Rigidity of structures

András Recski
Budapest University of Technology and Economics

Toronto, 2014

			1982 Montréal		

					
Walter Whiteley	Henry Crapo				

COLLOQUIA MATHEMATICA
SOCIETATIS JÁNOS BOLYAI, 40.

MATROID THEORY

Edited by:

L. LOVÁSZ
and
A. RECSKI

NORTH-HOLLAND
AMSTERDAM—OXFORD—NEW YORK

SCIENTIFIC PROGRAM

Monday, August 30

1982

MORNING

- 10^{00} Opening address
- 10^{15} Coffee break
- 10^{45} J. Edmonds–M. Las Vergnas: Oriented matroids I
- 11^{45} C. Benzaken–P.L. Hammer: Matroidal decomposition of independence systems: a first approach

AFTERNOON

- 14^{30} H. Crapo: The combinatorial theory of structures I
- 15^{30} R. Cordovil: On simplical matroids and index lemma
- 16^{00} Coffee break
- 17^{30} R. Euler: On perfect independence systems

Tuesday, August 31

MORNING

- 9^{00} J. Edmonds–M. Las Vergnas: Oriented matroids II
- 10^{00} Coffee break
- 10^{30} E.L. Lawler: Polymatroidal network flows with super-modular lower bounds
- 11^{00} U. Zimmermann: Shortest augmenting path methods for submodular flow problems
- 11^{30} N. Tomizawa: Hypermatroids, polymatroids, quasi-matroids and matroids

RNING

10^{00}	Opening address
10^{15}	Coffee break
10^{45}	J. Edmonds-M. Las Vergnas: Oriented matroids I
11^{45}	C. Benzaken-P.L. Hammer: Matroidal decomposition of independence systems: a first approach

TERNOON

14^{30}	H. Crapo: The combinatorial theory of structures I
15^{30}	R. Cordovil: On simplical matroids and index lemma
16^{00}	Coffee break
17^{30}	R. Euler: On perfect independence systems

Gian-Carlo Rota

1969 Balaton-füred

Walter Whiteley

Henry Crapo

Balatonfüred, Hungary, 1969

Erdős, Gallai, Rényi, Turán

Berge, Guy, van Lint, Milner, Nash-Williams, Rado, Rota, Sachs, Seidel, Straus, van der Waerden, Wagner, Zykov

1968

On the Foundations of Combinatorial Theory:

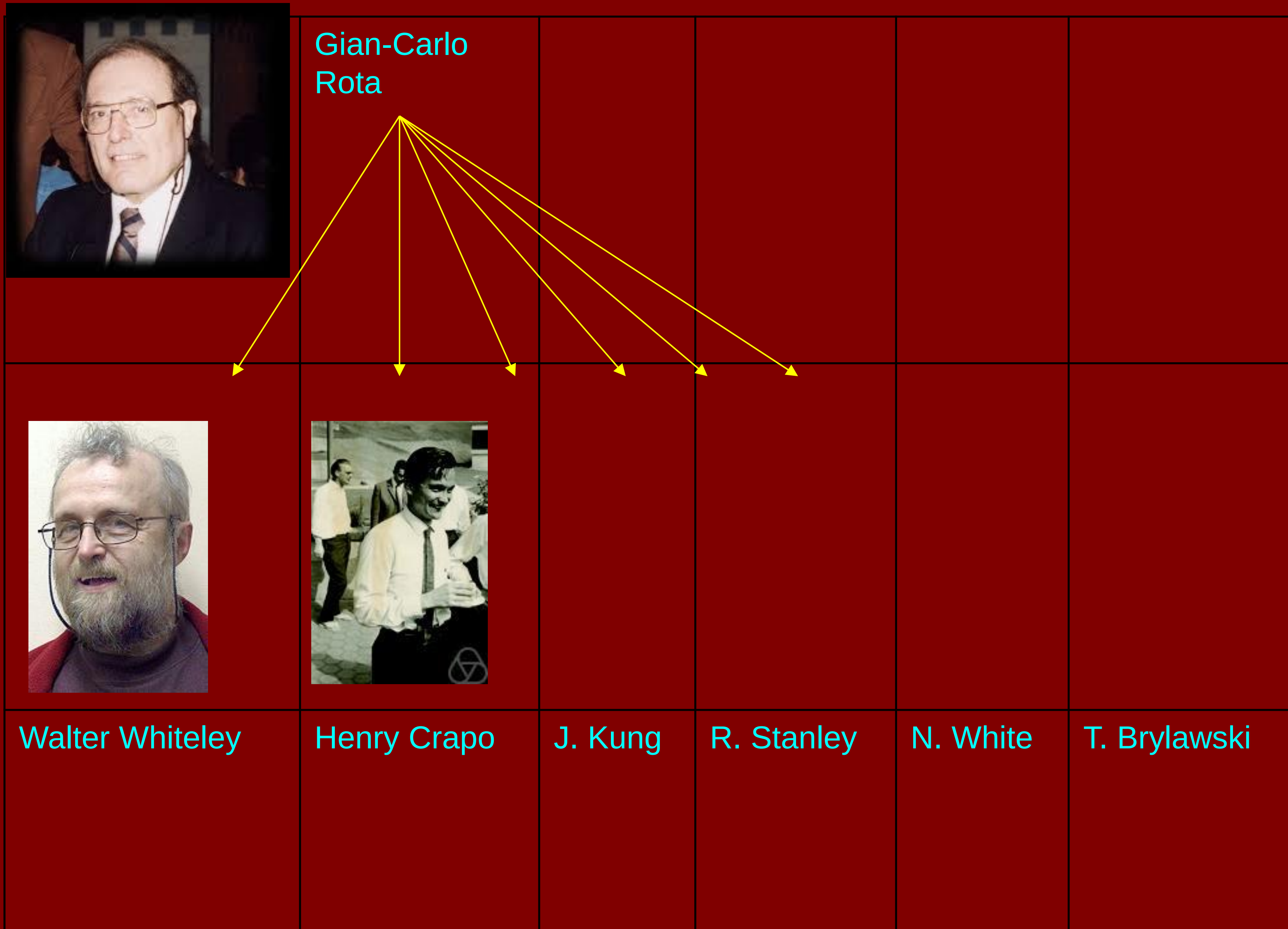
Combinatorial Geometries

by

Henry H. Crapo
University of Waterloo

and

Gian-Carlo Rota
M. I. T.


Gian-Carlo
Rota
Walter Whiteley
Henry Crapo
J. Kung
R. Stanley
N. White
T. Brylawski

	A. Kästner	
	J. Pfaff	
	J. Bartels	
	N. Lobachevsky	
	N. Brashman	
	P. Chebysev	
	A. Markov	
	J. Tamarkin	
	N. Dunford	
	J. Schwartz	
	G.-C. Rota	
	W. Whiteley	

	A. Kästner	
F. Bolyai	J. Pfaff	
J. Bolyai	J. Bartels	C. Gauß
	N. Lobachevsky	
	N. Brashman	
	P. Chebysev	
	A. Markov	
	J. Tamarkin	
	N. Dunford	
	J. Schwartz	
	G.-C. Rota	
	W. Whiteley	

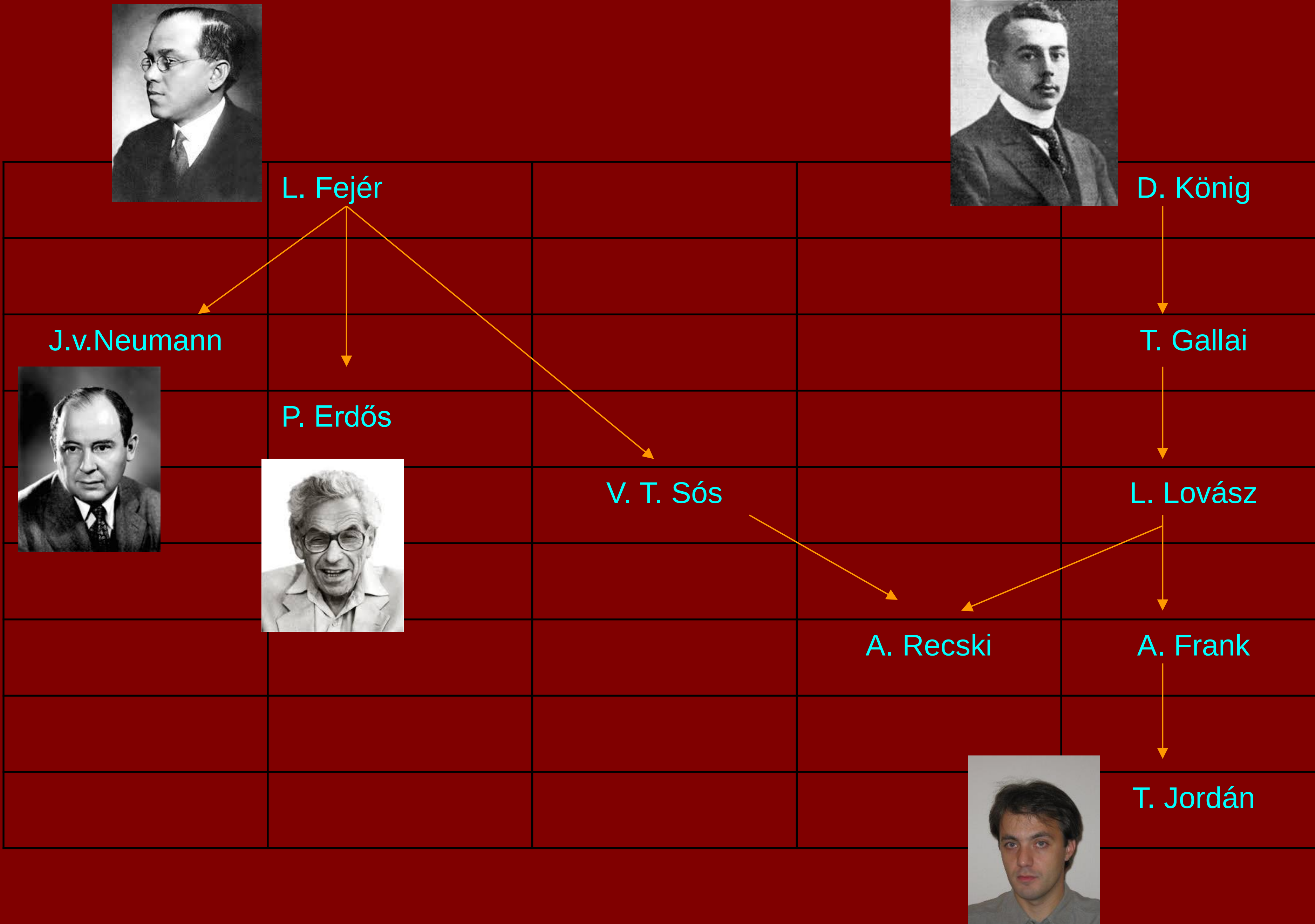
	A. Kästner	
F. Bolyai	J. Pfaff	
J. Bolyai	J. Bartels	C. Gauß
	N. Lobachevsky	
	N. Brashman	
	P. Chebysev	
	A. Markov	
	J. Tamarkin	
	N. Dunford	
	J. Schwartz	
	G.-C. Rota	
	W. Whiteley	

	A. Kästner	
F. Bolyai	J. Pfaff	
J. Bolyai	J. Bartels	C. Gauß
	N. Lobachevsky	
	N. Brashman	
	P. Chebysev	
	A. Markov	
	J. Tamarkin	
	N. Dunford	
	J. Schwartz	
	G.-C. Rota	
	W. Whiteley	

A. Kästner					L. Euler	
J. Pfaff					J. Lagrange	
J. Bartels	C. Gauß			J. Fourier		S. Poisson
N. Lobachevsky		C. Gerling			G. Dirichlet	
N. Brashman	F. Bessel		J. Plücker		R. Lipschitz	
P. Chebysev	H. Schwartz			C. Klein		
A. Markov	E. Kummer				C. Lindemann	
J. Tamarkin	H. Schwartz				H. Minkowski	
N. Dunford	L. Fejér				D. König	
J. Schwartz	J.v.Neumann	P. Erdős			T. Gallai	
G.-C. Rota			V. T. Sós		L. Lovász	
W. Whiteley				A. Recski	A. Frank	
					T. Jordán	

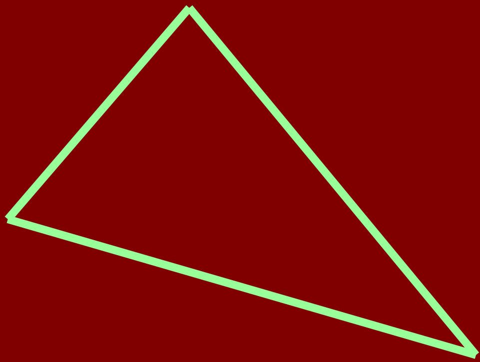
A. Kästner					L. Euler	
J. Pfaff					J. Lagrange	
J. Bartels	C. Gauß			J. Fourier		S. Poisson
N. Lobachevsky		C. Gerling			G. Dirichlet	
N. Brashman	F. Bessel		J. Plücker		R. Lipschitz	
P. Chebysev	H. Schwartz			C. Klein		
A. Markov	E. Kummer				C. Lindemann	
J. Tamarkin	H. Schwartz				H. Minkowski	
N. Dunford	L. Fejér				D. König	
J. Schwartz	J.v.Neumann	P. Erdős			T. Gallai	
G.-C. Rota			V. T. Sós		L. Lovász	
W. Whiteley				A. Recski	A. Frank	
					T. Jordán	

