A microscopic approach to Souslin trees constructions

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This is joint work with Ari M. Brodsky, and still *in progress*..

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e.g., $\mathrm{succ}_3(\omega_1 \setminus \omega) = \{\omega + 1, \omega + 2, \omega + 3\}$.

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Definition

- A κ-tree is a tree of height κ whose levels are of size $< \kappa$;
- A κ-Aronszajn tree is a κ-tree having no branches of size κ;
- A κ-Souslin tree is a κ-Aronszajn tree having no antichains of size κ.

The role of κ

Aronszajn and Souslin trees are useful objects that give rise to rich counterexamples in mathematics.

The literature concerning these trees splits roughly into two:

- Papers that deal with the construction of Aronszajn/Souslin trees with some additional features.
- Papers that deal with the construction of the trees from weaker and weaker hypotheses, or consistency results concerning non-existence.

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We shall now dedicate a few minutes to review some known results, highlighting that the behavior of κ-Aronszajn and κ-Souslin trees depends heavily on the identity of κ.

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There exists no $\aleph_0$*-Aronszajn tree.*

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Theorem (Specker, 1949. $\lambda = \omega$ is due to Aronszajn, 1935)

If λ is regular and $\lambda^{<\lambda} = \lambda$, then there exists a λ^+-Aronszajn tree.

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Theorem (Erdős-Taski, 1943)

If κ is weakly compact, then there exists no κ-Aronszajn tree.

λ^+-Souslin trees

Definition (Jensen, 1972)

For $S \subseteq \kappa$, $\diamondsuit(S)$ asserts the existence of a sequence $\langle A_\alpha \mid \alpha \in S \rangle$ such that $\{\alpha \in S \mid A \cap \alpha = A_\alpha\}$ is stationary for all $A \subseteq \kappa$.

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If $\lambda^{<\lambda} = \lambda$ and $\diamondsuit(E^{\lambda^+}_\lambda)$ holds, then there exists a λ-complete λ^+-Souslin tree.

This gives a method to construct Souslin tree at the level of successor of regulars. How to handle successor of singulars?

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$\square_\lambda(S)$ asserts the existence of a sequence $\langle C_\delta \mid \delta < \lambda^+ \rangle$ such that for all limit $\delta < \lambda^+$:

- C_δ is a club in δ of order-type $\leq \lambda$;
- if $\beta \in \mathrm{acc}(C_\delta)$, then $\beta \notin S$ and $C_\delta \cap \beta = C_\beta$.

Write $\square_\lambda$ for $\square_\lambda(\emptyset)$.

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Theorem (Jensen, 1972)

If there exists $S \subseteq \lambda^+$ for which $\square_\lambda(S) + \diamondsuit(S)$ holds, then there exists a λ^+-Souslin tree.

Special and specializable λ^+-trees

Definition

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- A λ^+-Souslin tree is non-special.

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Remark

Aronszajn's and Specker's constructions from $\lambda^{<\lambda} = \lambda$ may be steered to yield a special λ^+-tree.

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Theorem (Baumgartner-Mailtz-Reinhardt, 1970)

An $\aleph_1$-tree is Aronszajn iff it is specializable.

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Theorem (Baumgartner-Mailtz-Reinhardt, 1970)
An $\aleph_1$-tree is Aronszajn iff it is specializable.

Theorem (implicit in David, 1990)
If $V = L$, then for every regular λ, the canonical λ-complete λ^+-Souslin tree constructed using fine structure, is specializable.

Non-specializable λ^+-Souslin trees

Theorem (Baumgartner, 1970's, building on Laver)

$\square_{\aleph_1}$ *entails a non-specializable* $\aleph_2$*-Aronszajn tree.*

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$GCH + \square_{\aleph_1}$ *entails a non-specializable* $\aleph_2$*-Souslin tree.*

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If λ is a singular cardinal of countable cofinality, $\square_\lambda + \mathrm{CH}_\lambda$ and $\mu^{\aleph_1} < \lambda$ for all $\mu < \lambda$, then there exists a non-specializable λ^+-Souslin tree.

