Retrospective workshop on Forcing and its applications

Parametrized ♢-principles and canonical models

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Contents

Weak diamond

Definition (Devlin-Shelah 1978)

The weak diamond principle Φ is the following assertion:

$$\forall F : 2^{<\omega_1} \to 2 \ \exists g : \omega_1 \to 2 \ \forall f \in 2^{\omega_1}$$

$$\{\alpha < \omega_1 : F(f \restriction \alpha) = g(\alpha)\} \text{ is stationary.}$$

Theorem (Devlin-Shelah 1978)

Φ is equivalent to $2^\omega < 2^{\omega_1}$.

Parametrized weak diamonds

An invariant is a triple $(A, B, \to)$ where $\to \subseteq A \times B$ is such that
(1) $\forall a \in A\ \exists b \in B\ a \to b$, and
(2) $\forall b \in B\ \exists a \in A\ a \not\to b$.
Given an invariant $(A, B, \to)$ the evaluation of $(A, B, \to)$ is

$$||A, B, \to|| = \min\{|B'| : B' \subseteq B\ \forall a \in A\ \exists b \in B'\ a \to b\}$$

We abbreviate $(A, A, \to)$ as $(A, \to)$.

Definition $\Phi(A, B, \to)$

$$\forall F : 2^{<\omega_1} \to A\ \exists g : \omega_1 \to B\ \forall f \in 2^{\omega_1}$$

$$\{\alpha < \omega_1 : F(f \restriction \alpha) \to g(\alpha)\} \text{ is stationary.}$$

Disadvantage: $\Phi(A, B, \to)$ implies $2^{\omega} < 2^{\omega_1}$.

Parametrized diamonds - Moore-H.-Džamonja

We restrict to Borel invariants - require A, B and $\to$ to be Borel subsets of Polish spaces.

Definition (MHD 2004) $\diamondsuit(A, B, \to)$

$$\forall F : 2^{<\omega_1} \to A \text{ Borel } \exists g : \omega_1 \to B \ \forall f \in 2^{\omega_1}$$

$$\{\alpha < \omega_1 : F(f \restriction \alpha) \to g(\alpha)\} \text{ is stationary.}$$

F is Borel if $F \restriction 2^\alpha$ is Borel for every $\alpha < \omega_1$.

Easy observations:

- $\diamondsuit(A, B, \to) \Rightarrow \|A, B, \to\| \leq \omega_1$,
- $\diamondsuit \Leftrightarrow \diamondsuit(\mathbb{R}, =)$,
- $(A, B, \to) \leq_{GT} (A', B', \to')$ and $\diamondsuit(A', B', \to') \Rightarrow \diamondsuit(A, B, \to)$.

... and the point is ...

Theorem (MHD 2004)

If W is a canonical model and $(A, B, \to)$ is a Borel invariant then $W \models \diamondsuit(A, B, \to)$ if and only if $||A, B, \to|| \leq \omega_1$.

By a canonical model we mean a model which is the result of a CSI of length ω_2 of a single sufficiently definable (e.g. Suslin) and sufficiently homogeneous ($\mathbb{P} \simeq \{0, 1\} \times \mathbb{P}$) proper forcing $\mathbb{P}$.

Results from (MHD)

- $\diamondsuit(non(\mathcal{M}))$ $\Rightarrow$ There is a Suslin tree.
- $\diamondsuit(\mathfrak{s}^\omega)$ $\Rightarrow$ There is an Ostaszewski space.
- $\diamondsuit(\mathfrak{b})$ $\Rightarrow$ There is a non-trvial coherent sequence on ω_1 which can not be uniformized.
- $\diamondsuit(2,=)$ $\Rightarrow \mathfrak{p} = \omega_1$.
- $\diamondsuit(2,=)$ $\Rightarrow$ There are no uncountable Q-sets.
- $\diamondsuit(2,=)$ $\Rightarrow$ Every ladder system on ω_1 has a non-uniformizable coloring.
- $\diamondsuit(\mathfrak{b})$ $\Rightarrow$ There is a MAD family of size ω_1.
- $\diamondsuit(\mathfrak{r})$ $\Rightarrow$ There is a P-point of character ω_1.
- $\diamondsuit(\mathfrak{r}_{nwd})$ $\Rightarrow$ There is a maximal independent family of size ω_1.
- CH + "Almost no diamonds" hold is consistent.

Further results

- (Yorioka, 2005) $\diamondsuit(non(\mathcal{M})) \Rightarrow$ There is a ccc destructible Hausdorff gap.
- (Minami 2005) Separated ◇'s for invariants in the Cichoń diagram under CH.
- (Kastermans-Zhang 2006) $\diamondsuit(non(\mathcal{M})) \Rightarrow$ There is a maximal cofinitary group of size ω_1.
- (Minami 2008) Parametrized diamonds hold in FSI iterations of Suslin ccc forcings.
- (Mildenberger, Mildenberger-Shelah 2009-2011) No other diamonds in the Cichoń diagram imply the existence of a Suslin tree (all are consistent with "all Aronszajn trees are special").
- (Cancino-H.-Meza 2014) $\diamondsuit(\mathfrak{r}) \Rightarrow$ There is a countable irresolvable space of weight ω_1.
- (H.–Ramos-García 2014) $\diamondsuit(2,=) \Rightarrow$ There is a separable Fréchet non-metrizable group.
- (Chodounský 2014) $\diamondsuit(2,=) \Rightarrow$ There is a tight Hausdorff gap of functions.