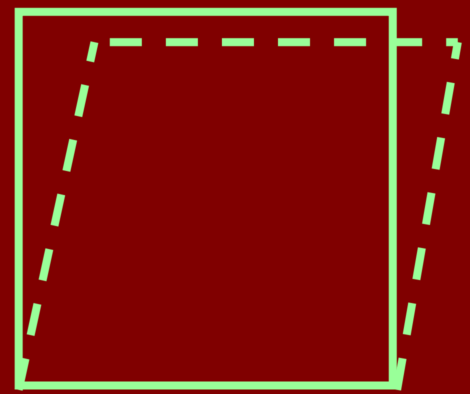
A. Kästner					L. Euler	
J. Pfaff					J. Lagrange	
J. Bartels	C. Gauß			J. Fourier		S. Poisson
N. Lobachevsky		C. Gerling			G. Dirichlet	
N. Brashman	F. Bessel		J. Plücker		R. Lipschitz	
P. Chebysev	H. Schwartz			C. Klein		
A. Markov	E. Kummer				C. Lindemann	
J. Tamarkin	H. Schwartz				H. Minkowski	
N. Dunford	L. Fejér				D. König	
J. Schwartz	J.v.Neumann	P. Erdős			T. Gallai	
G.-C. Rota			V. T. Sós		L. Lovász	
W. Whiteley				A. Recski	A. Frank	
					T. Jordán	


L. Fejér
D. König
J.v.Neumann
T. Gallai
P. Erdős
V. T. Sós
L. Lovász
A. Recski
A. Frank
T. Jordán

			1982 Montréal		

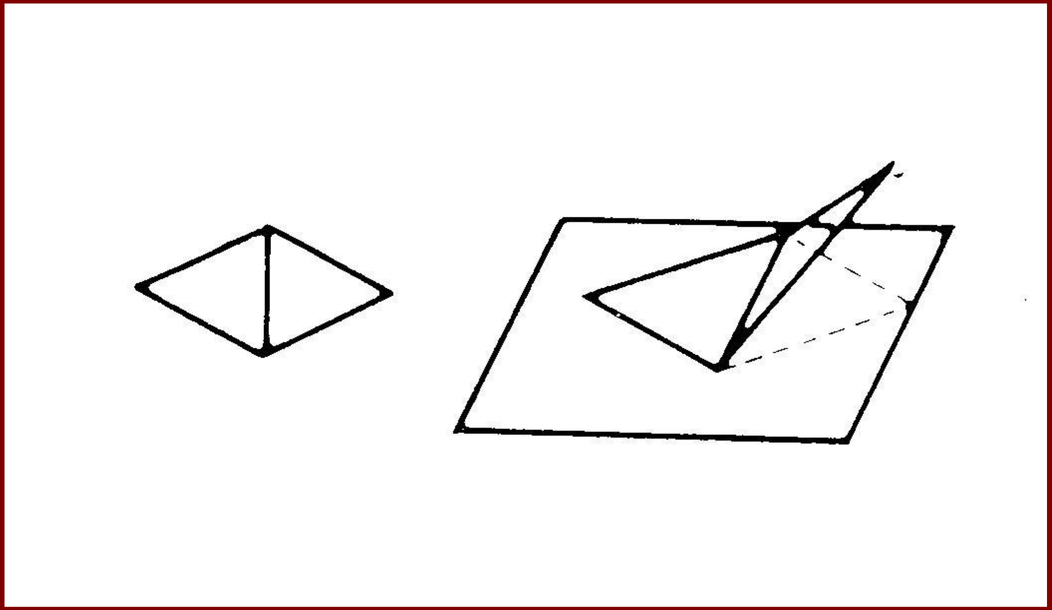
Bar and joint frameworks





Rigid

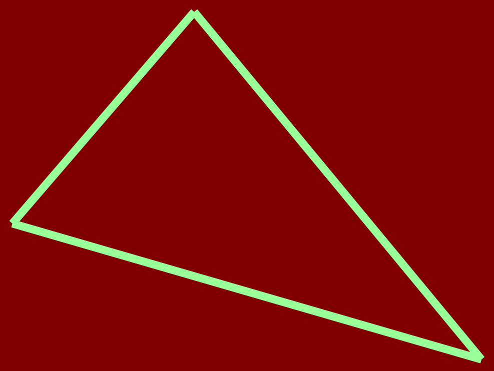
Non-rigid (mechanism)

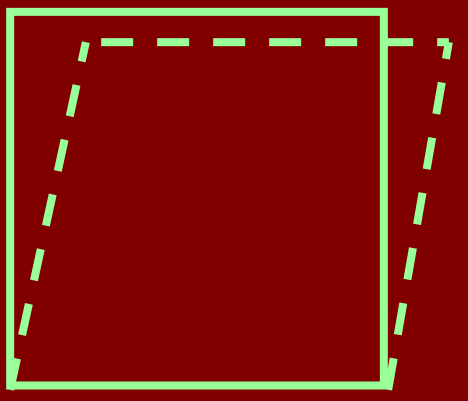


Rigid in the plane

Non-rigid in the space

Bar and joint frameworks





Rigid

Non-rigid (mechanism)

How can we describe the difference?

What is the effect of a rod?

i ———————— j

$$\sqrt{(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2} = \text{constant}$$

$$(x_i(t) - x_j(t))^2 + (y_i(t) - y_j(t))^2 + (z_i(t) - z_j(t))^2 = c_{ij}$$

What is the effect of a rod?

i ———————— j

$$\sqrt{(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2} = \text{constant}$$

$$(x_i(t) - x_j(t))^2 + (y_i(t) - y_j(t))^2 + (z_i(t) - z_j(t))^2 = c_{ij}$$

$$(x_i(t) - x_j(t))(\dot{x}_i(t) - \dot{x}_j(t)) + (y_i(t) - y_j(t))(\dot{y}_i(t) - \dot{y}_j(t)) + $$
$$+ (z_i(t) - z_j(t))(\dot{z}_i(t) - \dot{z}_j(t)) = 0,$$

$$(x_i(t) - x_j(t))\dot{x}_i(t) + (x_j(t) - x_i(t))\dot{x}_j(t) + \dots$$
$$\dots + (z_i(t) - z_j(t))\dot{z}_i(t) + (z_j(t) - z_i(t))\dot{z}_j(t) = 0.$$

$$(x_i(t) - x_j(t))\dot{x}_i(t) + (x_j(t) - x_i(t))\dot{x}_j(t) + \ldots$$
$$\ldots + (z_i(t) - z_j(t))\dot{z}_i(t) + (z_j(t) - z_i(t))\dot{z}_j(t) = 0.$$

Au=0

The matrix *A* in case of K_4 in the 2-dimensional space

$$\begin{bmatrix} x_1-x_2 & x_2-x_1 & 0 & 0 & y_1-y_2 & y_2-y_1 & 0 & 0 \\ x_1-x_3 & 0 & x_3-x_1 & 0 & y_1-y_3 & 0 & y_3-y_1 & 0 \\ x_1-x_4 & 0 & 0 & x_4-x_1 & y_1-y_4 & 0 & 0 & y_4-y_1 \\ 0 & x_2-x_3 & x_3-x_2 & 0 & 0 & y_2-y_3 & y_3-y_2 & 0 \\ 0 & x_2-x_4 & 0 & x_4-x_2 & 0 & y_2-y_4 & 0 & y_4-y_2 \\ 0 & 0 & x_3-x_4 & x_4-x_3 & 0 & 0 & y_3-y_4 & y_4-y_3 \end{bmatrix}$$

$$Au=0$$

has a mathematically trivial solution $u=0$

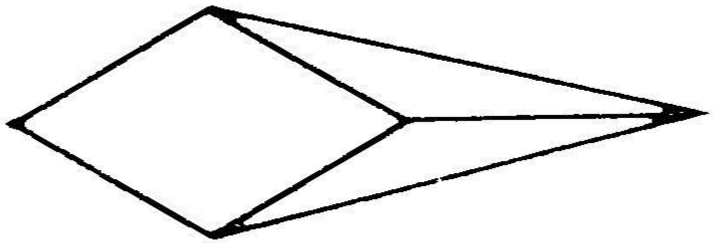
$$Au=0$$

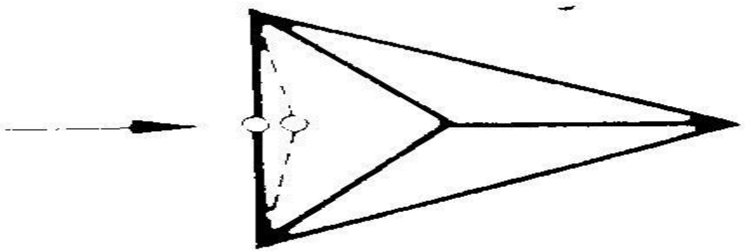
has a mathematically trivial solution $u=0$ and a lot of further solutions which are trivial from the point of view of statics.

A framework with n joints in the d-dimensional space is defined to be (infinitesimally) rigid if

$$r(A) = nd - d(d+1)/2$$

In particular: $r(A) = n - 1$ if $d = 1$, $r(A) = 2n - 3$ for the plane and $r(A) = 3n - 6$ for the 3-space.

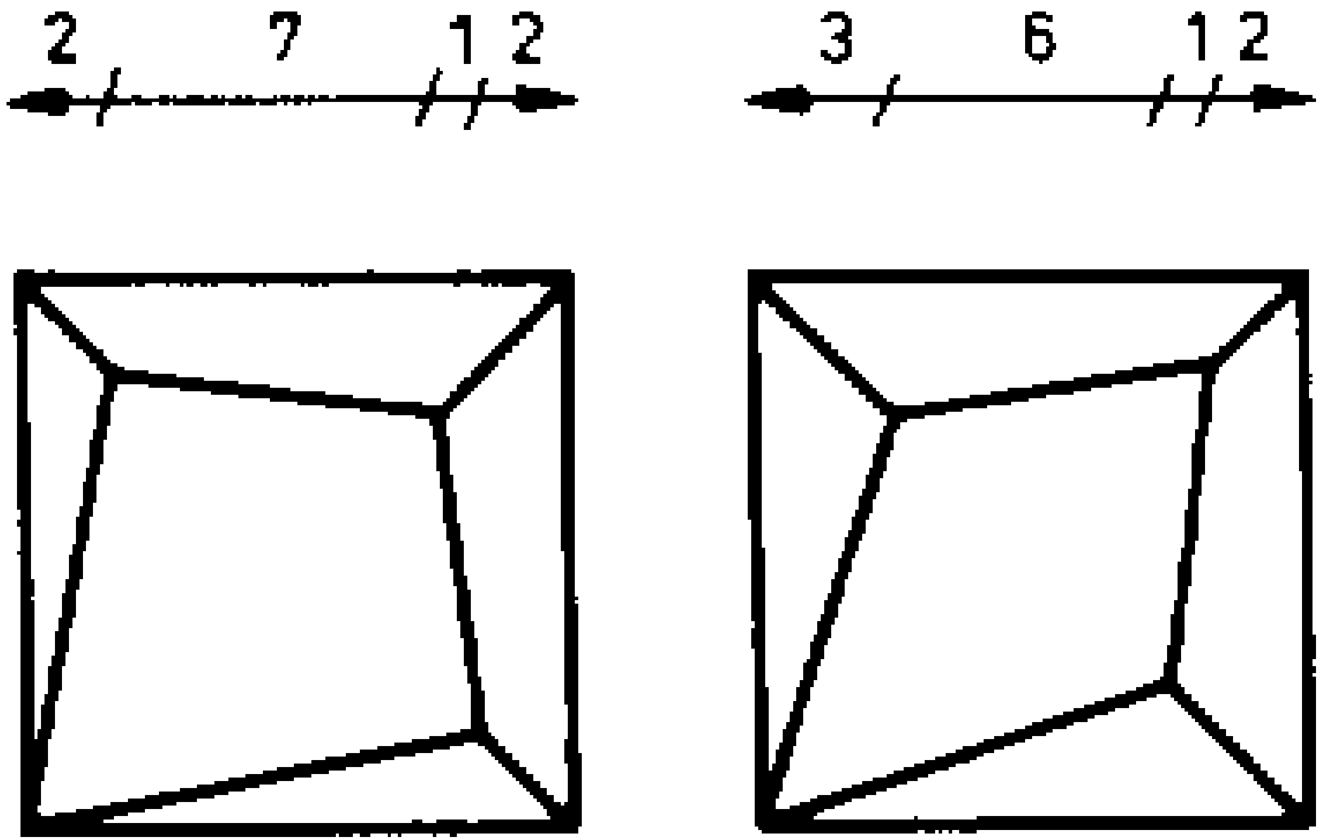




Rigid

Non-rigid (has an *infinitesimal* motion)

(although the graphs of the two frameworks are isomorphic)


