Blass–Shelah Forcing Revisited

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A part of a larger project

F1420 by Blass, Mildenberger, Shelah

A Simple $P_{\aleph_1}$-Point and a Simple $P_{\aleph_2}$-Point

Preserving an ultrafilter

Definition

Let $\mathbb{P}$ be a notion of forcing. Let $\mathcal{U}$ be an ultrafilter over I. We say $\mathbb{P}$ preserves $\mathcal{U}$ if

$$\Vdash_{\mathbb{P}} \text{``}(\forall X \subseteq I)(\exists Y \in \mathcal{U})(Y \subseteq X \vee Y \subseteq I \setminus X)\text{''}.$$

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Remark

Preservation of P-points. Some development ...

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Remark

Preservation of P-points. Some development ...

Theorem, Shelah 1994

Any forcing adding a real destroys an ultrafilter over ω.

Countable support iteration

Theorem, Blass, Shelah

Let $\mathcal{E}$ be a P-point. Let $\langle \mathbb{P}_\alpha, \mathbb{Q}_\beta : \beta < \gamma, \alpha \leq \gamma \rangle$ be a countable support iteration such that each $\mathbb{P}_\alpha$ is proper. If each $\mathbb{P}_\alpha$, $\alpha < \gamma$, preserves $\mathcal{E}$, then also $\mathbb{P}_\gamma$ preserves $\mathcal{E}$.

A request to destroy a P-point

Suppose:

(1) $\langle \mathbb{P}_\alpha, \mathbb{Q}_\beta : \beta < \omega_2, \alpha \leq \omega_2 \rangle$ is a countable support iteration of proper iterands, and

(2) in $V^{\mathbb{P}_{\omega_2}}$ there is a simple $P_{\aleph_2}$-point $\mathcal{U}$.

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Then there is an ω_1-club of stages at which $\mathcal{U} \cap V^{\mathbb{P}_\alpha}$ is a P-point.

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We take such a stage α, and consider the least $\beta > \alpha$ such that there is $X \in \mathcal{U} \setminus V^{\mathbb{P}_\alpha}$. $\mathcal{U} \cap V^{\mathbb{P}_{<\beta}}$ is destroyed by $\mathbb{P}_\beta$ (and complemented) later in the iteration.

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So a forcing destroying some P-points and keeping others is requested.

The Rudin–Blass order

Let $\mathcal{H}, \mathcal{H}' \subseteq [\omega]^\omega$ be closed under almost supersets. We write $\mathcal{H} \leq_{\mathrm{RB}} \mathcal{H}'$ and say $\mathcal{H}$ is Rudin-Blass-below $\mathcal{H}'$ iff there is a finite-to-one f such that $f(\mathcal{H}) \subseteq f(\mathcal{H}')$. Here $f(\mathcal{H}) = \{X : f^{-1}[X] \in \mathcal{H}\}$.

Subforcings of Matet forcing

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Theorem, Eisworth, 2002

Let $\mathcal{U}$ be a stable ordered-union ultrafilter over the set of blocks. The Matet forcing $\mathbb{M}(\mathcal{U})$ preserves $\mathcal{E}$ iff $\Phi(\mathcal{U}) \not\leq_{\mathrm{RB}} \mathcal{E}$.

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However, Matet forcing not add an unsplit real.

Definition

A real $X \in V[G] \setminus V$ is called an unsplit real if every $Y \in [\omega]^\omega \cap V$ we have

$$Y \subseteq^* X \vee Y \subseteq^* \omega \setminus X.$$

Sets of possibilities

So we work with suborders of Blass-Shelah forcing, for example the one of [BsSh:242]. For today we take a forgetful version.

Definition

(1) A finite ω is called a block. A set of possibilities is a subset of the power set of a block that contains the empty set. We denote by $\mathcal{P}$ the set of all sets of possibilities. Typically we use variables s, t for blocks and a, b, c for sets of possibilities.

(2) Let a be a set of possibilities and $Y \subseteq \omega$. We let $a \restriction Y = \{s \cap Y \ : \ s \in a\}$.