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Theorem (Cummings, 1997)

If λ is a singular cardinal of uncountable cofinality, $\square_\lambda + \mathrm{CH}_\lambda$ and $\mu^{\aleph_0} < \lambda$ for all $\mu < \lambda$, then there exists a non-specializable λ^+-Souslin tree.

To sum up

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Question

Do one really have to come up with such a long list of variations each time that a fundamental construction is discovered? Isn't there any automatic translation between the different cardinals?

An idea

Find a proxy!

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2. Prove that this operational principle is a consequence of the "usual" hypotheses.

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2. Prove that this operational principle is a consequence of the "usual" hypotheses. This part is done only once, and then will be utilized each time that a new construction is discovered.

The proxy principle

Goal

The proxy principle will allow to translate constructions from one cardinal to another, to calibrate the hypotheses needed to carry a construction, and will capture all known $\diamondsuit$-based constructions.

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$P(\kappa,\mu,\mathcal{R},\theta,\mathcal{S},\nu,\sigma,\varpi)$ asserts that $\diamondsuit(\kappa)$ holds, and so is the corresponding $P^-(\kappa,\mu,\mathcal{R},\theta,\mathcal{S},\nu,\sigma,\varpi)$.

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- if $C \in \mathcal{C}_\delta$ and $\beta \in \mathrm{acc}(C)$, then $\exists D \in \mathcal{C}_\beta$ with $(D, C) \in \mathcal{R}$;

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$\sqsubseteq$, where $D \sqsubseteq C$ iff $\exists\beta$ such that $D = C \cap \beta$.

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$\sqsubseteq_\chi$, where $D \sqsubseteq_\chi C$ iff $D \sqsubseteq C$ or $\mathrm{otp}(C) < \chi$.

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$\sqsubseteq^*$, where $D \sqsubseteq^* C$ iff $\exists\alpha < \sup(D)$ with $D \setminus \alpha \sqsubseteq C \setminus \alpha$.

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${}_{\chi}\sqsubseteq^*$, where $D\,{}_{\chi}\sqsubseteq^* C$ iff $\mathrm{cf}(\sup(D)) < \chi$ or $D \sqsubseteq^* C$.

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- for every sequence $\langle A_i \mid i < \theta \rangle$ of cofinal subsets of κ, and every $S \in \mathcal{S}$, there exists $\delta \in S$ with $|\mathcal{C}_\delta| < \nu$ such that:

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Recall

$\mathrm{succ}_\sigma(C) = \{\alpha \in C \mid \mathrm{otp}(C \cap \alpha) = j + 1 \text{ for some } j < \sigma\}$.

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- if $C \in \mathcal{C}_\delta$ and $\beta \in \mathrm{acc}(C)$, then $\exists D \in \mathcal{C}_\beta$ with $(D, C) \in \mathcal{R}$;
- for every sequence $\langle A_i \mid i < \theta \rangle$ of cofinal subsets of κ, and every $S \in \mathcal{S}$, there exists $\delta \in S$ with $|\mathcal{C}_\delta| < \nu$ such that:
  - $\forall i < \min\{\delta, \theta\} \forall C \in \mathcal{C}_\delta \sup\{\beta \in C \mid \mathrm{succ}_\sigma(C \setminus \beta) \subseteq A_i\} = \delta$;

The proxy principle

Recall

$\mathrm{succ}_{\varpi}(C) = \{\alpha \in C \mid \mathrm{otp}(C \cap \alpha) = j + 1 \text{ for some } j < \varpi\}$.

Definition

$P^{-}(\kappa, \mu, \mathcal{R}, \theta, \mathcal{S}, \nu, \sigma, \varpi)$ asserts the existence of $\langle \mathcal{C}_\delta \mid \delta < \kappa \rangle$ s.t.:

- for every limit $\delta < \kappa$, $\mathcal{C}_\delta$ is a collection of club subsets of δ;
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Default values

Don't worry, we have some default values!
Whenever omitted, let $\theta = 1, \mathcal{S} = \{\kappa\}, \nu = 2, \sigma = 1, \varpi = 0$.