2 7 1 2
3 6 1 2

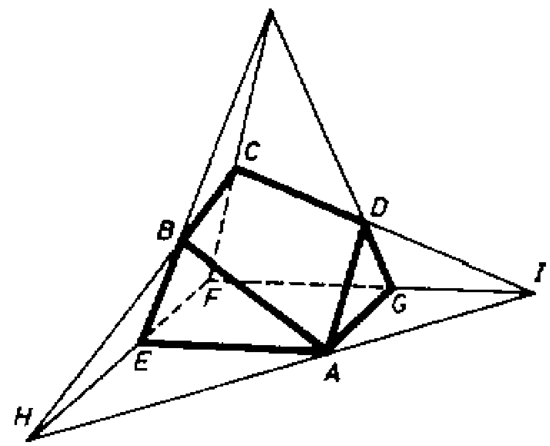


Fig. S.14.12

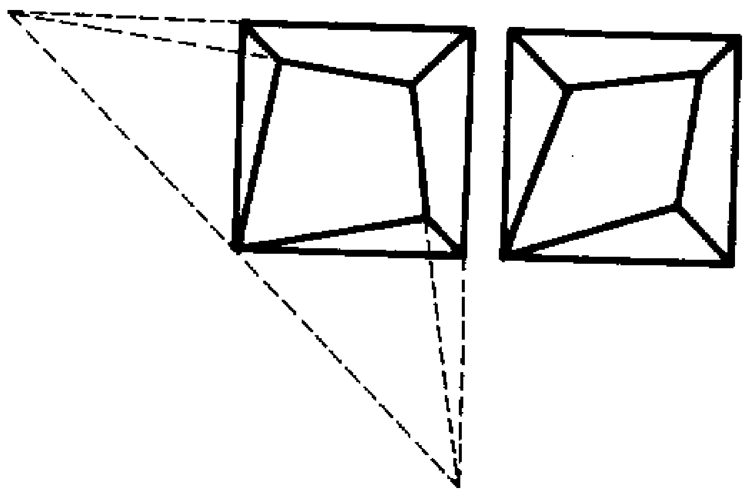
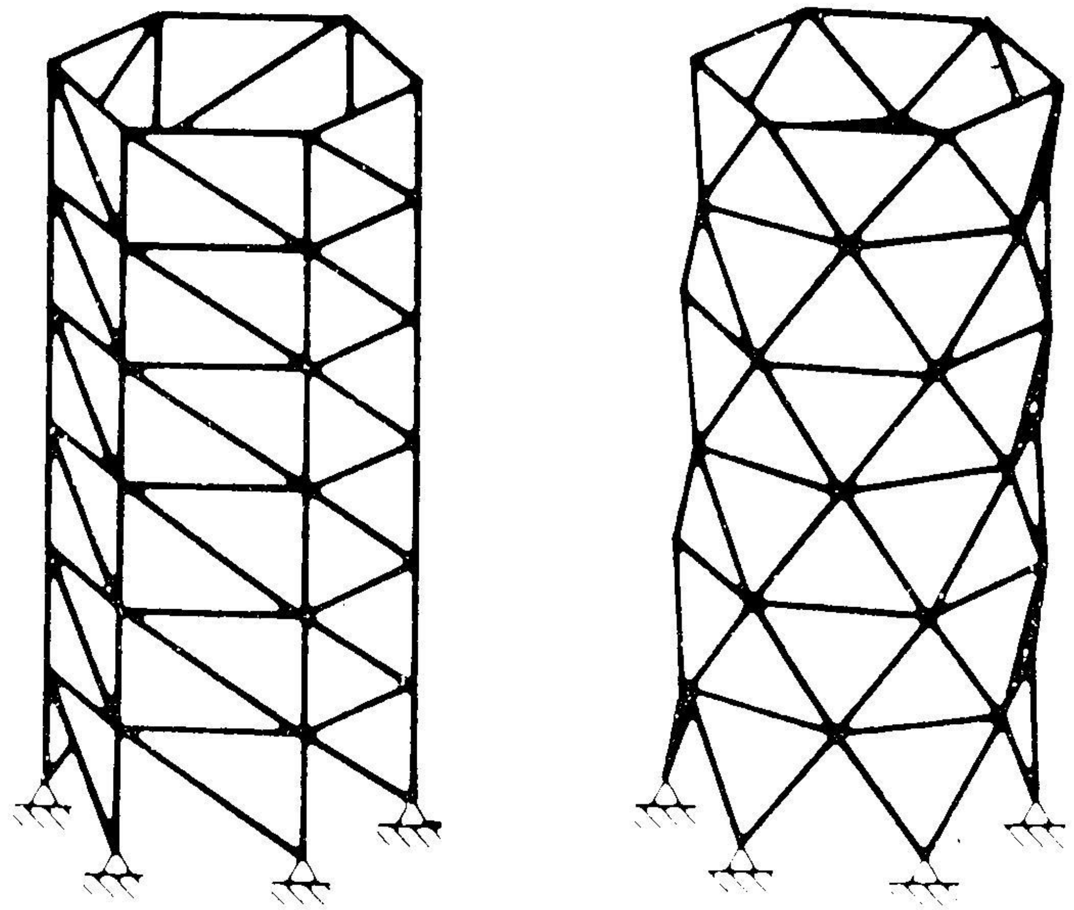
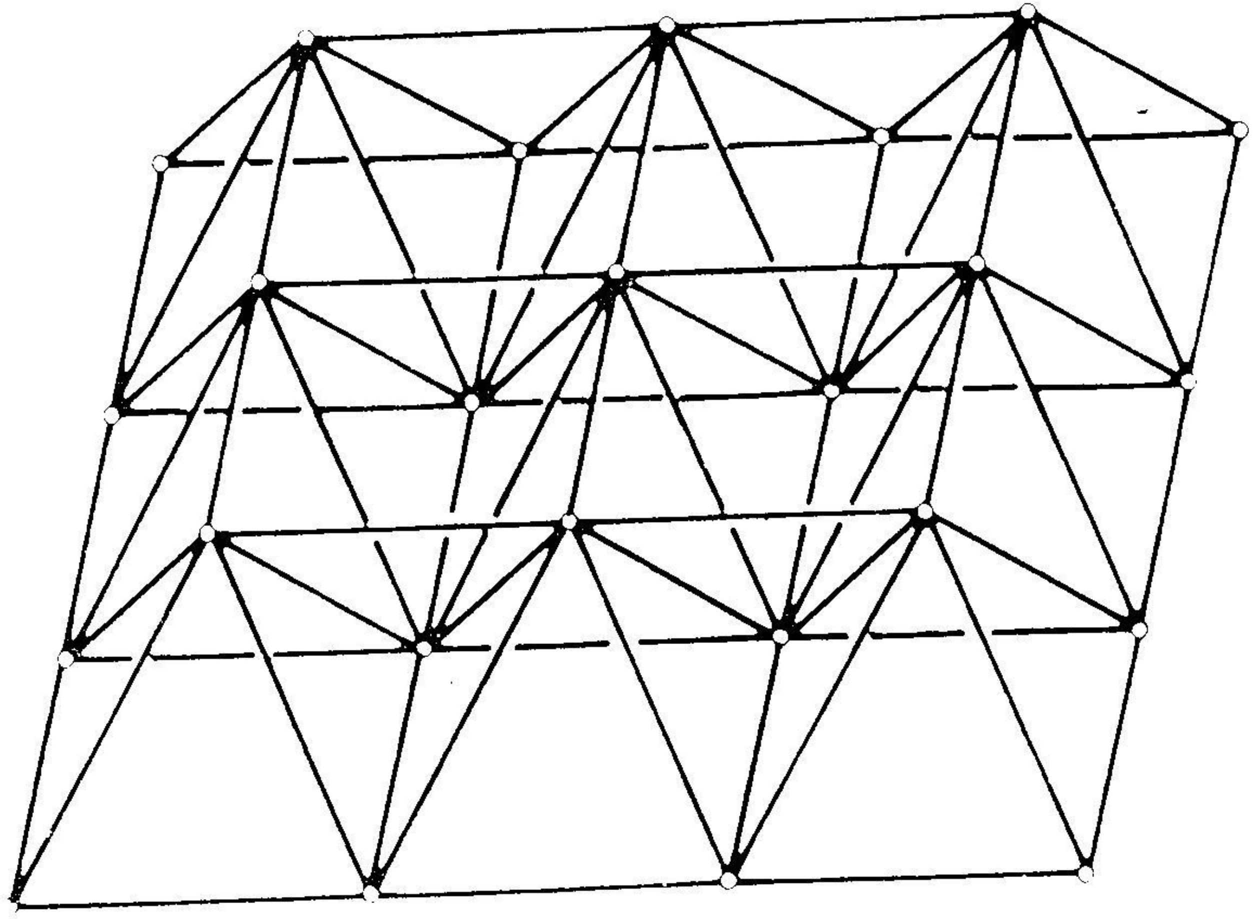


Fig. S.14.13



Rigid Non-rigid



When is *this* framework rigid?

- For certain graphs (like C_4) every realization leads to nonrigid frameworks.
- For others, some of their realizations lead to rigid frameworks.

These latter type of graphs are called *generic rigid.*

- Deciding the rigidity of a frame-work (that is, of an actual real-ization of a graph) is a problem in linear algebra.

- Deciding whether a graph is generic rigid is a combinatorial problem.

- Deciding the rigidity of a frame-work (that is, of an actual real-ization of a graph) is determining r(A) over the field of the reals.
- Deciding whether a graph is generic rigid is determining r(A) over a commutative ring.

- Special case: minimal generic rigid graphs (when the deletion of any edge destroys rigidity).

- In this case the number of rods must be $r(A) = nd - d(d+1)/2$

- Special case: minimal generic rigid graphs (when the deletion of any edge destroys rigidity).

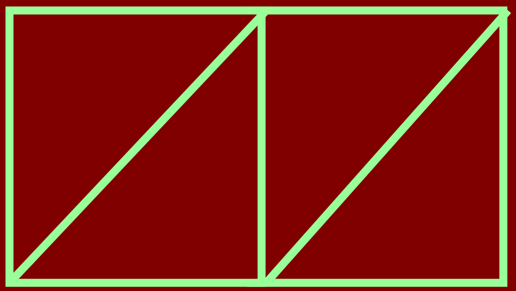
- In this case the number of rods must be $r(A) = nd - d(d+1)/2$

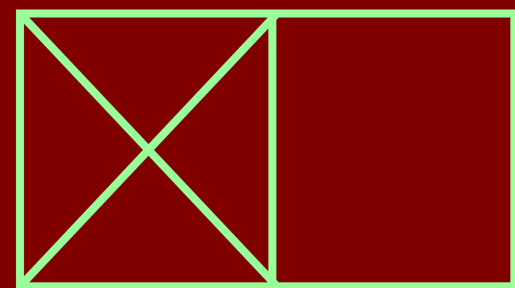
- Why minimal?

A famous minimally rigid structure:

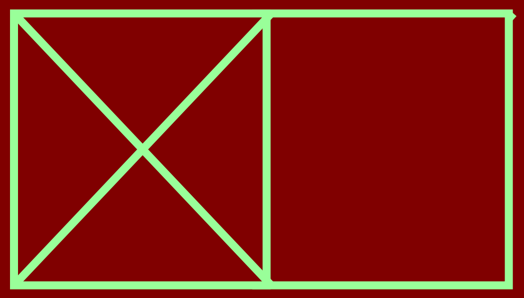
Szabadság Bridge, Budapest

Does $e = 2n-3$ imply that a planar framework is minimally rigid?





Certainly not:



If a part of the framework is „overbraced", there will be a nonrigid part somewhere else…

Maxwell (1864):

If a graph G is minimal generic rigid in the plane then,

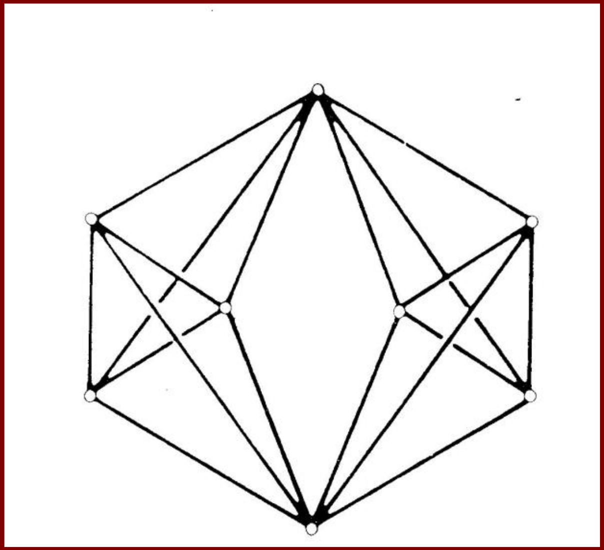
in addition to $e = 2n - 3$,

the relation $e' \leq 2n' - 3$ must hold for every (induced) sub-graph G' of G.

Laman (1970):

A graph G is minimal generic rigid in the plane if and only if $e = 2n - 3$ and the relation $e' \leq 2n' - 3$ holds for every (induced) subgraph G' of G.

However, the 3-D analogue of Laman's theorem is not true:



The double banana graph
(Asimow – Roth, 1978)

Laman (1970):

A graph G is minimal generic rigid in the plane if and only if $e = 2n - 3$ and the relation $e' \leq 2n' - 3$ holds for every (induced) subgraph G' of G.

This is a „good characterization” of minimal generic rigid graphs in the plane, but we do not wish to check some 2^n subgraphs…

Lovász and Yemini (1982):

A graph G is minimal generic rigid in the plane if and only if

$e = 2n - 3$ and doubling any edge the resulting graph, with $2(n\text{-}1)$ edges, is the union of two edge-disjoint trees.

A slight modification (R.,1984):

A graph G is minimal generic rigid in the plane if and only if

$e = 2n - 3$ and joining any two vertices with a new edge the resulting graph, with $2(n-1)$ edges, is the union of two edge-disjoint trees.

A (not particularly interesting) corollary in pure graph theory:

Let G be a graph with n vertices and $e = 2n - 3$ edges. If joining any two *adjacent* vertices, the resulting graph, with $2(n\text{-}1)$ edges, is the union of two edge-disjoint trees then joining *any two vertices* with a new edge leads to a graph with the same property.

Université de Montréal
Centre de recherches mathématiques
(CRM)

SPECIAL SEMESTER ON STRUCTURAL RIGIDITY

Centre de recherches mathématiques
Université de Montréal
January - May 1987

MEMBERS FOR THE SEMESTER

Janos Baracs, Robert Connelly, Henry Crapo, Ivo Rosenberg, Walter Whiteley.

SCHEDULE OF WORKSHOPS AND CURRENTLY ARRANGED VISITS.

