Approximate Ramsey properties of finite dimensional normed spaces.

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Research supported by the FAPESP project 13/24827-1
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March 31st, 2015

Outline

1. (Approximate) Ramsey Properties
   - The main results
   - Consequences
   - Borsuk-Ulam like reformulation
2. Partitions. Dual Ramsey and concentration of Measure
   - ℓ_∞^n's
   - Polyhedral spaces
   - Arbitrary spaces
   - ℓ_p^n's, $p \neq \infty$

Definition

Let $1 \leq p \leq \infty$, $n \in \mathbb{N}$. The p-norm $\|\cdot\|_p$ on $\mathbb{R}^n$ is defined for $(a_i)_{i<n}$ by

$$\|(a_i)_{i<n}\|_p := (\sum_{i<n} |a_i|^p)^{\frac{1}{p}} \text{ for } p < \infty$$

$$\|(a_i)_{i<n}\|_\infty := \max_{i<n} |a_i|$$

$$\ell_p^n := (\mathbb{R}^n, \|\cdot\|_p).$$

Same definition for $0 < p < 1$, but $\|\cdot\|_p$ is then a quasi-norm (the triangle inequality fails).

Definition

Given two Banach spaces X and Y, by a (linear isometric) embedding from X into Y we mean a linear operator $T : X \to Y$ such that

$$\|T(x)\|_Y = \|x\|_X \text{ for all } x \in X.$$

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Let

$$\mathrm{Emb}(X, Y)$$

be the collection of all embeddings from X *into* Y*. Then* $\mathrm{Emb}(X, Y)$ *is a metric space with the norm distance*

$$d(T, U) := \|T - U\| := \sup_{x \in S_X} \|T(x) - U(x)\|.$$

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An r-coloring of a set X is just a mapping $c : X \to r$. When (X, d) is a metric space, a ε-monochromatic set of c is a subset Y of X such that

$$Y \subseteq (c^{-1}(i))_\varepsilon \text{ for some } i < r,$$

where $(Z)_\varepsilon := \{x \in X \,:\, d(x, Z) < \varepsilon\}$ is the ε-fattening of Z.

Definition

We say that a collection of Banach spaces $\mathcal{F}$ has the Approximate Ramsey Property (ARP) when for every $F, G \in \mathcal{F}$, $r \in \mathbb{N}$ and $\varepsilon > 0$ there exists $H \in \mathcal{F}$ containing a (linear) **isometric** *copy of G such that every r-coloring of*

$$\mathrm{Emb}(F, G)$$

has a ε-monochromatic set of the form $\gamma \circ \mathrm{Emb}(F, G)$ for some

$$\gamma \in \mathrm{Emb}(G, H).$$

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Theorem (Ferenczi, LA, Mbombo and Todorcevic 15')

The ℓ_p^n's have the ARP for every $0 < p < \infty$.

Using the *approximate ultrahomogeneity* of $\mathbb{G}$,

Corollary (Bartosova, LA and Mbombo)

The group of (linear) isometries $\mathrm{Iso}(\mathbb{G})$ *of the Gurarij space* $\mathbb{G}$*, with the pointwise topology is extremely amenable.*

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This is a consequence of the ARP of ℓ_∞^n's and **positive** embeddings.

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The group of linear isometries of the Lebesgue spaces $L_p[0,1]$ is extremely amenable.

Here we use the following

Proposition

For $p < \infty$, $\theta \geq 1$ and $\varepsilon > 0$, every θ-embedding γ from an isometric copy X of ℓ_p^n into $L_p[0,1]$ there is an isometry g of $L_p[0,1]$ such that $\|g \restriction X - \gamma\|_p < \theta + \varepsilon$.

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Let (X, d) be a metric space, $\varepsilon > 0$. We say that an open covering $\mathcal{U}$ of X is ε-fat when $\mathcal{U}_{-\varepsilon} := \{U_{-\varepsilon}\}_{U \in \mathcal{U}}$ is a covering of X, where

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It is not difficult to see that if X is compact, then every open covering is ε-fat for some $\varepsilon>0$.

