer by adding a strongly fixed letter B :

$$X_B = \{0, 1, B\} \quad zB = B, \quad z|_B = e$$

$X_B^* \rtimes \mathbb{Z}$ satisfies (H), but is not cancelative.

$\mathcal{Q}(X_B^* \rtimes \mathbb{Z})$ is again simple and purely infinite.

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