Week of February 2	RIGIDITY AND SPHERE PACKING*	T. Tarnai, Budapest; J. Papadopoulos, Ithaca; Z. Gàspàr, Hungary; A. Bezdek, Budapest.
Week of February 9	TENSEGRITY*	T. Tarnai, Budapest; R. Motro, Montpellier, France; B. Roth, Wyoming; J. Graver, Syracuse; T. Havel, LaJolla Z. Gàspàr, Hungary, A. Bezdek, Budapest
Week of February 16	RIGIDITY OF TRIANGULATED SURFACES*	J. Graver, Syracuse; E. Kann, New York; B. Roth, Wyoming; A. Fogelsanger, Ithaca B. Servetius, Syracuse
Week of March 9	GEOMETRY OF 4-SPACE	H. Stachel, Vienna; T. Havel, LaJolla; D. Avis, Montréal
Week of March 16	COMBINATORIAL ANGLE AND DISTANCE DETERMINATION	T. Havel, La Jolla
Week of April 13	RIGIDITY OF GRIDS* , BIPARTITE FRAMEWORKS* (and introductory chapter)	A. Recski, Budapest; B. Roth, Wyoming; J. Graver, Syracuse; T-S.Tay,Singapore; E. Bolker, Boston
Week of April 20	GENERIC RIGIDITY*	A. Recski, Budapest; B. Roth, Wyoming; J. Graver, Syracuse; T-S. Tay, Singapore, N.White, Gainesville, B. Servetius, Syracuse, A. Dress, Bielefeld
Week of April 27	PROJECTIVE GEOMETRY OF FRAMEWORKS*	N. White, Gainseville; T-S. Tay, Singapore;
Week of May 11	RIGIDITY AND POLYHEDRAL COMBINATORICS*	L. Billera, Ithaca; M. Bayer, Boston; C. Lee, Louiseville, T-S. Tay, Singapore; A. Weiss, Toronto; B.Monson, Fredericton.

Workshops will run from Monday 10 a.m. through Friday noon. Current plans include a gathering at 10 a.m. each morning for informal discussion and planning, and a working session each afternoon around 3 p.m.. A regular colloquium will be held each Thursday afternoon. * indicates that a chapter of a forthcoming collective book on the

E. Bolker, Boston

Week of April 20 — GENERIC RIGIDITY*

A. Recski, Budapest; B. Roth, Wyoming; J. Graver, Syracuse; T-S. Tay, Singapore, N.White, Gainesville, B. Servetius, Syracuse, A. Dress, Bielefeld

Week of April 27 — PROJECTIVE GEOMETRY OF FRAMEWORKS*

N. White, Gainseville; T-S. Tay, Singapore;

Week of May 11 — RIGIDITY AND POLYHEDRAL COMBINATORICS*

L. Billera, Ithaca; M. Bayer, Boston; C. Lee, Louiseville, T-S. Tay, Singapore; A. Weiss, Toronto; B.Monson, Fredericton.

Workshops will run from Monday 10 a.m. through Friday noon. Current plans include a gathering at 10 a.m. each morning for informal discussion and planning, and a working session each afternoon around 3 p.m.. A regular colloquium will be held each Thursday afternoon. * indicates that a chapter of a forthcoming collective book on the Geometry of Rigid Structures will be reviewed and revised as part of the workshop. The chapter will be circulated to participants prior to the workshop.

For more information contact:

The Organizing Committee
Special Semester on Structural Topology
Centre de recherches mathématiques

C.P. 6128, succursale A
Montréal (Québec)
H3C 3J7

(514) 343-7501

March 1987

Week of March 16 COMBINATORIAL ANGLE AND DISTANCE DETERMINATION
T. Havel, La Jolla

Week of April 13 RIGIDITY OF GRIDS* , BIPARTITE FRAMEWORKS* (and introductory chapter)
A. Recski, Budapest; B. Roth, Wyoming;
J. Graver, Syracuse; T-S.Tay,Singapore;
E. Bolker, Boston

Week of April 20 GENERIC RIGIDITY*
A. Recski, Budapest; B. Roth, Wyoming;
J. Graver, Syracuse; T-S. Tay, Singapore,
N.White, Gainesville, B. Servetius,
Syracuse, A. Dress, Bielefeld

Week of April 27 PROJECTIVE GEOMETRY OF FRAMEWORKS*
N. White, Gainseville; T-S. Tay, Singapore;

Week of May 11 RIGIDITY AND POLYHEDRAL COMBINATORICS*
L. Billera, Ithaca; M. Bayer, Boston;
C. Lee, Louiseville, T-S. Tay, Singapore;
A. Weiss, Toronto; B.Monson, Fredericton.

Workshops will run from Monday 10 a.m. through Friday noon. Current plans include a gathering at 10 a.m. each morning for informal discussion and planning, and a working session each afternoon around 3 p.m.. A regular colloquium will be held each Thursday afternoon. * indicates that a chapter of a forthcoming collective book on the Geometry of Rigid Structures will be reviewed and revised as part of the workshop. The chapter will be circulated to participants prior to the workshop.

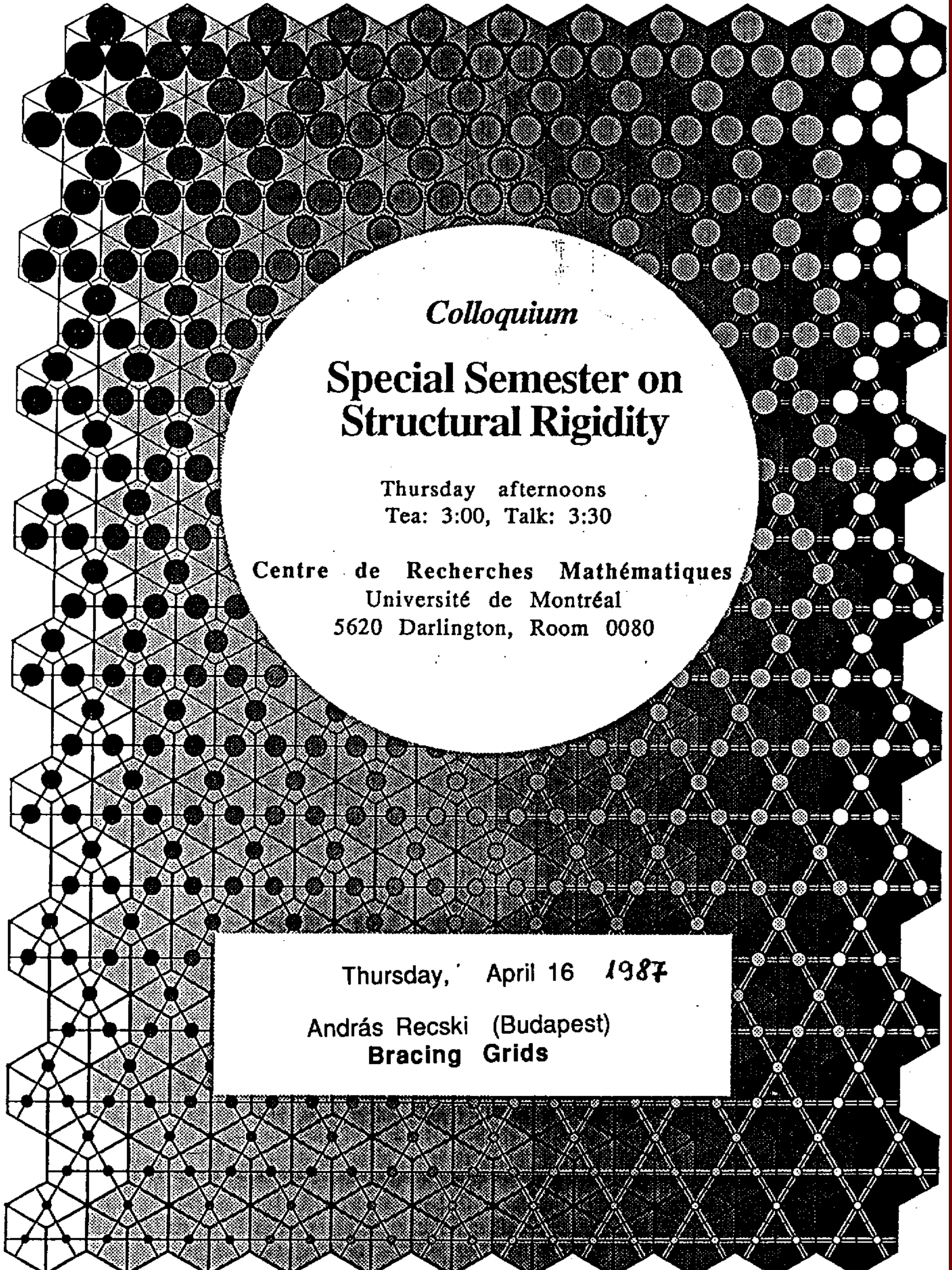
For more information contact:

The Organizing Committee
Special Semester on Structural Topology
Centre de recherches mathématiques

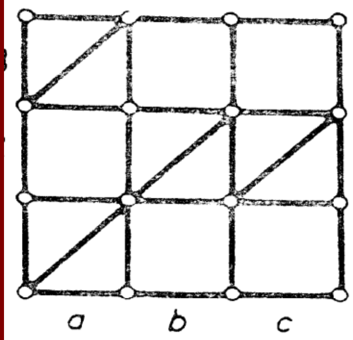
C.P. 6128, succursale A
Montréal (Québec)
H3C 3J7

(514) 343-7501

March 1987

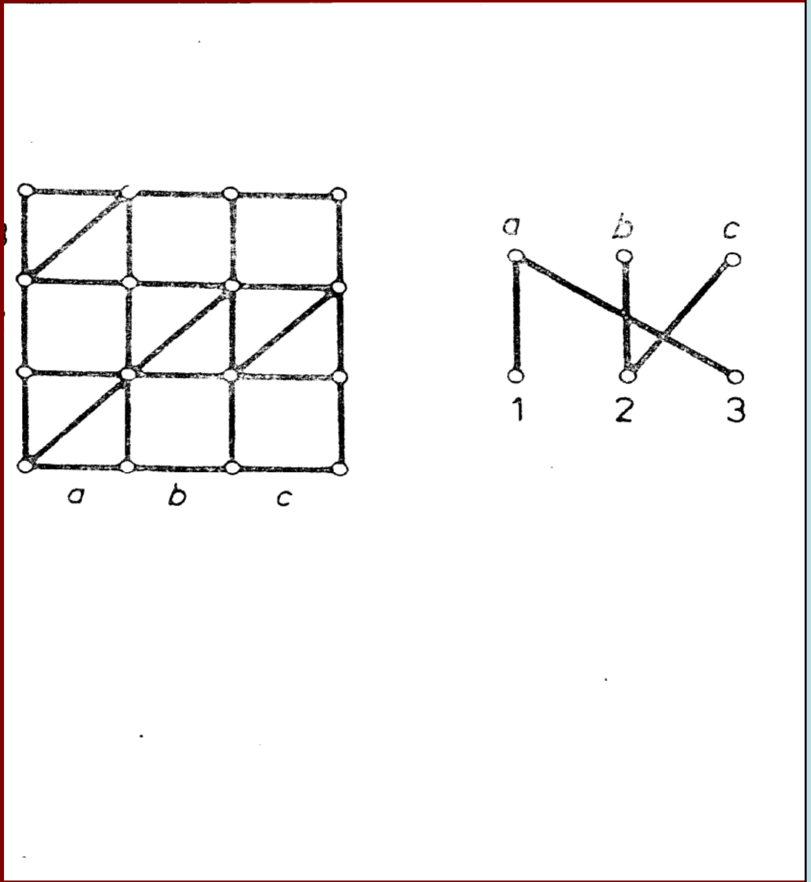

Colloquium
Special Semester on Structural Rigidity
Thursday afternoons
Tea: 3:00, Talk: 3:30
Centre de Recherches Mathématiques
Université de Montréal
5620 Darlington, Room 0080
Thursday, April 16 1987
András Recski (Budapest)
Bracing Grids

Square grids with diagonals



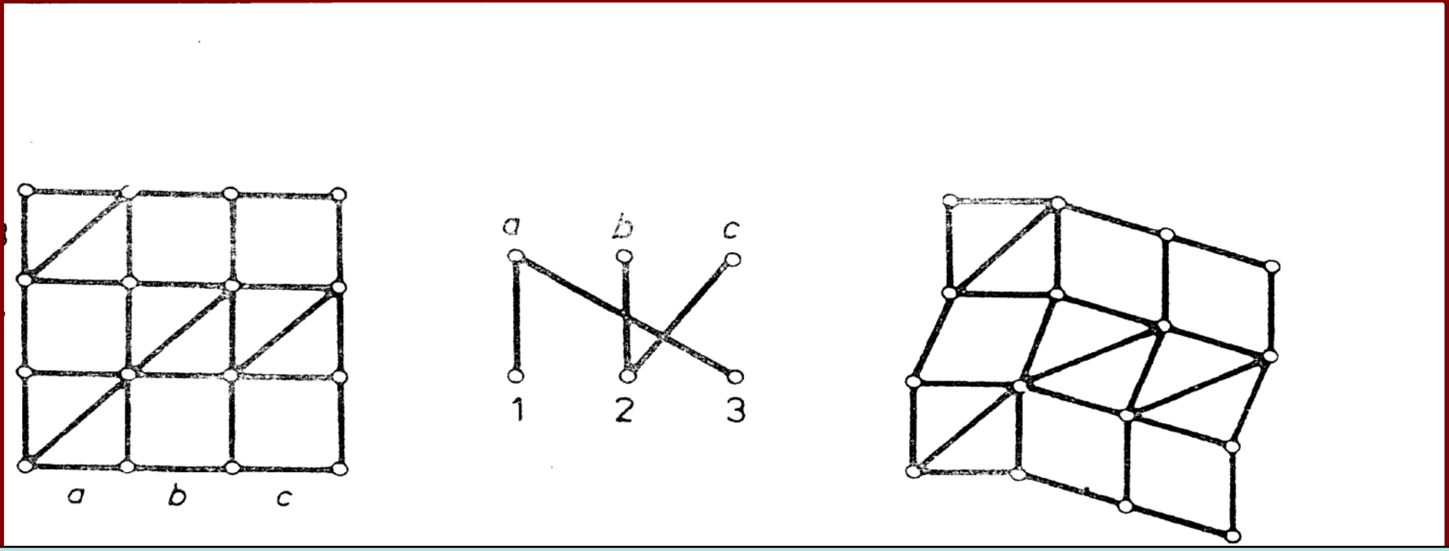
How many diagonals do we need (and where) to make a square grid rigid?