A cut and choose game and a norm

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If $\mathrm{nor}(a) \geq 1$, then a contains a non-empty set.

Conditions in the pure part of a forcing order

Definition

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(4) Let $\bar{a} \in (\mathcal{P})^\omega$. We write $a \in \bar{a}$ for $a \in \{a_n : n \in \omega\}$.

The $\leq$-relation on the pure part

Definition

For sequences $\bar{a}$, $\bar{b} \in (\mathcal{P})^\omega$ we write $\bar{b} \leq \bar{a}$ or "$\bar{b}$ stronger than $\bar{a}$" iff there is a strictly increasing function $g\colon \omega \to \omega$ such that for any n,

$$b_n \subseteq a_{g(n)} \circ \cdots \circ a_{g(n+1)-1},$$

and $a \circ b = \{s \cup t \,:\, s \in a, t \in b\}$.

Projection to subsets of ω

The next two notions connect generated sets in $(\mathcal{P})^\omega$ with semifilters over ω.

Definition

(1) For $\bar{a} \in (\mathcal{P})^\omega$ we let $\mathrm{set}_2(\bar{a}) = \bigcup\{\bigcup a_n \,:\, n \in \omega\}$. We write 2 to distinguish it from notions that are used in Matet forcing.

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(2) Let $\mathcal{H} \subseteq (\mathcal{P})^\omega$. The projection of $\mathcal{H}$ into $[\omega]^\omega$ is $\Phi_2(\mathcal{H}) = \{\mathrm{set}_2(\bar{a}) : \bar{a} \in \mathcal{H}\}$.

An unsplit real

There is a connection to adding a real that is not split by any real in the ground model and to ultrafilters over ω:

Lemma

If $\bar{a} \in (\mathcal{P})^\omega$ and $X \subseteq \omega$ then there is $\bar{b} \leq \bar{a}$ such that $\mathrm{set}_2(\bar{b}) \subseteq X$ or $\mathrm{set}_2(\bar{b}) \subseteq (\omega \setminus X)$.

Diagonal lower bounds

Definition

Let $\bar{a} \in (\mathcal{P})^\omega$, $n \in \omega$. We write $(\bar{a} \text{ past } n)$ for $\langle a_i : i \in [k, \omega) \rangle$, where k is the minimal number such that $n \leq \min \bigcup a_k$.

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Definition

Let $\langle \bar{a}_n : n \in \omega \rangle$ be a $\leq$-descending sequence in $(\mathcal{P})^\omega$. A sequence $\bar{b} \in (\mathcal{P})^\omega$ is a diagonal lower bound of $(\bar{a}_n)_n$ iff

$$(\forall b \in \bar{b})((\bar{b} \operatorname{past} \max(b)) \leq \bar{a}_{\max(b)}).$$

Trees of finite sets

Definition

Let $\bar{a} \in (\mathcal{P})^\omega$, s a finite set, $s < \min(a_0)$.

$\mathrm{Lev}_{\leq k}(s, \bar{a}) = \bigcup \{\{s\} \circ a_{i_0} \circ \cdots \circ a_{i_{n-1}} \,:\, i_0 < \cdots < i_{n-1} \leq k\}$.

$T(s, \bar{a}) = \bigcup \{\mathrm{Lev}_{\leq k}(\bar{a}) \,:\, k \in \omega.\}$.

Large monochromatic trees

Theorem, Blass, Shelah

For any $C\colon [\omega]^{<\omega} \to 2$ *and any* $\bar{a} \in (\mathcal{P})^{\omega}$ *there is* $\bar{b} \leq \bar{a}$ *such that* $C \restriction T(\bar{b})$ *is constant.*

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Now we fix a P-point $\mathcal{E}$ and assume CH.

