Applications of Ramsey theory in topological dynamics

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Forcing and its Applications
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Outline

(KPT) Topological dynamics and Ramsey theory

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($\mathbb{G}$) Gurarij space

- group of linear isometries
- approximate Ramsey propety for finite dimensional normed spaces

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- group of affine homeomorphisms
- approximate Ramsey propety for finite dimensional simplexes

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(L) Lelek fan

- group of homeomorphism
- exact Ramsey propety for sequences in FIN_k

Topological dynamics

G-flow

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G is extremely amenable $\longleftrightarrow$ its universal minimal flow is a singleton ($\longleftrightarrow$ every G-flow has a fixed point).

Structural Ramsey property

Theorem (Ramsey)

For every $k \leq m$ *and* $r \geq 2$, *there exists* n *such that for every colouring of k-element subsets of* n *with* r*-many colours there is a subset* X *of* n *of size* m *such that all k-element subsets of* X *have the same colour.*

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for every colouring of copies of A in C by r colours, there is a copy B' of B in C, such that all copies of A in B' have the same colour.

Ramsey classes and extremely amenable groups

Ramsey classes

- finite linear orders (Ramsey)
- finite linearly ordered graphs (Nešetřil and Rödl)
- finite linearly ordered metric spaces (Nešetřil)
- finite Boolean algebras (Graham and Rothschild)

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Extremely amenable groups

- $\mathrm{Aut}(\mathbb{Q},<)$ (Pestov)
- $\mathrm{Aut}(\mathcal{OR})$ – $\mathcal{OR}$ the random ordered graph (Kechris, Pestov & Todorčević)
- $\mathrm{Iso}(\mathbb{U},d)$ (Pestov)
- $\mathrm{Homeo}(C,\mathcal{C})$ – $(C,\mathcal{C})$ the Cantor space with a generic maximal chain of closed subsets (KPT; Glasner & Weiss)

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Theorem (KPT; NvT)

$\mathrm{Aut}(\mathcal{A})$ *is extremely amenable* $\longleftrightarrow$ *finitely-generated substructures of* $\mathcal{A}$ *satisfy the Ramsey property and are rigid.*

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Structure $\mathcal{A}$	$M(\mathrm{Aut}(\mathcal{A}))$	authors
$\mathbb{N}$	linear orders on $\mathbb{N}$	Glasner and Weiss
random graph $\mathcal{R}$	linear orders on $\mathcal{R}$	KPT
Cantor space C	maximal chains of closed subsets of C	Glasner and Weiss

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Structure...	...homogeneous w.r.t.
$\mathbb{N}$, $\mathcal{R}$	embeddings
Gurarij space	linear isometric embeddings
Lelek fan	epimorphisms
Poulsen simplex	affine epimorphisms

Gurarij space $\mathbb{G}$

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$$\|i - f \restriction E\| < \varepsilon$$

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Conditions (1),(2),(3) uniquely define $\mathbb{G}$ up to a linear isometry.

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KUBIŚ-SOLECKI; HENSON

Simple proof - metric Fraïssé theory.

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Katětov construction

Approximate Ramsey property for finite-dimensional normed spaces

E, F - finite dimensional spaces
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$$\mathrm{Emb}_\theta(E, F) = \{T : E \longrightarrow F : T \text{ embedding } \& \, \|\mathrm{T}\| \left\|\mathrm{T}^{-1}\right\| \leq \theta\}$$

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Theorem (B-LA-M)

r - number of colours, $\varepsilon > 0$ $\longrightarrow$ $\exists H$ f.d. with $\mathrm{Emb}(F,H) \neq \emptyset$ *such that for every*

$$c : \mathrm{Emb}_\theta(E,H) \longrightarrow \{0, 1, \ldots, r-1\}$$

$\exists i \in \mathrm{Emb}_\theta(F,H)$ *and $\alpha < r$ such that*

$$i \circ \mathrm{Emb}_\theta(E,F) \subset (c^{-1}(\alpha))_{\theta - 1 + \varepsilon}$$

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Theorem (Pestov)

TFAE

- *G is extremely amenable,*
- *every $f : G \longrightarrow \mathbb{R}$ bounded left-uniformly continuous is finite oscillation stable.*

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Theorem (B-LA-M)

$\mathrm{Iso}_l(\mathbb{G})$ *is extremely amenable.*

Urysohn space $\mathbb{U}$

Theorem

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Corollary (Pestov)

$\mathrm{Iso}(\mathbb{U})$ *is extremely amenable.*

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Properties (1),(2) and (3) uniquely determine P up to an affine homeomorphism.

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The set of extreme points of P is dense in P.

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FACT

$T : \{0,1\}^{\mathbb{Z}} \longrightarrow \{0,1\}^{\mathbb{Z}}$ the shift $\Rightarrow$ T-invariant probability measures form P

A projective characterization of P

$S_n :=$ positive part of the unit ball of l_1^n – finite-dimensional simplex with $n+1$ extreme points

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(U) $\forall n \ \exists \phi : P \longrightarrow S_n$ – continuous affine surjection

(APU) $\forall \varepsilon > 0 \ \forall n \ \forall \phi_1, \phi_2 : P \longrightarrow S_n \ \exists f \in AH(P)$ with $d(\phi_1, \phi_2 \circ f) < \varepsilon$

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Theorem (B-LA-M)

(U) + (APU) characterize P among non-trivial metrizable simplexes up to affine homeomorphism.