Square grids with diagonals



The edge *{a,1}* indicates that column *a* and row *1* will move together.

Square grids with diagonals



This is nonrigid, since the associated bipartite graph is disconnected.

Rigidity of square grids

- Bolker and Crapo, 1977: A set of diagonal bars makes a k X ℓ square grid rigid if and only if the corresponding edges form a ***connected*** subgraph in the bipartite graph model.

Rigidity of square grids

- Bolker and Crapo, 1977: A set of diagonal bars makes a k X ℓ square grid rigid if and only if the corresponding edges form a ***connected*** subgraph in the bipartite graph model.

- Baglivo and Graver, 1983: In case of diagonal cables, ***strong connectedness*** is needed in the (directed) bipartite graph model.

Minimum # diagonals needed:

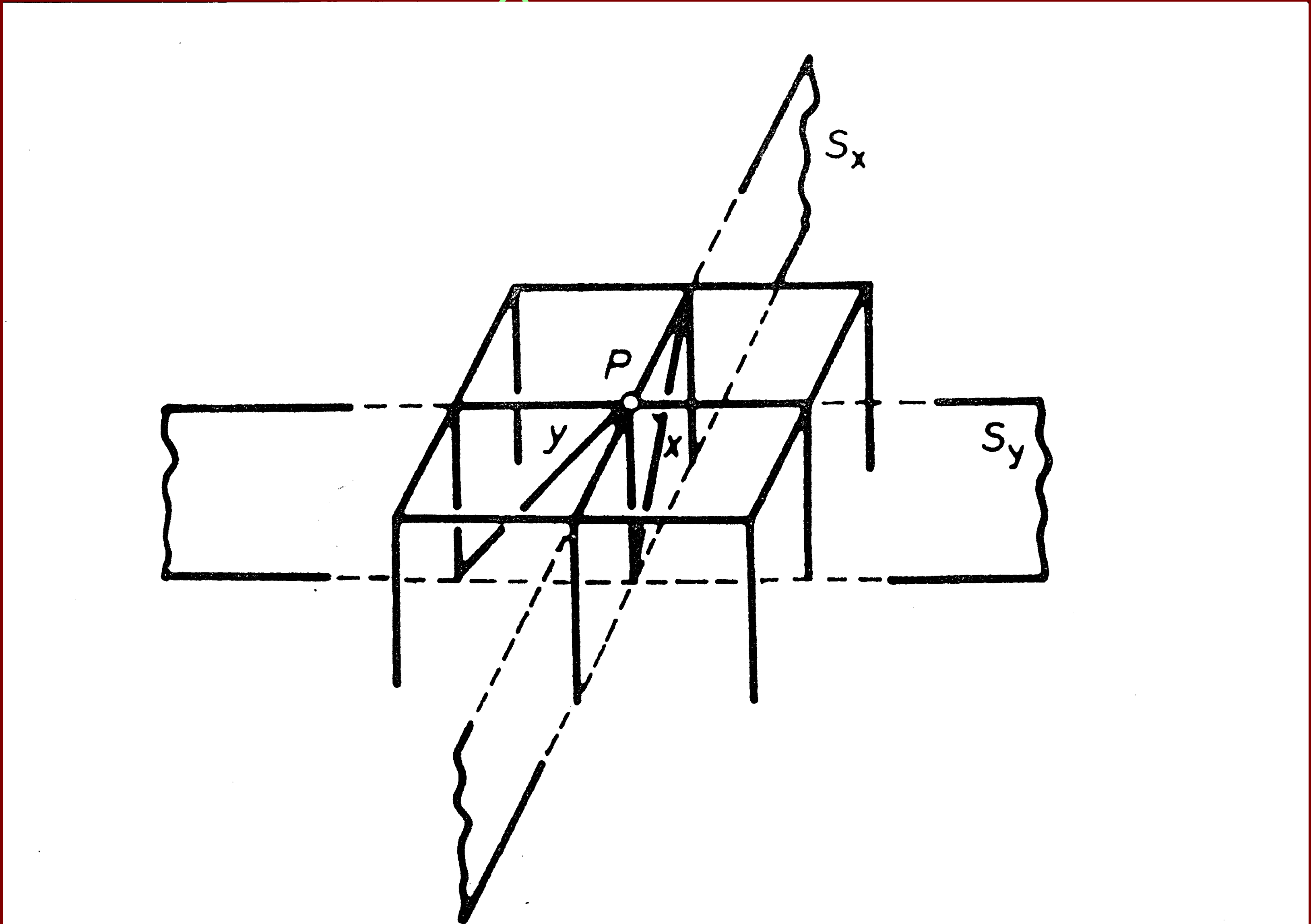
$B = k + \ell - 1$ diagonal bars

$C = 2 \cdot \max(k, \ell)$ diagonal cables

(If $k \neq \ell$ then $C - B > 1$)

In case of a one-story building some squares in the ***vertical*** walls should also be braced.

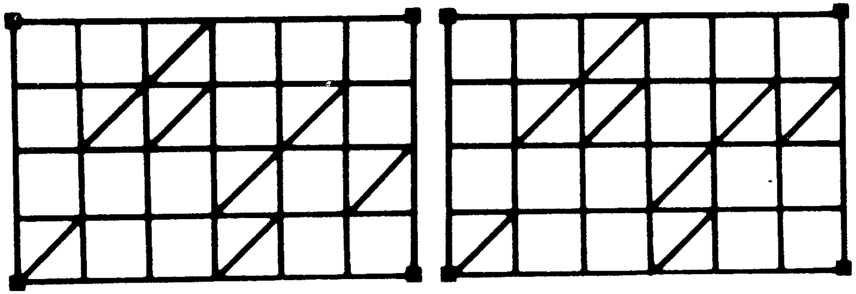
Such a diagonal *x* prevents motions of that plane S_x along itself.



Rigidity of one-story buildings

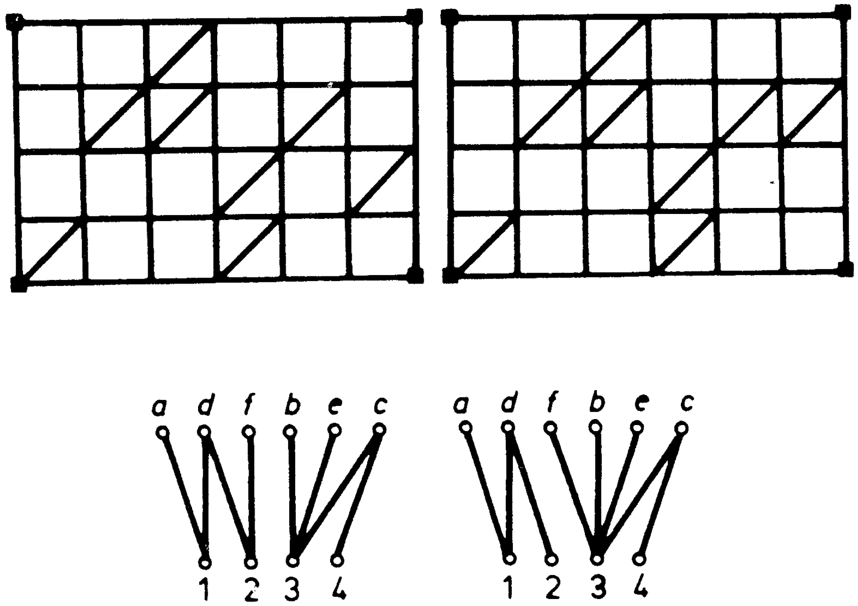
Bolker and Crapo, 1977:

If each external vertical wall contains a diagonal bar then instead of studying the roof of the building one may consider a $k \times \ell$ square grid ***with its four corners pinned down.***

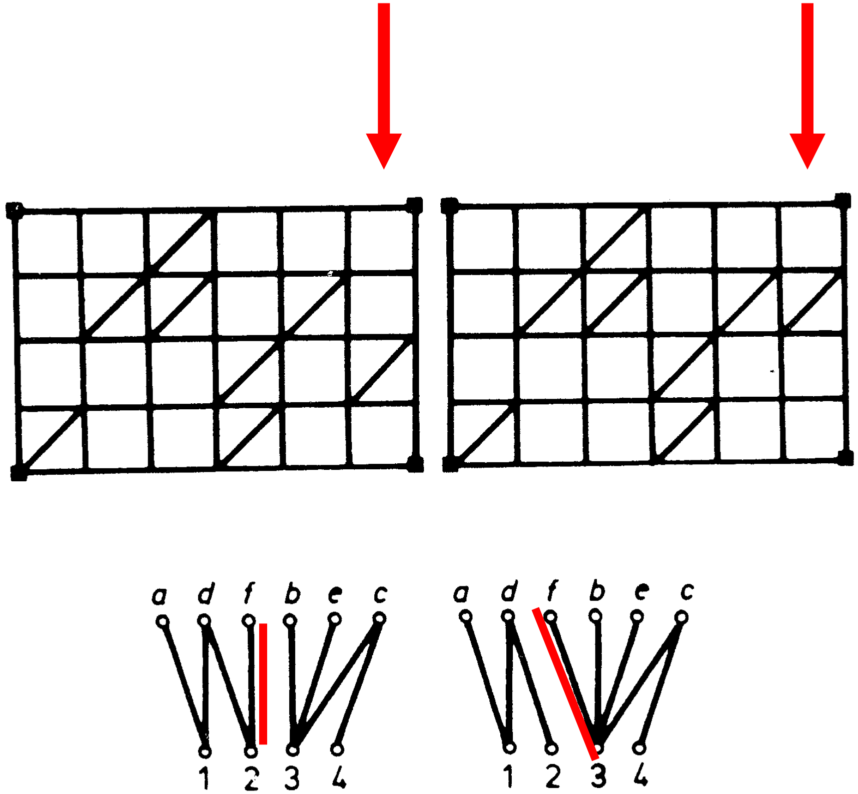


?

One of these 4 X 6 grids is rigid (if the four corners are pinned down), the other one has an (infinitesimal) motion. Both have 4 + 6 – 2 = 8 diagonals.



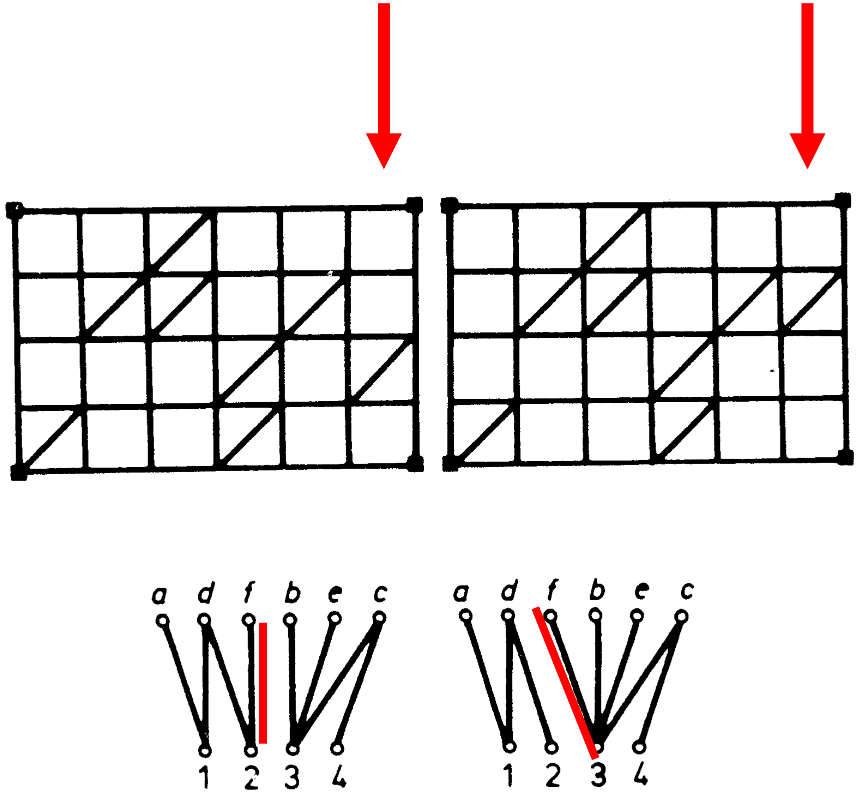
In the bipartite graph model we have 2-com-ponent forests



3 3 2 4

2 2 2 2

In the bipartite graph model we have 2-com-ponent forests



3 3 2 4

2 2 2 2

symmetric asymmetric

In the bipartite graph model we have 2-com-ponent forests

Rigidity of one-story buildings

Bolker and Crapo, 1977: A set of diagonal bars makes a k X ℓ square grid (with corners pinned down) rigid if and only if the corresponding edges in the bipartite graph model form either a ***connected*** subgraph or a ***2-component asymmetric forest***.

For example, if $k = 4$, $\ell = 6$, $k' = 2$, $\ell' = 3$, then the 2-component forest is symmetric ($L = K$, where $\ell' / \ell = L$, $k' / k = K$).

$k \times \ell$ square grid	$k \times \ell$ 1-story building
$k + \ell - 1$ diagonal bars	$k + \ell - 2$ diagonal bars
$2 \cdot \max(k, \ell)$ diagonal cables	

$k \times \ell$ square grid	$k \times \ell$ 1-story building
$k + \ell - 1$ diagonal bars	$k + \ell - 2$ diagonal bars
$2 \cdot \max(k, \ell)$ diagonal cables	How many diagonal cables?