Suitable sets of pure parts of conditions

Definition

A set $\mathcal{H} \subseteq (\mathcal{P})^\omega$ is called a suitable set if the following hold:

(1) (Upwards Closure) $\mathcal{H} \subseteq (\mathcal{P})^\omega$, and $\bar{a} \in \mathcal{H}$ and $\bar{b} \geq^* \bar{a}$ implies $\bar{b} \in \mathcal{H}$,

(2) (Existence of Diagonal Lower Bounds) If $\langle \bar{a}_n : n \in [\omega]^{<\omega} \rangle$ is a $\leq$-descending sequence of elements of $\mathcal{H}$ then there is $\bar{b} \in \mathcal{H}$ such that $(\forall b \in \bar{b})((\bar{b} \text{ past } \max(b)) \leq \bar{a}_{\max(b)})$.

(3) (Fullness) For any $Y \subseteq \omega$, there is $\bar{a} \in \mathcal{H}$ such that $\mathrm{set}_2(\bar{a}) \subseteq Y$ or $\mathrm{set}_2(\bar{a}) \subseteq Y^c$.

(4) (Ramsey property: Monochromatic trees of possibilities) For any $C\colon [\omega]^{<\omega} \to 2$ and any $\bar{a} \in \mathcal{H}$ there is $\bar{b} \leq \bar{a} \in \mathcal{H}$ such that $C \restriction T(\bar{b})$ is constant.

(5) (Avoiding $\mathcal{E}$) We require $\Phi_2(\mathcal{H}) \not\leq_{\mathrm{RB}} \mathcal{E}$.

A full Blass–Shelah forcing

Definition

In the forcing order $\mathbb{BS}$, conditions are pairs $(s, \bar{a})$ such that $s \in \mathscr{P}_{<\omega}(\omega)$ and $\bar{a} \in (\mathcal{P})^\omega$ and $s < a_0$. The forcing order is $(t, \bar{b}) \leq (s, \bar{a})$ (recall the stronger condition is the smaller one) iff

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(2) there is $k \in \omega$ such that $t \setminus s \in a_0 \circ \cdots \circ a_{k-1}$ and

(3) $\bar{b} \leq \langle a_n : n \geq k \rangle$.

Subforcings of Blass–Shelah forcing

Definition

Given a suitable set $\mathcal{H}$ in $(\mathcal{P})^\omega$, the notion of forcing $\mathbb{BS}(\mathcal{H})$ consists of all pairs $(s, \bar{a})$ such that $\bar{a} \in \mathcal{H}$ and $s < \min(a_0)$. The order relation is as in $\mathbb{BS}$.

We name the generic reals:

Definition

Let G be $\mathbb{BS}(\mathcal{H})$-generic over V. We call

$$W = \bigcup \{s \,:\, \exists \bar{a}(s, \bar{a}) \in G\}$$

the $\mathbb{BS}(\mathcal{H})$-generic real.

Properties of the forcing

Remark

If $\Phi_2(\mathcal{H})$ *is an ultrafilter, then* $\mathbb{BS}(\mathcal{H})$ *destroys the ultrafilter* $\Phi_2(\mathcal{H})$*, since* W *diagonalises* $\Phi_2(\mathcal{H})$*. Moreover,* $\mathbb{BS}(\mathcal{H})$ *destroys any ultrafilter* $\mathcal{U}$ *such that* $\Phi_2(\mathcal{H}) \leq_{\mathrm{RB}} \mathcal{U}$*.*

The generic real is not split by any set in the ground model:

Lemma

Let $\mathcal{H}$ *be a suitable set in* $(\mathcal{P})^\omega$*. If* $X \subseteq \omega$*,* $X \in V$ *then after forcing with* $\mathbb{BS}(\mathcal{H})$ *we have* $W \subseteq^* X$ *or* $W \subseteq^* (\omega \setminus X)$*.*

Proposition

(Prop. 2.9 in BsSh:242) Let $\underset{\sim}{A}$ *be a* $\mathbb{BS}(\mathcal{H})$*-name for a subset of* ω*. Then every condition* $(s, \bar{a})$ *has a 0-extension* $(s, \bar{b})$ *with the following property. If* $\ell \in \omega$*, if* $n = n(\ell)$ *is the number such that* $b_\ell \subseteq n$*, if* $t \in b_0 \circ \cdots \circ b_{\ell-1}$*, and if* $i < n(\ell - 1)$*, then* $(t, \bar{b} \operatorname{past} n(\ell - 1))$ *decides whether* $i \in \underset{\sim}{A}$*.*