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Corollary

For every $0 < p \leq \infty$, every d, m and r, and $\varepsilon > 0$ there exists n such that in every ε-fat covering $\mathcal{U}$ of $\mathrm{Emb}(\ell_p^d, \ell_p^n)$ of cardinality at most r there is $U \in \mathcal{U}$ containing $\gamma \circ \mathrm{Emb}(\ell_p^d, \ell_p^m)$ for some $\gamma \in \mathrm{Emb}(\ell_p^m, \ell_p^n)$.

For example, Borsuk-Ulam Theorem is the statement that for $p = 2$, $d = m = 1$, r and **all** $\varepsilon > 0$ such n is at most the number r of open sets of the covering:

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(iii) So, having $\gamma \circ \mathrm{Emb}(\ell_2^1, \ell_2^1)$ in one open set $U \in \mathcal{U}$ means that the point x determining γ satisfies that $\pm x \in U$.

We work with the unit bases of $\mathbb{R}^n$'s. Then given $0 < p \leq \infty$, let $\mathbb{E}^p_{m,n}$ be the collection of all matrices that determine an embedding between ℓ_p^d and ℓ_p^n.

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(iii) Given $0 < p < \infty$, $p \ne 2$, $A \in \mathbb{E}^p_{d,n}$ if and only if for every column vector c of A one has that $\|c\|_p = 1$ and every two column vectors have disjoint support.

We work with a type of matrices of $\mathbb{E}_{d,n}^{p}$, called Δ-*matrices*.

Definition

Let $\Delta \subseteq B_{\ell_{p*}^d}$, $1/p^* + 1/p = 1$. *We call a matrix* $A \in \mathcal{M}_{n\times d}$ Δ-*matrix, when there is a surjective* $F : n \to \Delta$ *such that for every* $v \in \Delta$ *and every* $i \in F^{-1}(v)$ *the* i^{th}-*row of* A *is*

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A Δ-matrix is determined by a mapping $F : n \to \Delta$. So, the collection of Δ-matrices can be canonically identified with the collection of all mappings from n to Δ.

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Theorem (Graham and Rothschild)

For every finite linearly ordered sets S and T, and $r \in \mathbb{N}$ there exists $n \geq \#T$ such that every r-coloring of $\mathrm{Epi}_{\mathrm{min}}(n, S)$ has a monochromatic set of the form $\mathrm{Epi}_{\mathrm{min}}(T, S) \circ \sigma$ for some $\sigma \in \mathrm{Epi}_{\mathrm{min}}(n, T)$.

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We order $B_{\ell_1^d}$ in a way that we preserve the ℓ_1-norm. Multiplication of an appropriate F-matrix that represents an embedding by an arbitrary matrix embedding is close to a composition of the F-matrix with another G-matrix.

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Proposition

Every polyhedral space F has an injective envelope. That is, there is some n and an isometric embedding $T_F : F \to \ell_\infty^n$ such that for any other isometric embedding $U : F \to \ell_\infty^k$ there is an isometric embedding $\Theta : \ell_\infty^n \to \ell_\infty^k$ such that $U = \Theta \circ T_F$.

Polyhedral spaces are dense in the class of finite dimensional spaces. So, an isometric embedding T between two f.d. spaces X and Y will induce a θ-embedding T' ($\theta^{-1}\|x\| \le \|T'x\| \le \theta\|x\|$) between polyhedral spaces X' and Y' appropriately closed to X and Y.

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Proposition

Let $(X_i)_{i\le n}$ be f.d. spaces, and $1 < \theta < \tau$. Then there is a f.d. space Y having isometric copies of each X_i and an isometric embedding $J : X_n \to Y$ such that for every θ-embedding $T : X_i \to X_n$ there is an isometric embedding $I : X_i \to Y$ such that $\|I - J \circ T\| < \tau - 1$.

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Theorem

For every F, G, r, $\varepsilon > 0$ and $\theta \geq 1$ there is H containing an isometric copy of G such that every r-coloring of $\mathrm{Emb}_\theta(F, H)$ has a $(\theta - 1 + \varepsilon)$-monochromatic set of the form $\gamma \circ \mathrm{Emb}(F, G)$ for some $\gamma \in \mathrm{Emb}(G, H)$.