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Special case

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- if $C \in \mathcal{C}_\delta$ and $\beta \in \operatorname{acc}(C)$, then $\exists D \in \mathcal{C}_\beta$ with $(D, C) \in \mathcal{R}$;
- for every cofinal $A \subseteq \kappa$, there exists $\delta < \kappa$ with $\mathcal{C}_\delta = \{C_\delta\}$, such that $\sup(\operatorname{nacc}(C_\delta) \cap A) = \delta$.

A Souslin tree from the weakest principle

Let κ denote a regular uncountable cardinal.

Proposition

$P(\kappa, \kappa, \sqsubseteq^*, 1, \{\kappa\}, \kappa)$ entails a κ-Souslin tree.

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Sanity check #1

Let λ denote an uncountable cardinal.

Theorem (Jensen, 1972)

If $\lambda^{<\lambda} = \lambda$ and $\diamondsuit(E^{\lambda^+}_\lambda)$ holds, then there exists a λ-complete λ^+-Souslin tree.

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$\diamondsuit(E^{\lambda^+}_\lambda)$ *entails* $P(\lambda^+, 2, {}_\lambda\sqsubseteq, \{E^{\lambda^+}_\lambda\})$.

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Sanity check #2

Let λ denote an uncountable cardinal.

Theorem (Jensen, 1972)

If there exists $S \subseteq \lambda^+$ for which $\square_\lambda(S) + \diamondsuit(S)$ holds, then there exists a λ^+-Souslin tree.

Sanity check #2

Let λ denote an uncountable cardinal.

Theorem (Jensen, 1972)

If there exists $S \subseteq \lambda^+$ for which $\square_\lambda(S) + \diamondsuit(S)$ holds, then there exists a λ^+-Souslin tree.

Theorem

$\square_\lambda + \mathrm{CH}_\lambda$ *entails* $P(\lambda^+, 2, \sqsubseteq, \{E^{\lambda^+}_{\geq\theta} \mid \theta < \lambda\})$.

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$\square_\lambda + \mathrm{CH}_\lambda$ *entails* $P(\lambda^+, 2, \sqsubseteq, \{E^{\lambda^+}_{\geq\theta} \mid \theta < \lambda\})$.

Corollary

If $\square_\lambda + \mathrm{CH}_\lambda$ holds, then for every $\chi < \lambda$ with $\lambda^{<\chi} = \lambda$, there exists a χ-complete λ^+-Souslin tree.

Sanity check #3

Let λ denote an uncountable cardinal.

Theorem (Gregory, 1976)

If $\lambda^{<\lambda} = \lambda, 2^\lambda = \lambda^+$ and exists a nonreflecting stationary subset of $E^{\lambda^+}_{<\lambda}$, then there exists a λ^+-Souslin tree.

Sanity check #3

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If $2^{\lambda} = \lambda^{+}$ and there exists a nonreflecting stationary subset of $E^{\lambda^+}_{<\lambda}$, then $P(\lambda^+, 2, {}_{\lambda}\sqsubseteq^*, \{E^{\lambda^+}_{\lambda}\})$ holds.

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Corollary (Kojman-Shelah, 1993)

If $\lambda^{<\lambda} = \lambda, 2^\lambda = \lambda^+$ and there exists a nonreflecting stationary subset of $E^{\lambda^+}_{<\lambda}$, then there exists a λ-complete λ^+-Souslin tree.

Sanity check #4

Let λ denote an uncountable cardinal.

Theorem (Shelah, 1984)

If $2^{\aleph_0} = \aleph_1$, $NS_{\aleph_1}$ is saturated, then there exists an $\aleph_2$-Souslin tree.

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Corollary

If $2^{\aleph_0} = \aleph_1$, $NS_{\aleph_1}$ is saturated, then there exists an $\aleph_1$-complete $\aleph_2$-Souslin tree.

And so on..

And so on..

Okay, so you seem to found a way to redirect all $\diamondsuit$-based constructions of Souslin trees through a single construction. You haven't yet shown me anything new!

κ-trees which cohere modulo finite

Definition

A subtree T of ${}^{<\kappa}\kappa$ is said to be coherent if for all $\delta < \kappa$:

- if $x, y \in T_\delta$, then $\{\alpha < \delta \mid x(\alpha) \neq y(\alpha)\}$ is finite;
- if $x, y \in {}^{\delta}\kappa$, and $\{\alpha < \delta \mid x(\alpha) \neq y(\alpha)\}$ is finite, then $x \in T_\delta$ iff $y \in T_\delta$.