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Theorem

Let $\mathcal{H}$ be an suitable set, and let $\mathcal{E}$ be a P-point such that $\Phi_2(\mathcal{H}) \not\leq_{\mathrm{RB}} \mathcal{E}$. Then $\mathcal{E}$ continues to generate an ultrafilter after we force with $\mathbb{BS}(\mathcal{H})$.

Proof sketch

$T(s, \bar{b})$ monochromatic as in Proposition.

Let for $\ell \in \omega$, $n(\ell) = \max(\bigcup b_\ell)$.

$A(t) = \{i : (\exists \ell)(i < n(\ell) \wedge (t, (\bar{b} \operatorname{past} n(\ell))) \Vdash i \in \underset{\sim}{A})\}$.

Assume all $A(t) \in \mathcal{E}$, and $B \subseteq^* A(t)$, $B \in \mathcal{E}$.

Inductively define a sequence $\langle \zeta(n) : n \in \omega \rangle$ of natural numbers, starting with $\zeta(0) = 0$, and increasing so rapidly that, if $t \in \operatorname{Lev}_{\leq \zeta(k)}(s, \bar{b})$, then

(i) $B \setminus A(t) \subseteq \zeta(k+1)$, and

(ii) if $i \in A(t)$ and $i < n(\zeta(k))$, then
$(t, (\bar{b} \operatorname{past} \max(b_{\zeta(k)}))) \Vdash i \in A$.

Find $\bar{c} \in \mathcal{H}$, such that $\operatorname{set}_2(\bar{c})$ avoids B in a strong sense.

Then $(s, \bar{c}) \Vdash B \cap X_2 \subset \underset{\sim}{A}$.

Conjecture of an Induction Lemma

There is a countable support iteration of proper forcings $\langle \mathbb{P}_\alpha, \mathbb{Q}_\beta : \beta < \omega_2, \alpha \leq \omega_2 \rangle$ such that for each $\beta < \omega$ there is a $\mathbb{P}_\beta$-name $\mathcal{H}_\beta$ for suitable sets such that for any $\alpha \leq \omega_2$, the initial segment $\langle \mathbb{P}_\gamma, \mathbb{Q}_\beta, \mathcal{H}_\beta : \beta < \alpha, \gamma \leq \alpha \rangle$ fulfils:

(P1) For all $\gamma < \alpha$,

$$\Vdash_{\mathbb{P}_\gamma} \text{“}\underset{\sim}{\mathbb{Q}}_\gamma = \mathbb{BS}(\mathcal{H}_\gamma) \text{ for a suitable set } \mathcal{H}_\gamma \text{ adding } W_\gamma \ \wedge\ \Phi_2(\mathcal{H}_\gamma) \not\leq_{\mathrm{RB}} \mathcal{E}\text{”}.$$

(P2) $\mathbb{P}_\alpha$ is proper and

$$\mathbb{P}_\alpha \Vdash \text{“}\mathcal{E} \text{ generates an ultrafilter”}.$$

(P3) $\mathbb{P}_\alpha \Vdash \mathcal{H}_\alpha = \{\bar{a} \in (\mathcal{P})^\omega : (\forall \gamma < \alpha)(\exists \bar{b} \in (\mathcal{P})^\omega)(\bar{b} \leq \bar{a} \wedge \mathrm{set}_2(\bar{b}) \subseteq^* W_\gamma)\}$.

Revising the choice of $\mathcal{H}_\alpha$

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We name the generic maximal centred set:

Definition

We denote by $\mathcal{C}_{\mathcal{H}}$ the $(\mathcal{H}, \leq)$ generic filter.