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A mapping $T : n \to \Delta$ is called an ε-equipartition, $\varepsilon \geq 0$ when

$$\frac{n}{\#\Delta}(1-\varepsilon) \leq \#F^{-1}(\delta) \leq \frac{n}{\#\Delta}(1+\varepsilon)$$

for every $\delta \in \Delta$. Let $\mathrm{Equi}_\varepsilon(n, \Delta)$ be the set of all ε-equipartions, and $\mathrm{Equi}(n, \Delta)$ be the min*-surjection (0-)equipartitions.*

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$$\frac{n}{\#\Delta}(1-\varepsilon) \leq \#F^{-1}(\delta) \leq \frac{n}{\#\Delta}(1+\varepsilon)$$

for every $\delta \in \Delta$. Let $\mathrm{Equi}_\varepsilon(n, \Delta)$ be the set of all ε-equipartions, and $\mathrm{Equi}(n, \Delta)$ be the min-*surjection (0-)equipartitions.*

Problem (Dual Ramsey for equipartitions)

Suppose that $d|m$, and r is arbitrary. Does there exist $m|n$ such that every r-coloring of $\mathrm{Equi}(n, d)$ has a monochromatic set of the form $\mathrm{Equi}(m, d) \circ \sigma$ for some $\sigma \in \mathrm{Equi}(n, m)$?

The Dual Ramsey Theorem is useless when $p < \infty$, up to now. We would need a version of it.

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Theorem

For every $d, r \in \mathbb{N}$ and $\varepsilon, \delta > 0$ there is an integer n such that every r-coloring of $\mathrm{Equi}_\varepsilon(n, d)$ has a δ-monochromatic set of the form

$$\mathcal{S}_d \circ F$$

for some $F \in \mathrm{Equi}_\varepsilon(n, d)$.

We prove the previous result by using concentration of measure of the *Hamming cube* Δ^n.

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$$\alpha_X(\varepsilon) := 1 - \inf\{\mu(A_\varepsilon) : \mu(A) \geq \frac{1}{2}\}.$$

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A sequence $(X_n)_n$ *of mm-spaces is called Lévy when*

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and normal Lévy when there are $c_1, c_2 > 0$ *such that*

$$\alpha_{X_n}(\varepsilon) \leq c_1 e^{-c_2 \varepsilon^2 n}.$$

It is known that

$$\alpha_{(\Delta^n,d,\mu)}(\varepsilon) \le e^{-\frac{1}{8}\varepsilon^2 n},$$

where d is the normalized Hamming distance

$$d(f,g) := \frac{1}{n}\#(f \neq g)$$

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Proposition

$(\mathrm{Equi}_\varepsilon(n,\Delta), d, \mu)_n$ *is asymptotically normal Lévy.*

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Proposition

For every $p \neq \infty$ and every p-adequate set Δ, the mapping assigning to each ε-equipartition $F : n \to \Delta$ the corresponding F-matrix in $\mathbb{E}_{d,n}^p$ is uniformly continuous with modulus of continuity independent of n.

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Proposition

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It suffices to prove the previous for $p = 1$, because all ℓ_p-spheres are uniformly homeomorphic ($p \neq \infty$).

Recall it is a classical result of Ribe that the *Mazur map*

$$M_{p,q}((a_i)_i) := (\mathrm{sgn}(a_i)|a_i|^{\frac{p}{q}})_i$$

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$$\omega_{p,q}(t) \leq \frac{p}{q} t \text{ if } p > q$$
$$\omega_{p,q}(t) \leq c t^{\frac{p}{q}} \text{ if } p < q.$$

Since all the ℓ_p-embeddings ($p \neq 2, \infty$) have the same "shape" (they have disjointly supported columns) it follows from Ribe's result that the ARP for ℓ_p^n's is equivalent to the ARP of ℓ_1^n's, for such p's.

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The case $p = 2$ is somehow easier because of the *ultrahomogeneity* of ℓ_2^n.