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Theorem (Jensen, 1970's)

$\diamondsuit(\aleph_1)$ *entails a coherent* $\aleph_1$*-Souslin tree.*

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In my talk at "Young Set Theory 2011" workshop, I asked about the consistency of a coherent λ^+-Souslin tree for λ singular.

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Theorem
$P(\kappa, 2, \sqsubseteq, \kappa)$ *entails a coherent* κ*-Souslin tree.*

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Corollary

If $V = L$*, then any regular uncountable* κ *is not weakly compact iff there exists a coherent* κ*-Souslin tree.*

A concept of "being productive" for Souslin trees

Definition

A κ-Souslin tree T is free, if for every nonzero $n < \omega$ and any sequence of distinct nodes $\langle t_i \mid i < n \rangle$ from a fixed level $\delta < \kappa$, the product tree of the upper cones $\bigotimes_{i<n} {t_i}^{\uparrow}$ is again κ-Souslin.

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Jensen construct the levels of the tree by recursion, where the nodes of limit level α are obtained by forcing with finite conditions over some countable elementary submodel that knows about the diamond sequence and the tree constructed so far.

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Genericity entails freeness.

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Observation

$\mathsf{CH} + \diamondsuit(E^{\aleph_2}_{\aleph_1})$ entails a free $\aleph_2$-Souslin tree.

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$\mathsf{CH} + \diamondsuit(E^{\aleph_2}_{\aleph_1})$ entails a free $\aleph_2$-Souslin tree.

Construct a $\aleph_1$-complete tree by recursion, where the nodes of level α of uncountable cofinality are obtained by forcing with countable conditions over some $\aleph_1$-sized elementary submodel that knows about anything relevant.

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The $\aleph_1$-completeness of the tree and the countable conditions are necessary for the existence of a generic over the $\aleph_1$-sized model.

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Question

How about free λ^+-Souslin tree for λ singular?

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Freeness requires that the generic meet λ many dense sets, but the tree cannot be λ-complete, and there cannot be a generic for the relevant poset over a model of size λ.

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Question

How about free λ^+-Souslin tree for λ singular?
Freeness requires that the generic meet λ many dense sets, but the tree cannot be λ-complete, and there cannot be a generic for the relevant poset over a model of size λ. But, there is another way:

Theorem

$P(\kappa, \mu, \sqsubseteq, \kappa)$ entails a μ-slim, free κ-Souslin tree.

A concept of “being productive” for Souslin trees

Theorem

$P(\kappa, \mu, \sqsubseteq, \kappa)$ entails a μ-slim, free κ-Souslin tree.

Corollary

If $\square_\lambda + \mathsf{CH}_\lambda$ holds for λ singular, then there exists a free λ^+-Souslin tree.

A concept of “being productive” for Souslin trees

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If $\square_\lambda + \mathsf{CH}_\lambda$ holds for λ singular, then there exists a free λ^+-Souslin tree.

Corollary

If $V = L$, then any regular uncountable κ is not weakly compact iff there exists a free κ-Souslin tree.

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A κ-Souslin tree T is χ-free, if for every nonzero $\nu < \chi$ and any sequence of distinct nodes $\langle t_i \mid i < \nu \rangle$ from a fixed level $\delta < \kappa$, the product tree of the upper cones $\bigotimes_{i<\nu} {t_i}^{\uparrow}$ is again κ-Souslin.

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From GCH-type assumption, we can also construct χ-free trees for uncountable χ. For instance:

Corollary

If $\square_\lambda + \mathsf{CH}_\lambda$ holds for λ singular, then there exists a $\log_\lambda(\lambda^+)$-free λ^+-Souslin tree.

Calibrating

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So, one would expect that the fact that free is weaker than coherent be reflected in the hypothesis needed to construct such.

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If $V = L$, then for every regular λ, there exists a λ^+-Souslin tree which is specializable.