Minimum # diagonals needed:

$B = k + \ell - 2$ diagonal bars

$C = k + \ell - 1$ diagonal cables
(except if $k = \ell = 1$ or $k = \ell = 2$)

(Chakravarty, Holman, McGuinness and R., 1986)

$k \times \ell$ square grid	$k \times \ell$ 1-story building
$k + \ell - 1$ diagonal bars	$k + \ell - 2$ diagonal bars
$2 \cdot \max(k, \ell)$ diagonal cables	$k + \ell - 1$ diagonal cables

Rigidity of one-story buildings

Which $(k + \ell - 1)$-element sets of cables make the k X ℓ square grid (with corners pinned down) rigid?

Let X, Y be the two colour classes of the directed bipartite graph. An XY-path is a directed path starting in X and ending in Y.

If X_0 is a subset of X then let $N(X_0)$ denote the set of those points in Y which can be reached from X_0 along XY-paths.

R. and Schwärzler, 1992:

A ($k + \ell - 1$)-element set of cables makes the k X ℓ square grid (with corners pinned down) rigid if and only if

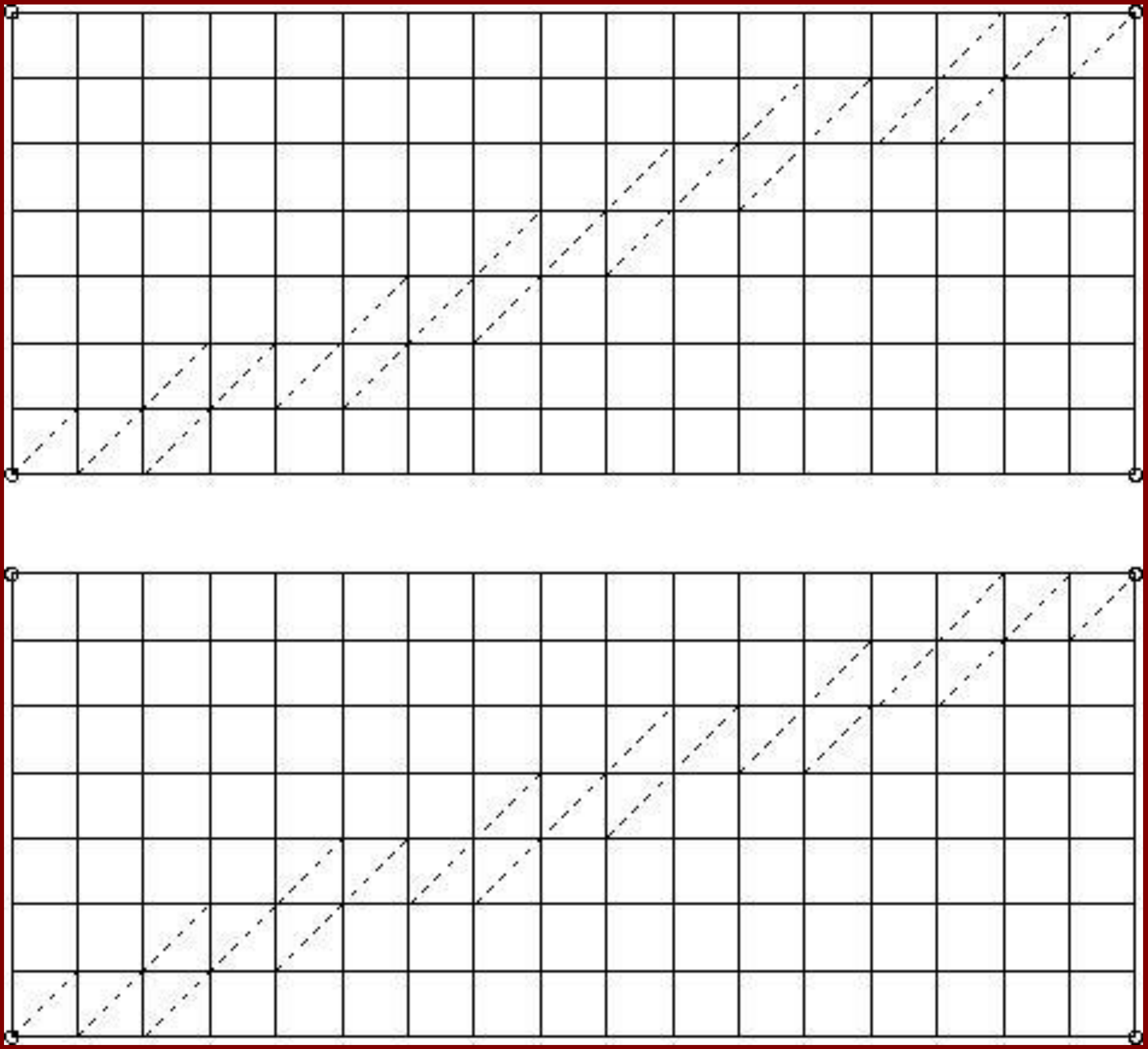
$$|N(X_0)| \cdot k > |X_0| \cdot \ell$$

holds for every proper subset X_0 of X and

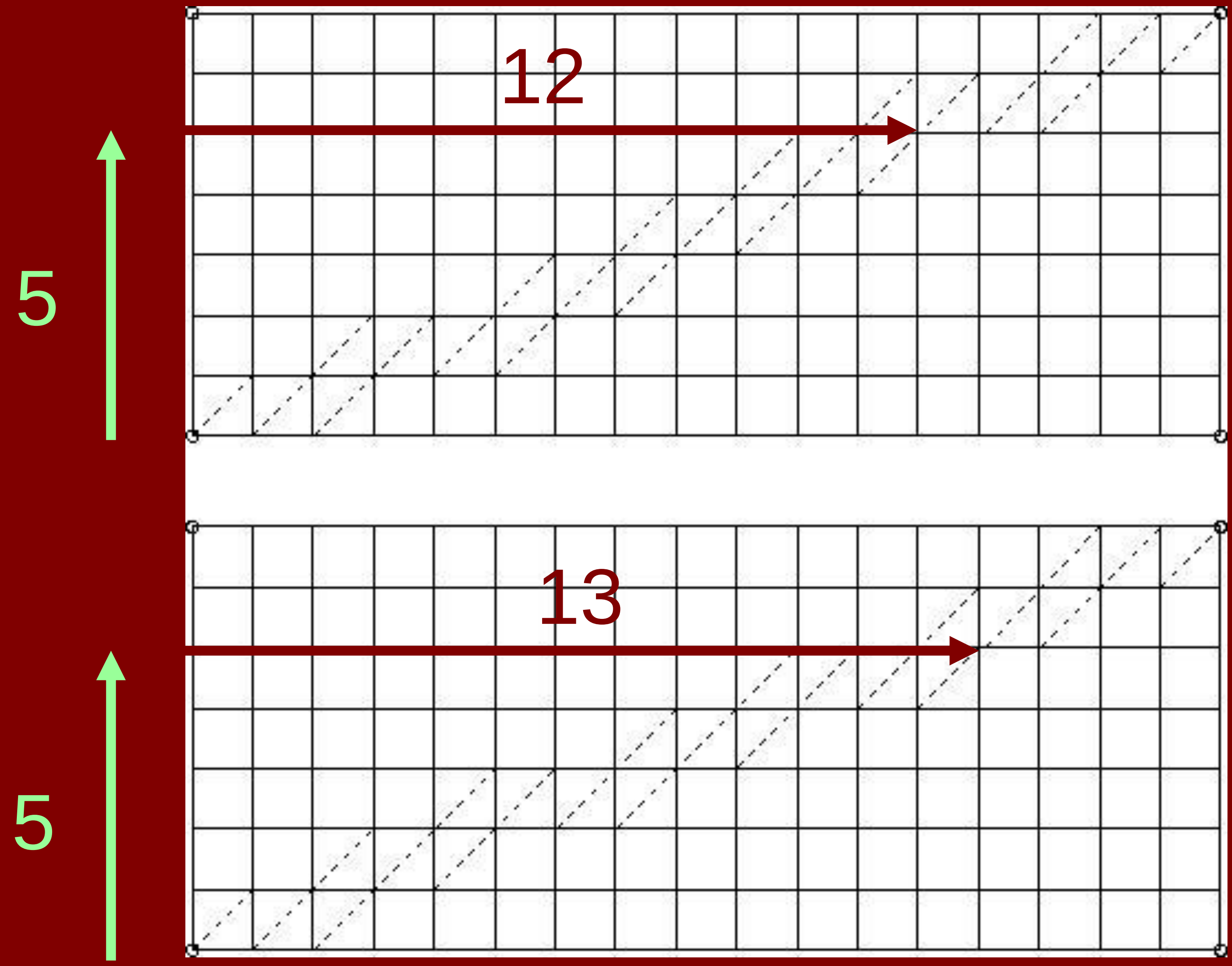
$$|N(Y_0)| \cdot \ell > |Y_0| \cdot k$$

holds for every proper subset Y_0 of Y.

Which one-story building is rigid?



Which one-story building is rigid?

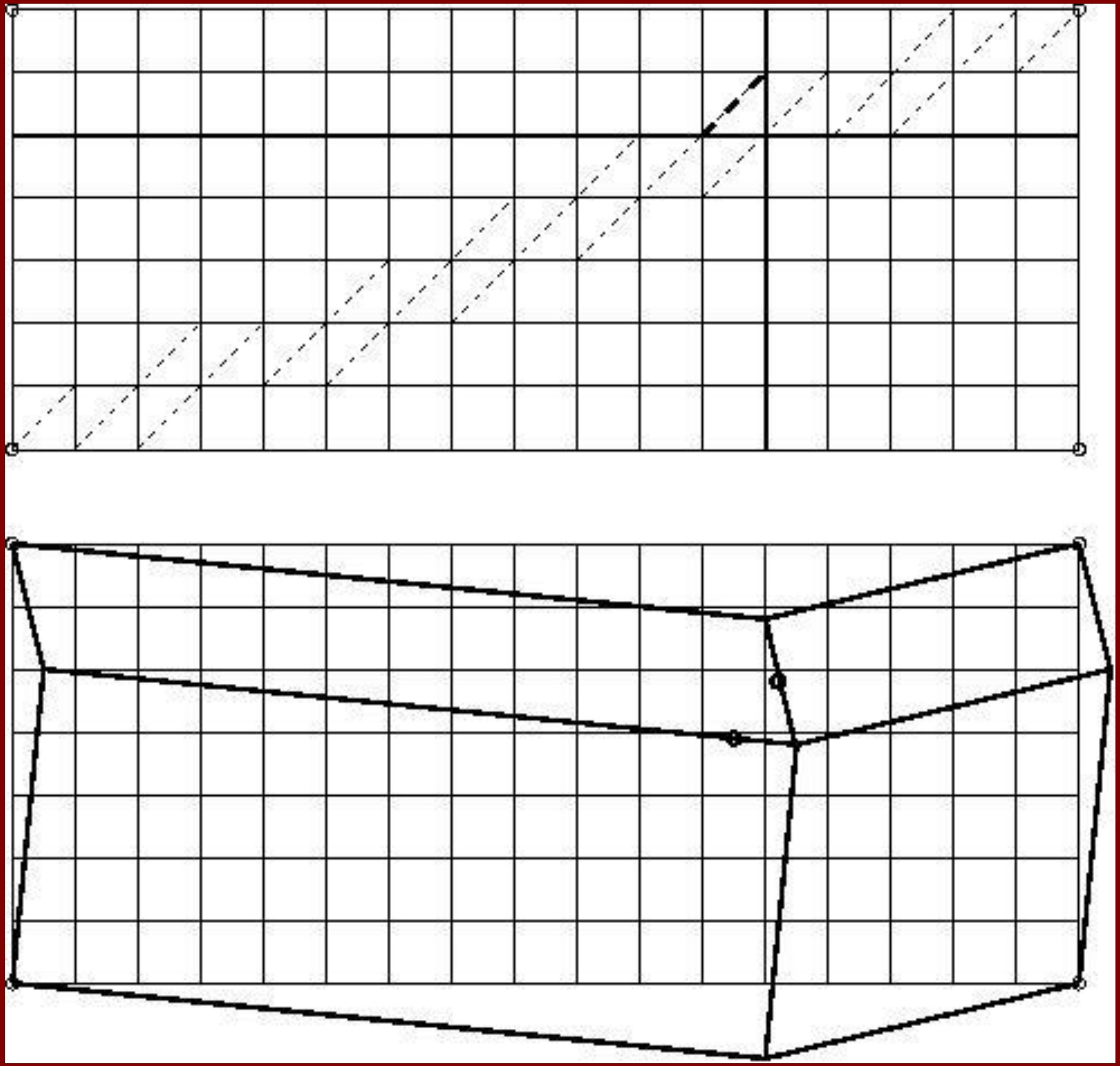


Solution:

Top: $k = 7$, $\ell = 17$, $k_0 = 5$, $\ell_0 = 12$, $L < K$
(0.7059 < 0.7143)

Bottom: $k = 7$, $\ell = 17$, $k_0 = 5$, $\ell_0 = 13$, $L > K$
(0.7647 > 0.7143)

where $\ell_0 / \ell = L$, $k_0 / k = K$.



Hall, 1935 (König, 1931):

A bipartite graph with colour classes X, Y has a perfect matching if and only if

$$|N(X_0)| \geq |X_0|$$

holds for every ~~proper~~ subset X_0 of X and

$$|N(Y_0)| \geq |Y_0|$$

holds for every ~~proper~~ subset Y_0 of Y.

Hetyei, 1964:

A bipartite graph with colour classes *X*, *Y* has perfect matchings ***and every edge is contained in at least one*** if and only if

$$|N(X_0)| > |X_0|$$

holds for every proper subset X_0 of *X* and

$$|N(Y_0)| > |Y_0|$$

holds for every proper subset Y_0 of *Y*.