Approximate Ramsey property for P

$\mathrm{Epi}_0(S_n, S_m)$ - continuous affine surjections preserving 0

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$d \leq m$ and r natural numbers and $\varepsilon > 0$ given $\longrightarrow$ $\exists n$ such that for every colouring

$$c : \mathrm{Epi}_0(S_n, S_d) \longrightarrow \{0, 1, \dots, r\}$$

there is $\pi \in \mathrm{Epi}_0(S_n, S_m)$ and $\alpha < r$ such that

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p - extreme point of P
$AH_p(P) = \{f \in AH(P) : f(p) = p\}$

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Theorem (B-LA-M)

$AH_p(P)$ is extremely amenable.

Universal minimal flow of $AH(P)$

Theorem (B-LA-M)

$M(AH(P)) \cong \widehat{AH(P)/AH_p}(P) \cong P$

Hilbert cube $\mathcal{Q} = [-1, 1]^{\mathbb{N}}$

PROBLEM
What is the universal minimal flow of Homeo($\mathcal{Q}$)?

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Aut($\mathcal{Q}$) *is topologically isomorphic to* $\{-1, 1\}^{\mathbb{N}} \times S_\infty$.

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What is the universal minimal flow of $\mathrm{Homeo}(\mathcal{Q})$?

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Theorem (B-LA-M)

$M(\mathrm{Aut}(\mathcal{Q})) = \{-1,1\}^{\mathbb{N}} \times LO(\mathbb{N})$.

Lelek fan L

= unique non-trivial subcontinuum of the Cantor fan with a dense set of endpoints (Bula-Oversteegen, Charatonik)

continuum = connected compact metric Hausdorff space

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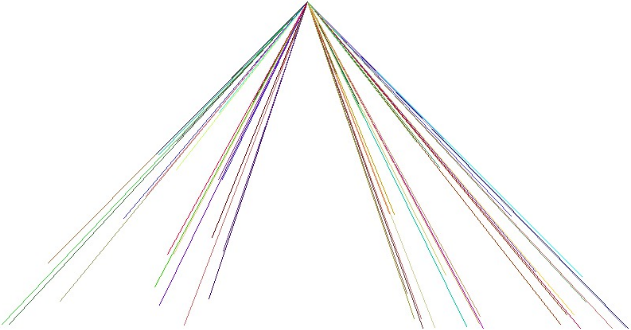

fan.jpg

Pre-Lelek fan

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$\mathcal{F} = \{\text{finite fans}\} + \text{surjective homomorphisms}$

Pre-Lelek fan

$(\mathbb{L}, R_s^{\mathbb{L}})$ - compact, 0-dim, $R_s^{\mathbb{L}} \subset \mathbb{L}^2$ closed with one or two element equivalence classes

$$\mathbb{L}/R_s^{\mathbb{L}} \cong L$$

$\mathcal{F} = \{\text{finite fans}\}$ + surjective homomorphisms

- (U) $T \in \mathcal{F} \rightsquigarrow \exists \phi : (\mathbb{L}, R^{\mathbb{L}}) \longrightarrow T$ - continuous surjective homomorphism
- (R) X finite, $f : \mathbb{L} \longrightarrow X$ continuous $\rightsquigarrow \exists T \in \mathcal{F}$, $\phi : \mathbb{L} \longrightarrow T$ and $g : T \longrightarrow X$ such that $f = g \circ \phi$
- (PU) $T \in \mathcal{F}$, $\phi_1, \phi_2 : \mathbb{L} \longrightarrow T \rightsquigarrow \exists g : \mathbb{L} \longrightarrow \mathbb{L}$ automorphism with $\phi_1 = \phi_2 \circ g$

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$$\begin{array}{ccc} h & \mapsto & h^* \\ \pi \circ h & = & h^* \circ \pi . \end{array}$$

Ramsey property for L

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For every $A, B \in \mathcal{F}_<$ there exists $C \in \mathcal{F}_<$ such that for every colouring

$$c : \{C \longrightarrow A\} \longrightarrow \{1, 2, \ldots, r\}$$

there exists $f : C \longrightarrow B$ such that $\{B \longrightarrow A\} \circ f$ is monochromatic.

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Theorem (B-K)

Let $\mathbb{L}_<$ be the limit of $\mathcal{F}_<$. Then $\mathrm{Aut}(\mathbb{L}_<)$ is extremely amenable.

Universal minimal flow of Homeo(L)

Theorem (B-K)

- $M(\mathrm{Aut}(\mathbb{L})) \cong \mathrm{Aut}(\mathbb{L}\widehat{)/\mathrm{Aut}}(\mathbb{L}_<)$

Universal minimal flow of $\mathrm{Homeo}(L)$

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- $M(\mathrm{Aut}(\mathbb{L})) \cong \widehat{\mathrm{Aut}(\mathbb{L})/\mathrm{Aut}(\mathbb{L}_<)}$
- $M(\mathrm{Homeo}(L)) \cong \widehat{\mathrm{Homeo}(L)/\mathrm{Homeo}(L_<)}$

A question

Is there a non-trivial simplex with extremely amenable group of affine homeomorphisms?

THANK YOU

HAPPY FOOLS' DAY!