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If $\lambda^{<\lambda} = \lambda$, $P(\lambda^+, 2, \sqsubseteq, \lambda^+)$ entails a specializable λ^+-Souslin tree.

Specializable Souslin trees

Theorem

If $\lambda^{<\lambda} = \lambda$, $P(\lambda^+, \lambda^+, {}_\lambda\sqsubseteq^*, 1, \{E^{\lambda^+}_\lambda\}, \lambda^+, 1, 1)$ *entails a* λ*-complete, specializable* λ^+*-Souslin tree.*

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Let $\chi < \lambda$ denote infinite cardinals.

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A model of "all Aronszajn trees are nonspecial"

It is consistent that κ is supercompact, $\lambda = \kappa^{+\omega}$, and there exists a non-Specializable λ^+-Souslin tree.

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Some more

Generalizing Gregory's theorem to singular cardinals

Recall (Gregory, 1976)

If $\lambda^{<\lambda} = \lambda, \mathsf{CH}_\lambda$ and there exists a nonreflecting stationary subset of $E^{\lambda^+}_{<\lambda}$, then there exists a λ^+-Souslin tree.

Theorem

If $2^{<\lambda} = \lambda, \mathsf{CH}_\lambda + \square^*_\lambda$ and exists a nonreflecting stationary subset of $E^{\lambda^+}_{\neq \mathrm{cf}(\lambda)}$, then $P(\lambda^+, \lambda^+, \sqsubseteq)$ holds.

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Theorem (Cummings-Foreman-Magidor, 2001)

After Prikry forcing over a supercompact cardinal λ, $\square^*_\lambda$ holds, yet, any stationary subset of $E^{\lambda^+}_{\neq \mathrm{cf}(\lambda)}$ reflects.

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After Prikry forcing over a measurable cardinal λ satisfying CH_λ, $P(\lambda^+, \lambda^+, \sqsubseteq, \lambda^+)$ holds.

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After Prikry forcing over a measurable cardinal λ *satisfying* CH_λ*,* $P(\lambda^+, \lambda^+, \sqsubseteq, \lambda^+)$ *holds.*

Generalizing Gregory's theorem to singular cardinals

The derived trees

- $P(\lambda^+, \lambda^+, \sqsubseteq)$ entails a rigid λ^+-Souslin tree;
- $P(\lambda^+, \lambda^+, \sqsubseteq, \lambda^+)$ entails a free λ^+-Souslin tree;
- $P(\lambda^+, \lambda^+, \sqsubseteq, \lambda^+)$ entails an homogeneous λ^+-Souslin tree.

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Theorem

After Prikry forcing over a measurable cardinal λ satisfying CH_λ, $P(\lambda^+, \lambda^+, \sqsubseteq, \lambda^+)$ holds.

More results

Let $\lambda^{<\lambda} = \lambda$ denote a regular uncountable cardinal.

- If CH_λ, then adding a single λ-Cohen set entails $P(\lambda^+, \lambda^+, \sqsubseteq, \lambda^+, \{E^{\lambda^+}_\lambda\})$, and hence free/homogeneous/specializable trees.

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- If $\square_\lambda + \diamondsuit^*(\lambda^+)$, then there exists a (free) λ^+-Souslin tree T, whose ω-reduced power tree ${}^\omega T/\mathcal{U}$ is λ^+-Kurepa for any nonprincipal ultrafilter $\mathcal{U}$ over ω.

The microscopic approach

Diamond for H_κ

Recall that $P(\kappa, \cdots)$ asserts that $\diamondsuit(\kappa) + P^-(\kappa, \cdots)$ holds.

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Proposition

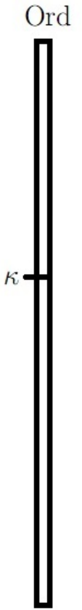
For κ regular uncountable, $\diamondsuit(\kappa)$ iff $\diamondsuit(H_\kappa)$.