An application in pure math

Bolker and Crapo, 1977: A set of diagonal bars makes a k X ℓ square grid (with corners pinned down) rigid if and only if the corresponding edges in the bipartite graph model form either a ***connected*** subgraph or a ***2-component asymmetric forest***.

Why should we restrict ourselves to bipartite graphs?

An application in pure math

Let $G(V, E)$ be an arbitrary graph and let us define a weight function $w: V \rightarrow \mathbb{R}$ so that $\Sigma\, w(v) = 0$. A 2-component forest is called asymmetric if the sums of the vertex weights taken separately for the two components are nonzero.

Theorem (R., 1987) The 2-component asymmetric forests form the bases of a matroid on the edge set *E* of the graph.

A side remark

The set of all 2-component forests form another matroid on the edge set of *E*.

A side remark

The set of all 2-component forests form another matroid on the edge set of *E*. This is the well known truncation of the usual cycle matroid of the graph.

A side remark

That is, the sets obtained from the spanning trees by deleting a single edge (and thus leading to the 2-component forests) form the bases of a new matroid.

A side remark

That is, the sets obtained from the spanning trees by deleting a single edge (and thus leading to the 2-component forests) form the bases of a new matroid.

Similarly, the sets obtained from the spanning trees by adding a single edge (and leading to a unique circuit of the graph) form the bases of still another matroid.

The sets obtained from the spanning trees by adding a single edge (and leading to a unique circuit of the graph) form the bases of still another matroid.

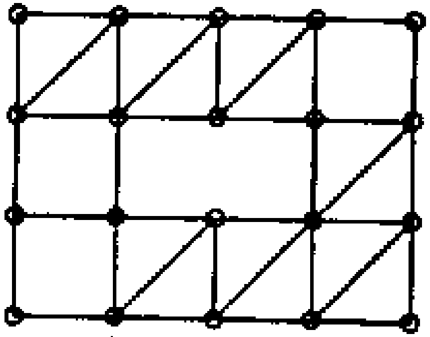
The sets obtained from the spanning trees by adding a single edge (and leading to a unique circuit of the graph) form the bases of still another matroid.

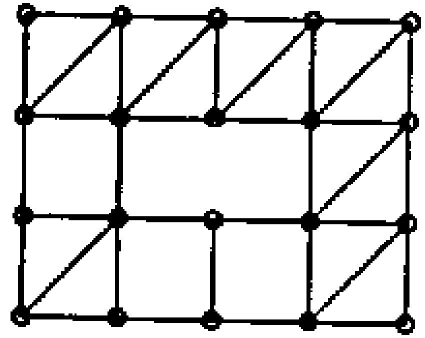
Let us fix a subset *V'* of the vertex set *V* of the graph and then permit the addition of a single edge if and only if the resulting unique circuit shares at least one vertex with *V'*.

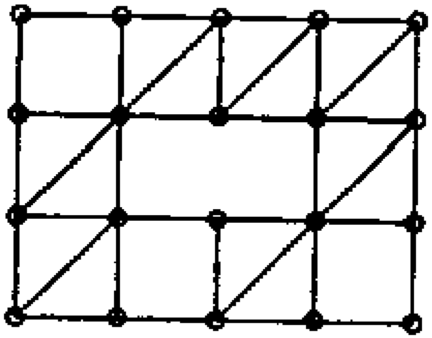
The sets obtained from the spanning trees by adding a single edge (and leading to a unique circuit of the graph) form the bases of still another matroid.

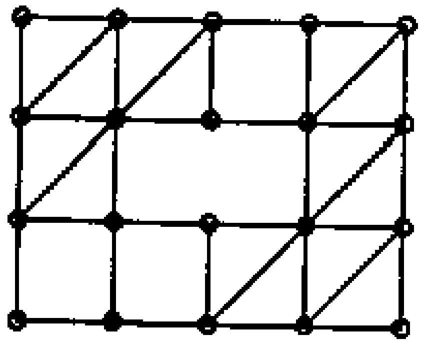
Let us fix a subset *V’* of the vertex set *V* of the graph and then permit the addition of a single edge if and only if the resulting unique circuit shares at least one vertex with *V’*.

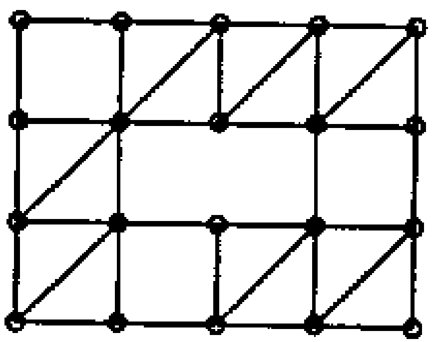
Theorem (R., 2002) The sets obtained in this way also form the bases of a matroid.

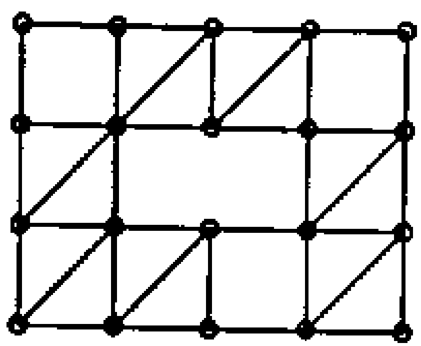


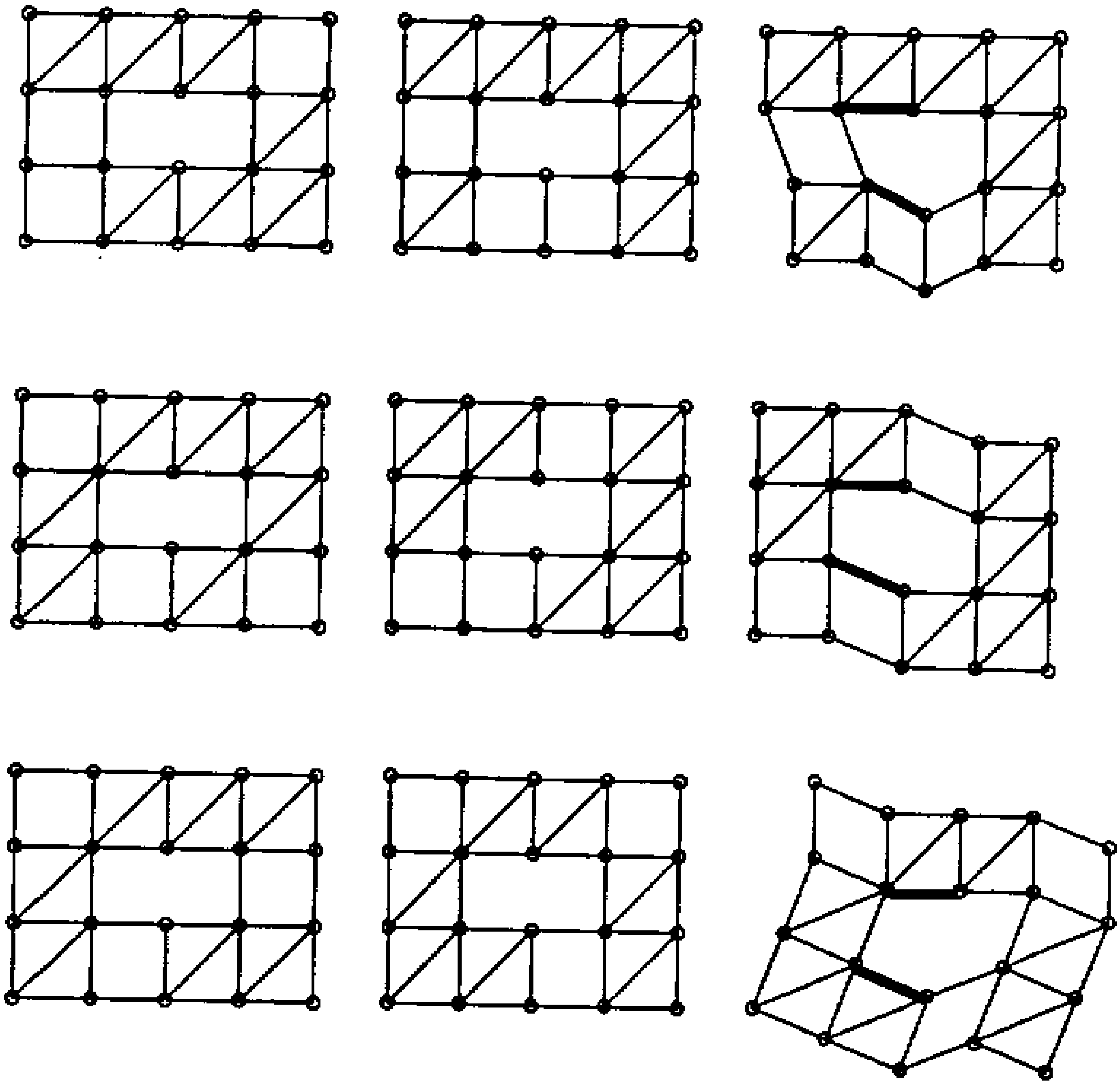












Rigid rods are resistant to compressions and tensions:

$$\|\mathbf{x}_i - \mathbf{x}_k\| = c_{ik}$$

Rigid rods are resistant to compressions and tensions:

$$\|\mathbf{x}_i - \mathbf{x}_k\| = c_{ik}$$

Cables are resistant to tensions only: $\|\mathbf{x}_i - \mathbf{x}_k\| \leq c_{ik}$

Rigid rods are resistant to compressions and tensions:

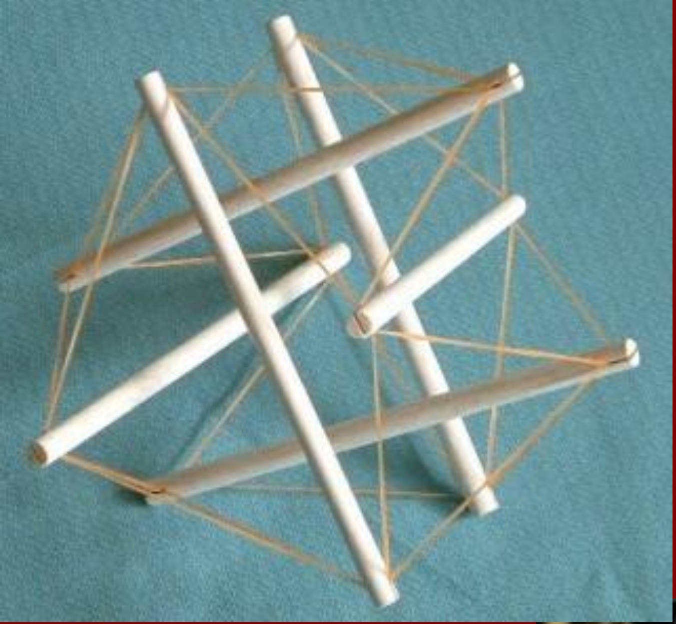
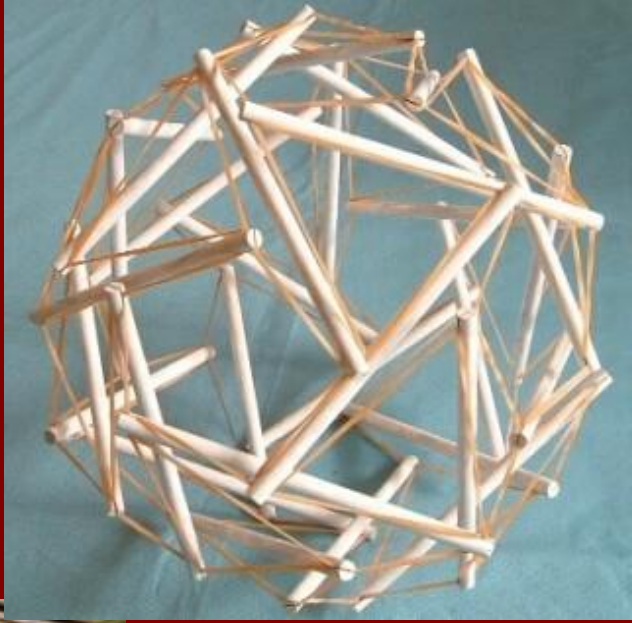
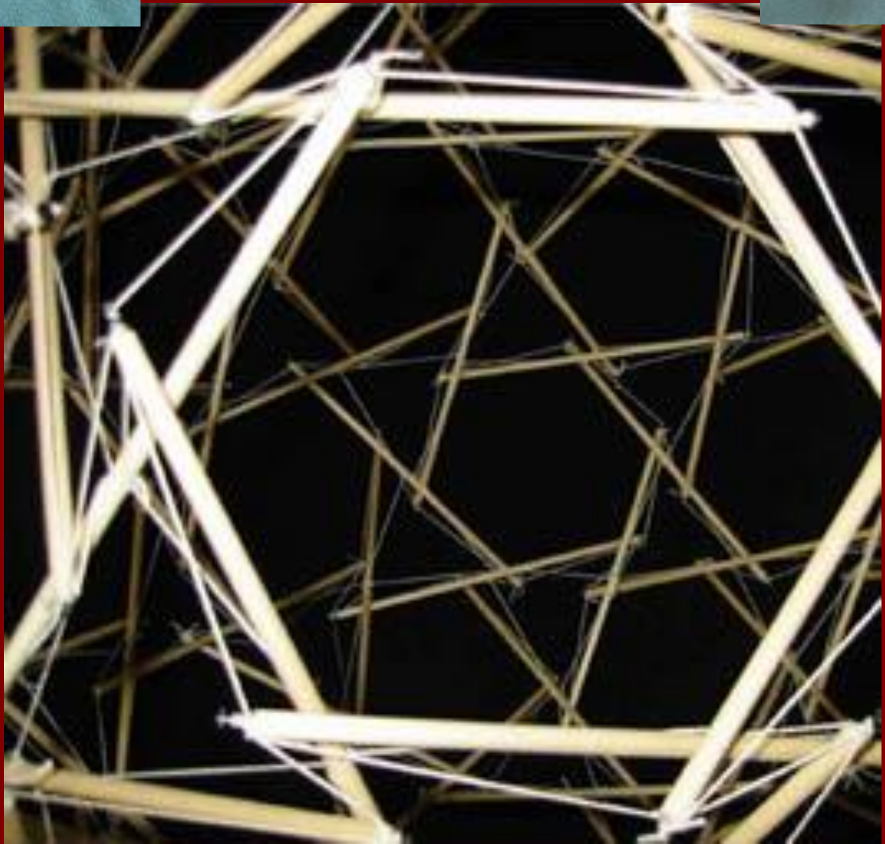
$$\|\mathbf{x}_i-\mathbf{x}_k\| = c_{ik}$$

Cables are resistant to tensions only: $\|\mathbf{x}_i-\mathbf{x}_k\| \leq c_{ik}$

Struts are resistant to compressions only:

$$\|\mathbf{x}_i-\mathbf{x}_k\| \geq c_{ik}$$

Frameworks composed from rods (bars), cables and struts are called *tensegrity frame-works*.