Factorisation

$\mathbb{BS}(\mathcal{H}) = (\mathcal{H}, \leq) * \mathbb{BS}(\mathcal{C}_{\mathcal{H}})$

A forcing with centred sets

Induction Conjecture

There is a countable support iteration of proper forcings $\langle \mathbb{P}_\alpha, \mathbb{Q}_\beta : \beta < \omega_2, \alpha \leq \omega_2 \rangle$ such that for each $\beta < \omega$ there is a $\mathbb{P}_\beta$-name $\mathcal{H}_\beta$ for suitable sets such that for any $\alpha \leq \omega_2$, the initial segment $\langle \mathbb{P}_\gamma, \mathbb{Q}_\beta, \mathcal{C}_\beta : \beta < \alpha, \gamma \leq \alpha \rangle$ fulfils:

(P1') For all $\gamma < \alpha$,
$\Vdash_{\mathbb{P}_\gamma}$ "$\mathbb{Q}_\gamma = \mathbb{BS}(\mathcal{C}_\gamma)$ for a maximal centred suitable set $\mathcal{C}_\gamma$"..
Let W_γ denote the $\mathbb{BS}(\mathcal{C}_\gamma)$-generic real.

(P2) $\mathbb{P}_\alpha$ is proper and $\mathbb{P}_\alpha \Vdash$ "$\mathcal{E}$ generates an ultrafilter".

(P3') $\mathbb{P}_\alpha \Vdash (\forall \gamma < \alpha)\big(\mathcal{C}_\gamma \subseteq \mathcal{C}_\alpha \wedge (\forall \bar{a} \in \mathcal{C}_\alpha)(\exists \bar{b} \in \mathcal{C}_\alpha)(\bar{b} \leq \bar{a} \wedge \mathrm{set}_2(\bar{b}) \subseteq^* W_\gamma)\big)$

Good news about the limit steps of uncountable cofinality

For $\alpha < \omega_2$ of uncountable cofinality, $\mathcal{C}_\alpha = \bigcup\{\mathcal{C}_\beta : \beta < \alpha\}$ is a maximal centred suitable set.

On the condition $\mathrm{set}_2(\bar{b}) \subseteq^* W_\alpha$

Definition

For a sequence $\bar{a} \in (\mathcal{P})^\omega$ and $X \subseteq \omega$ is such that $C = \{n : \mathrm{nor}(\bar{a}_n \restriction X) \geq n - 1\}$ is infinite, then we we write $\bar{a} \restriction X$ for $\langle \bar{a}_n \restriction X : n \in C \setminus \min(C) \rangle$.

Lemma

Let $(s, \bar{a}) \in \mathbb{BS}(\mathcal{C})$ and let G be $\mathbb{BS}(\mathcal{C})$-generic and let W denote the $\mathbb{BS}(\mathcal{C})$-generic real. Then $V[G] \models \bar{a} \restriction W \in (\mathcal{P})^\omega$.

Towards covering models

We define tuple $\langle R_{n,\alpha} : n \in \omega\rangle$ of relations such that there is a suitable set as in (P3) is equivalent to

$$V^{\mathbb{P}_\alpha} \models (\forall f \in \mathrm{dom}(R_{0,\alpha}))(\exists \bar{g} \in \mathrm{range}(R_{0,\alpha}))(\bigvee_{n\in\omega} f R_{n,\alpha} g).$$

We let $\bar{R} = \langle R_{n,\alpha} : \alpha \leq \omega_2, n \in \omega\rangle$, α being the stage, n the size of the "mistake" in properties of the kind "for all but finitely many".

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We let $\bar{R} = \langle R_{n,\alpha} : \alpha \leq \omega_2, n \in \omega \rangle$, α being the stage, n the size of the "mistake" in properties of the kind "for all but finitely many". $R_{n,\alpha}$ will be a closed relation in $\mathrm{dom}(R_{n,\alpha}) \times \mathrm{range}(R_{n,\alpha})$ in the Baire space topology. The ordinal α indicates the stage.