Definition

$\diamondsuit(H_\kappa)$ asserts the existence of $\varphi_0 : \kappa \to H_\kappa$ and $\varphi_1 : \kappa \to H_\kappa$ as follows. For every $a \in H_\kappa$, $A \subseteq H_\kappa$, and $p \in H_{\kappa^{++}}$, there exists an elementary submodel $\mathcal{M} \prec H_{\kappa^{++}}$ such that:

- $p \in \mathcal{M}$;
- $\mathcal{M} \cap \kappa \in \kappa$;
- $\varphi_0(\mathcal{M} \cap \kappa) = a$;
- $\varphi_1(\mathcal{M} \cap \kappa) = \mathcal{M} \cap A$.

Diamond for H_κ



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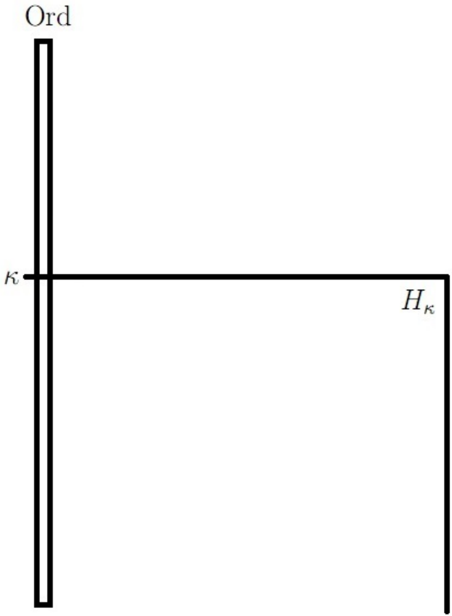

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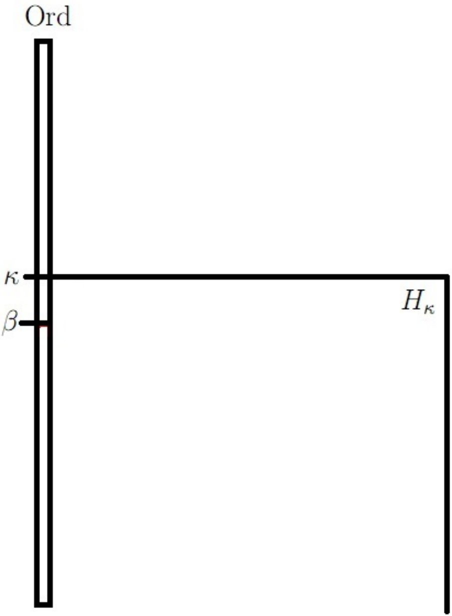

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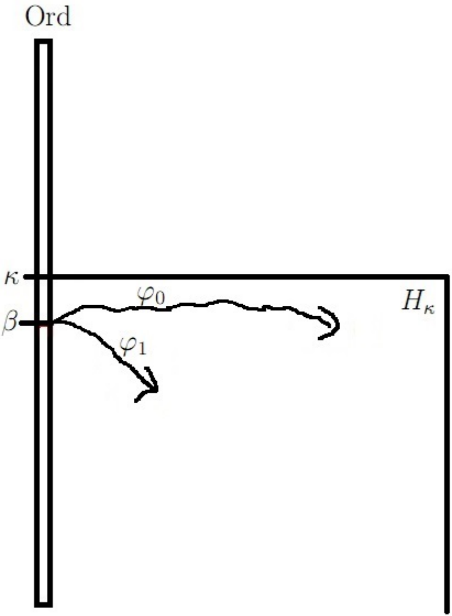

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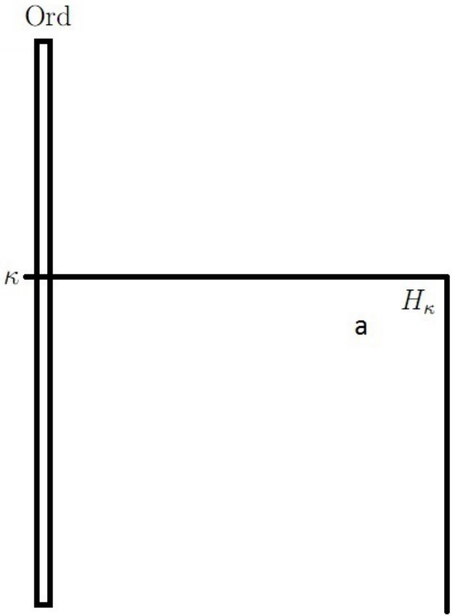