Frameworks composed from rods (bars), cables and struts are called *tensegrity frameworks*.

A more restrictive concept is the *r-tensegrity framework*, where rods are not allowed, only cables and struts. (The letter r means rod-free or restricted.)

We wish to generalize the above results for tensegrity frameworks:

When is a graph minimal generic rigid in the plane as a tensegrity framework (or as an r-tensegrity framework)?

Which is the more difficult problem?

Which is the more difficult problem?

If rods are permitted then why should one use anything else?

Which is the more difficult problem?

If rods are permitted then why should one use anything else?

„Weak" problem: When is a graph minimal generic rigid in the plane as an r-tensegrity framework?

„Strong" problem: When is a graph *with a given tripartition* minimal generic rigid in the plane as a tensegrity framework?

The 1-dimensional case is still easy

R. – Shai, 2005:

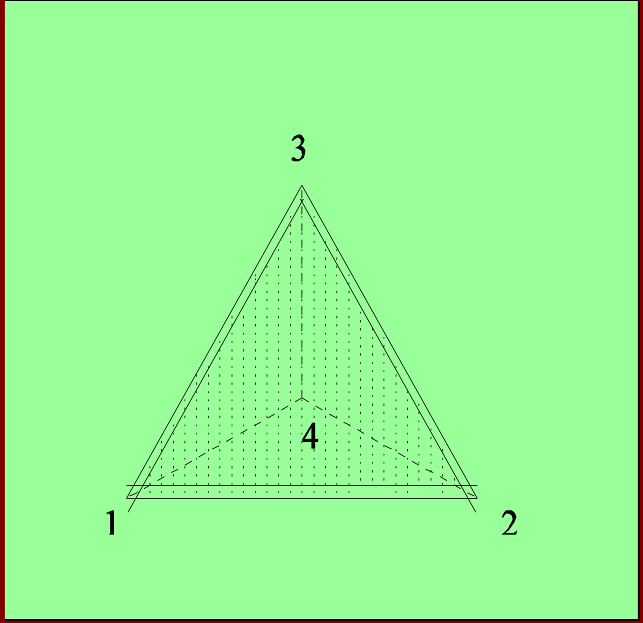
Let the cable-edges be red, the strut-edges be blue (and replace rods by a pair of parallel red and blue edges).

The graph with the given tripartition is realizable as a rigid tensegrity framework in the 1-dimensional space if and only if

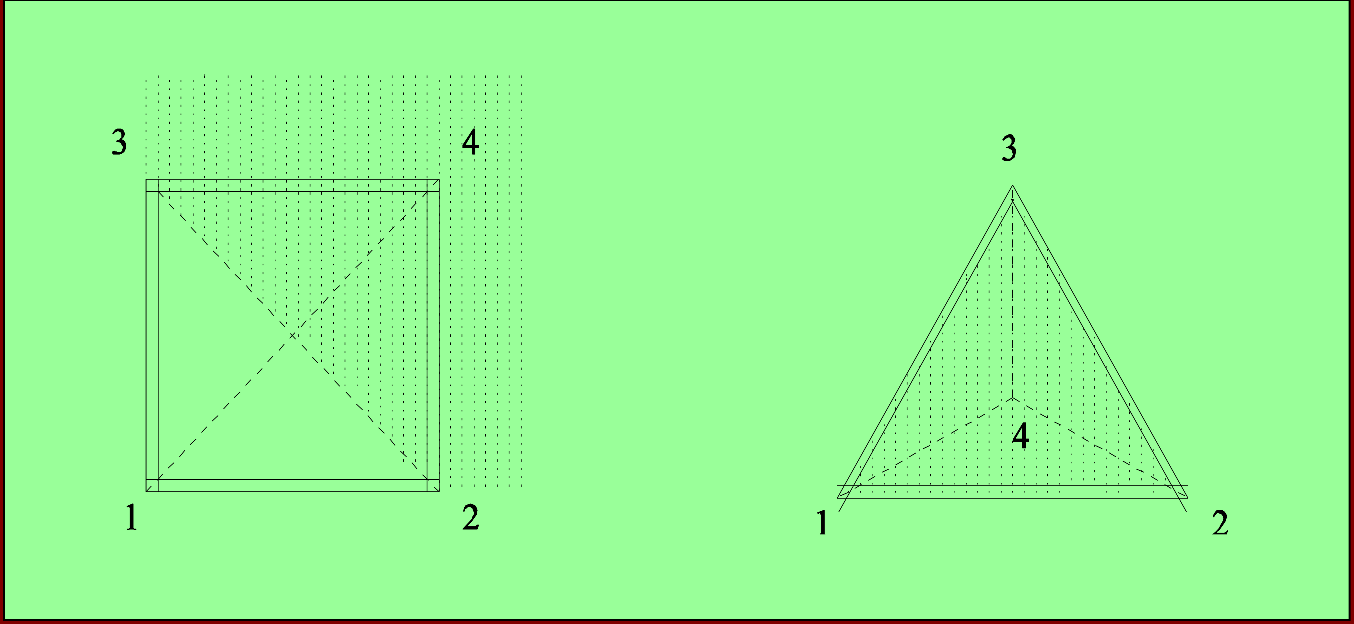
- it is 2-edge-connected and
- every 2-vertex-connected component contains edges of both colours.

An example to the

2-dimensional case:



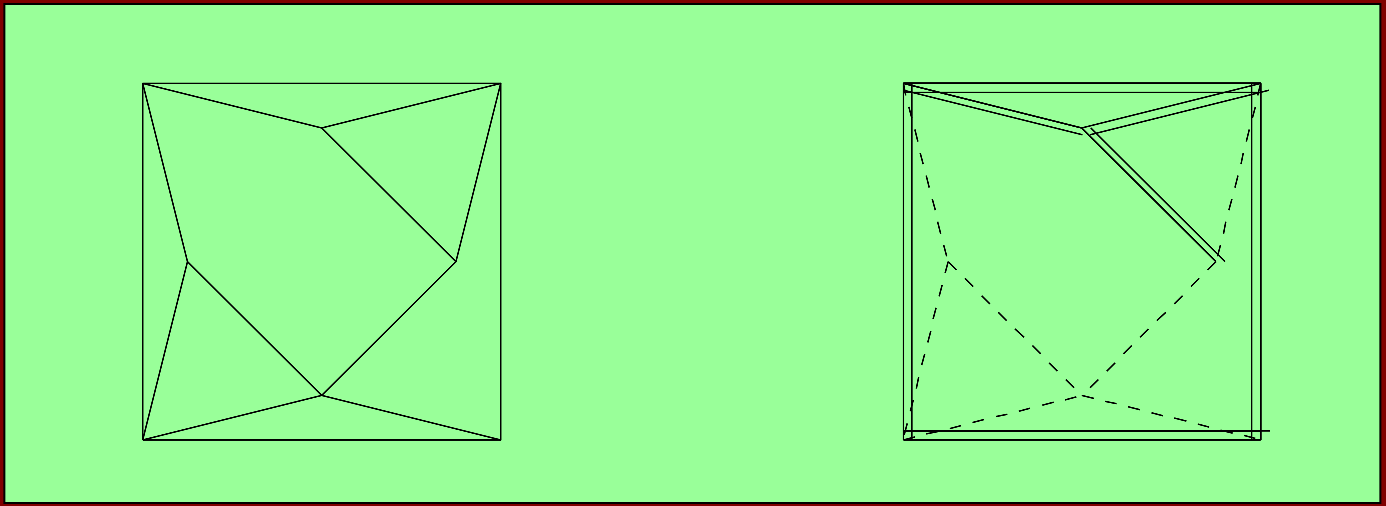
The graph K_4 can be realized as a rigid tensegrity framework with struts {1,2}, {2,3} and {3,1} and with cables for the rest (or *vice versa*) if ‘4’ is in the convex hull of {1,2,3} …



…or with cables for two independent edges and struts for the rest (or *vice versa*) if none of the joints is in the convex hull of the other three.

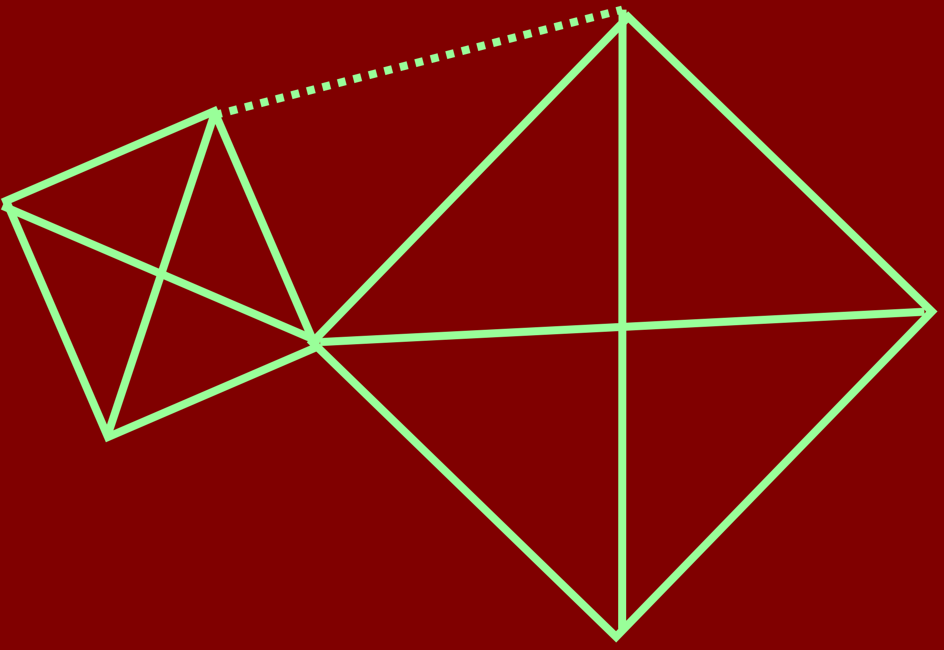
As a more difficult example, consider the emblem of the Sixth Czech-Slovak International Symposium on Combinatorics, Graph Theory, Algorithms and Applications

(Prague, 2006, celebrating the 60th birthday of Jarik Nešetřil).



If every bar must be replaced by a cable or by a strut then only one solution (and its reversal) is possible.

Critical rods cannot be replaced by cables or struts if we wish to preserve rigidity



Jordán – R. – Szabadka, 2007

A graph can be realized as a rigid *d*-dimensional r-tensegrity framework

if and only if

it can be realized as a rigid *d*-dimensional rod framework and none of its edges are critical.

Corollary (Laman – type):

A graph G is minimal generic rigid in the plane as an r-tensegrity framework if and only if

$e = 2n - 2$ and the relation $e' \leq 2n' - 3$ holds for every proper subgraph G' of G.

Corollary (Laman – type):

A graph G is minimal generic rigid in the plane as an r-tensegrity framework if and only if

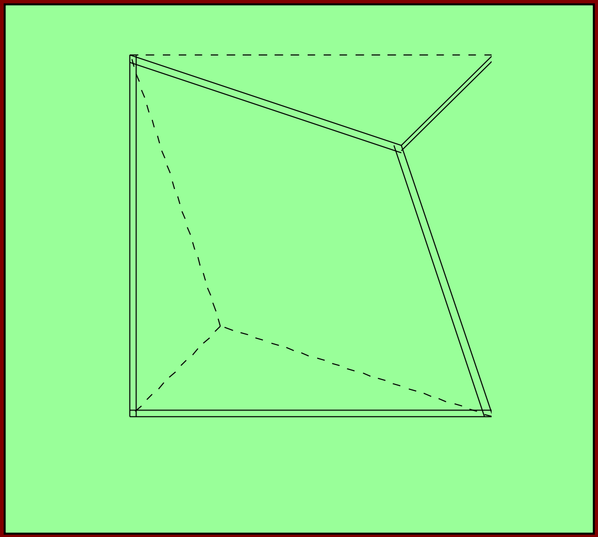
$e = 2n - 2$ and the relation $e' \leq 2n' - 3$ holds for every proper subgraph G' of G.

Corollary (Lovász-Yemini – type):

A graph is minimal generic rigid in the plane as an r-tensegrity framework if and only if it is the union of two edge-disjoint trees and remains so if any one of its edges is moved to any other position.

- A graph is generic rigid in the 1-dimensional space as an r-tensegrity framework if and only if it is 2-edge-connected.

- For the generic rigidity in the plane as an r-tensegrity framework, a graph must be 2-vertex-connected and 3-edge-connected. Neither 3-vertex-connectivity nor 4-edge-connectivity is necessary.



Happy Birthday, Walter

Thank you for your attention

recski@cs.bme.hu