Towards covering models

We define tuple $\langle R_{n,\alpha} : n \in \omega \rangle$ of relations such that there is a suitable set as in (P3) is equivalent to

$$V^{\mathbb{P}_\alpha} \models (\forall f \in \mathrm{dom}(R_{0,\alpha}))(\exists \bar{g} \in \mathrm{range}(R_{0,\alpha}))(\bigvee_{n \in \omega} f R_{n,\alpha} g).$$

We let $\bar{R} = \langle R_{n,\alpha} : \alpha \leq \omega_2, n \in \omega \rangle$, α being the stage, n the size of the "mistake" in properties of the kind "for all but finitely many". $R_{n,\alpha}$ will be a closed relation in $\mathrm{dom}(R_{n,\alpha}) \times \mathrm{range}(R_{n,\alpha})$ in the Baire space topology. The ordinal α indicates the stage.
The definition of the relation, and the proof that it is preserved, are carried on simultaneously by induction on $\alpha < \omega_2$.
Now suppose $\mathbb{P}_\alpha$, $\mathcal{C}_\beta$, $\beta < \alpha$ are defined and we work in $V^{\mathbb{P}_\alpha}$:

A relation

Definition

$f = (\langle \bar{a}_\ell : \ell \in \omega \rangle, X, C, h)$ is called a *task* iff

(1) $(\bar{a}_\ell)_\ell$ is a $\leq$-descending sequence in $(\mathcal{P})^\omega$,

(2) $C \colon [\omega]^\omega \to 2$,

(3) $h \colon \omega \to \omega$ a finite-to-one function.

Definition

Let f be a task. We say $\bar{g}$ answers the task, iff

(1) $\bar{g} \in (\mathcal{P})^\omega$.

(2) $\bar{g}$ is a diagonal lower bound of $(\bar{a}_\ell)_\ell$.

(3) $C \restriction T(\bar{g})$ is constant).

(4) $\exists E \in \mathcal{E} h[E] \cap h[\mathrm{set}_2(\bar{g})] = \emptyset$.

A relation, continued

Definition

Assume that $\langle \mathcal{H}_\gamma : \gamma < \alpha \rangle$ is an sequence of suitable sets $\mathcal{H}_\gamma \in V^{\mathbb{P}_\gamma}$ and in $\langle \mathcal{H}_\gamma : \gamma < \alpha \rangle \in V^{\mathbb{P}_\alpha}$.

(a) The domain of $R_{n,\alpha}$ is the set of $f = ((\bar{a}^\ell)_\ell, C, h)$ such that f is a task in $V^{\mathbb{P}_\alpha}$ and such that all $\bar{a}_\ell$ are compatible with $\mathcal{C}_\gamma$, $\gamma < \alpha$,

$$(\forall \ell)(\forall \gamma < \alpha)(\exists \bar{b}_\ell \leq \bar{a}_\ell)(\mathrm{set}_2(\bar{b}_\ell) \subseteq^* W_\gamma).$$

(b) The range of $R_{n,\alpha}$ is the set of $\bar{g}$ that are compatible such that $(\forall \gamma < \alpha)((\exists \bar{b} \leq \bar{g})(\mathrm{set}_2(\bar{b}) \subseteq^* W_\gamma)$.

(c) We write $f R_{n,\alpha} g$ iff

(1) $(\forall \ell \in \omega)\Big(((\bar{g} \text{ past } g_\ell), \text{ past } n+1) \leq \bar{a}_{\max(g_\ell)+1}\Big)$

(3) $C \restriction T(\bar{g} \text{ past } n)$ is constant).

(4) $\exists E \in \mathcal{E} h[\mathrm{set}_2(\bar{g})] \cap h[E] \subseteq n$.

The intended use

Lemma

There is a suitable set $\mathcal{C}_\alpha$ as in (P3') iff
$V^{\mathbb{P}_\alpha} \models (\forall f \in \mathrm{dom}(R_{0,\alpha}))(\exists \bar{g} \in \mathrm{range}(R_{0,\alpha}))(\bigvee_{n\in\omega} f R_{n,\alpha} \bar{g}))$.