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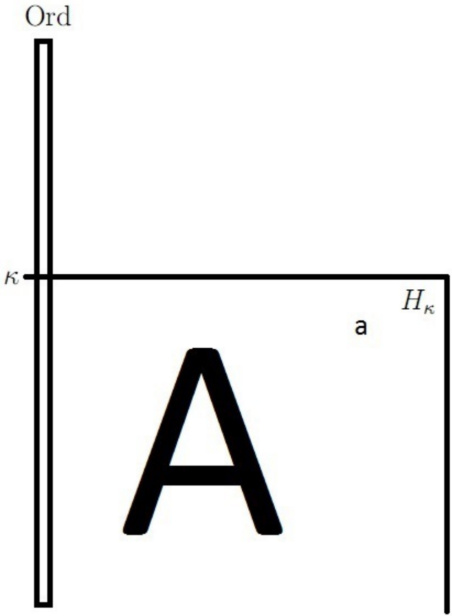

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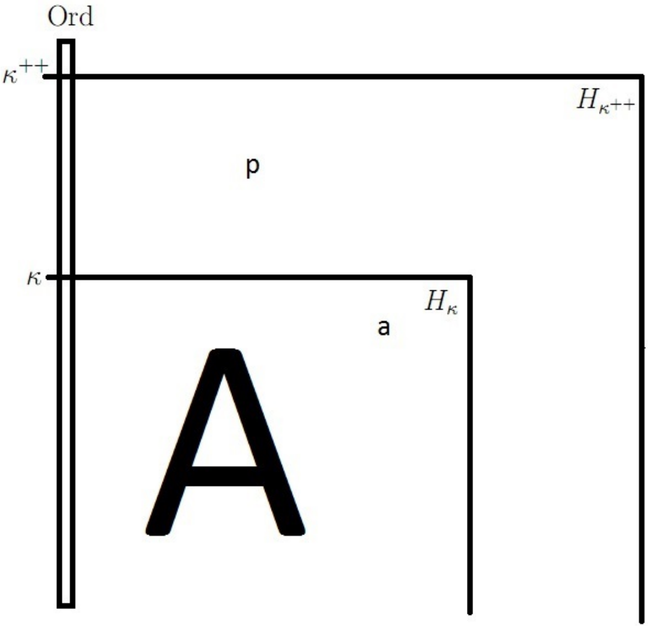

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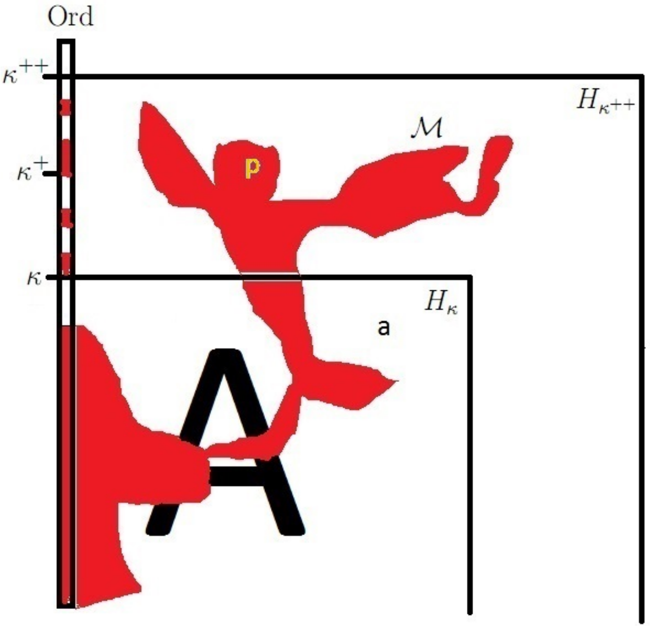

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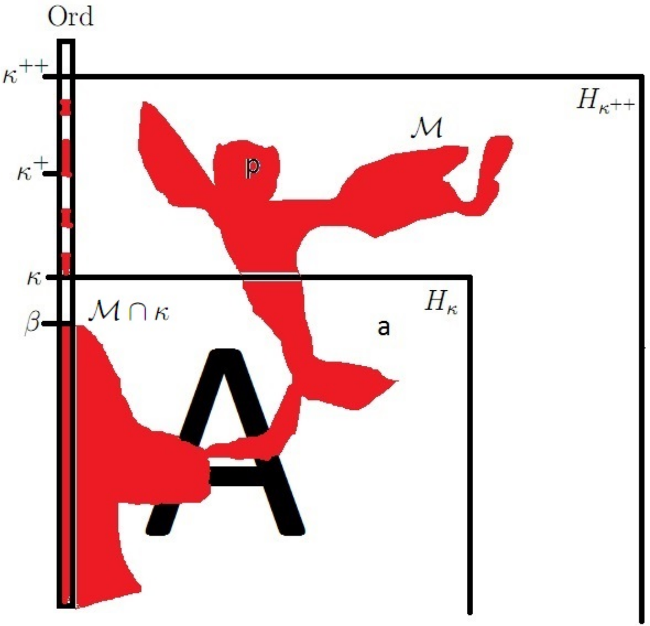

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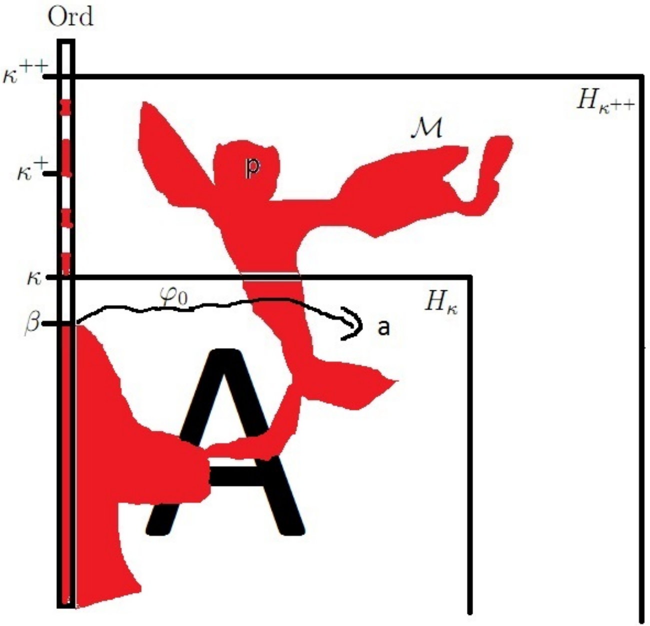

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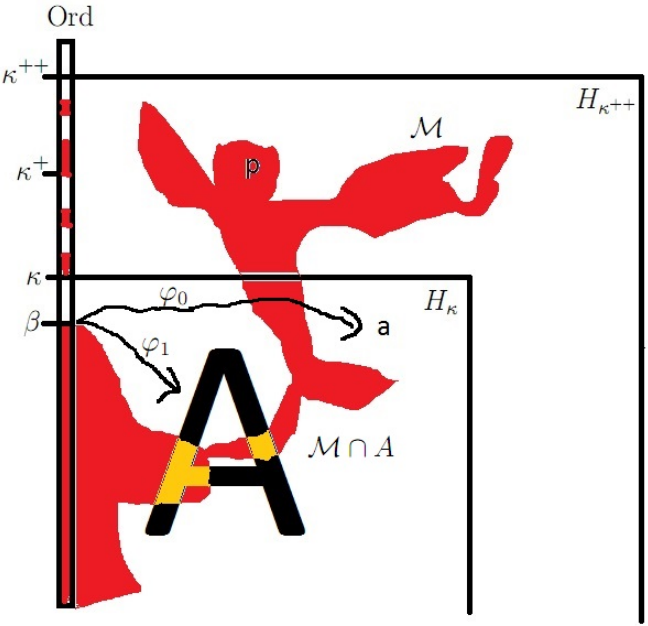

A construction á la microscopic approach

```
#include <NormalTree.h>
#include <SealAntichain.h>
//#include <Specialize.h>
#include <SealAutomorphism.h>
//#include <SealProductTree.h>
```