Contents

Cosmetic changes

Definition $\diamondsuit(A, B, \to)$

$$\forall F : 2^{<\omega_1} \to A \text{ Borel } \exists g : \omega_1 \to B \ \forall f \in 2^{\omega_1}$$

$$\{\alpha < \omega_1 : F(f \restriction \alpha) \to g(\alpha)\} \text{ is stationary.}$$

It turns out that the requirement that F be Borel is unnecessarily strong – can be replaced by $F \restriction 2^\alpha$ is definable from an ω_1-sequence of reals (or even an ω_1-sequence of ordinals), i.e. $F \restriction 2^\alpha \in L(\mathbb{R})[X]$, where X is an ω_1-sequence of ordinals, which we shall call ω_1-definable.

Definition $\diamondsuit^{\omega_1}(A, B, \to)$

$$\forall F : 2^{<\omega_1} \to A \ \omega_1\text{-definable } \exists g : \omega_1 \to B \ \forall f \in 2^{\omega_1}$$

$$\{\alpha < \omega_1 : F(f \restriction \alpha) \to g(\alpha)\} \text{ is stationary.}$$

The weakest weak diamond and failure of Baumgartner

$\diamondsuit^{\omega_1}(2,=)$ - the Weakest weak diamond

$$\forall F : 2^{<\omega_1} \to 2 \ \omega_1\text{-definable} \ \exists g : \omega_1 \to 2 \ \forall f \in 2^{\omega_1}$$

$$\{\alpha < \omega_1 : F(f \restriction \alpha) = g(\alpha)\} \text{ is stationary.}$$

Example.

$\diamondsuit^{\omega_1}(2,=) \Rightarrow$ Every $\aleph_1$-dense set of reals X contains an $\aleph_1$-dense set Y such that X and Y are not order isomorphic.

Proof.

Fix X and Z $\aleph_1$-dense subset of X such that $X \setminus Z$ is uncountable. Enumerate $X \setminus Z$ as $\{x_\alpha : \alpha < \omega_1\}$, and let $H : 2^\omega \to Aut(\mathbb{R})$ be Borel and onto. Let $F(s) = 0$ iff $|s| < \omega$ or $H(s \restriction \omega)(x_{|s|}) \in X$.

Given g, let $Y = Z \cup \{x_\alpha : g(\alpha) = 1\}$. Given an $h \in Aut(\mathbb{R})$ consider any $f \in 2^{\omega_1}$ such that $H(f \restriction \omega) = h$.

□

Sequential composition of invariants

Definition

Given $\mathfrak{i} = (A, B, \to)$ and $\mathfrak{j} = (A', B', \to')$, we define the sequential composition $\mathfrak{i};\mathfrak{j}$ of $\mathfrak{i}$ and $\mathfrak{j}$ by

$$\mathfrak{i};\mathfrak{j} = (A \times A'^B, B \times B', \to'') \text{ with } (a, h) \to'' (b, b') \text{ iff } a \to b \ \&\ h(b) \to' b'.$$

Remark: $||\mathfrak{i};\mathfrak{j}|| = \max\{||\mathfrak{i}||, ||\mathfrak{j}||\}$.

Recall

$$\mathfrak{r}_\sigma = \min\{|\mathcal{R}| : \mathcal{R} \subseteq [\omega]^\omega \ \forall \langle A_n : n \in \omega\rangle \subseteq [\omega]^\omega$$
$$\exists R \in \mathcal{R} \ \forall n \in \omega \ (R \subseteq^* A_n \text{ or } R \cap A_n =^* \emptyset)\}.$$

Monk's questions

Questions (D. Monk 2014)

1. Is it consistent that there is a maximal family of pairwise incomparable elements of $\mathcal{P}(\omega)/fin$ of size less than $\mathfrak{c}$?
2. Is it consistent that there is a maximal subtree of $\mathcal{P}(\omega)/fin$ of size less than $\mathfrak{c}$?
3. Can the two be consistently different?

Definition

A set $\mathcal{T} \subseteq [\omega]^\omega$ is a maximal tree if

1. $\mathcal{T}$ is a tree (ordered by reverse $\subseteq^*$), and
2. $\forall C \in [\omega]^\omega (\exists T \in \mathcal{T}$ such that $T \subseteq^* C$ or $\exists T_0, T_1 \in \mathcal{T}$ incomparable such that $C \subseteq^* T_0 \cap T_1)$.

Note that levels of the tree are incomparable families, not AD families.

The answers are NO, YES, YES.

Monk's questions

Theorem (Campero-Cancino-H.-Miranda 2015)

$\diamondsuit^{\omega_1}(\mathfrak{r}_\sigma;\mathfrak{d}) \Rightarrow$ There is a maximal tree in $\mathcal{P}(\omega)/fin$ of size ω_1.

Corollary.

It is consistent that here is a maximal tree in $\mathcal{P}(\omega)/fin$ of size less than $\mathfrak{c}$.

Recall

A set $\mathcal{T} \subseteq [\omega]^\omega$ is a maximal tree if

1. it is a tree (ordered by reverse $\subseteq^*$), and
2. $\forall C \in [\omega]^\omega (\exists T \in \mathcal{T}$ such that $T \subseteq^* C$ or $\exists T_0, T_1 \in \mathcal{T}$ incomparable such that $C \subseteq^* T_0 \cap T_1)$.

Further small changes - The strongest weak diamond

Definition $\diamondsuit_S^{\omega_1}(\omega_1, =)$ - the Strongest weak diamond

Let $S \subseteq \omega_1$ be stationary.

$$\forall F : 2^{<\omega_1} \to \omega_1 \ \omega_1\text{-definable} \ \exists g : \omega_1 \to \omega_1 \ \forall f \in 2^{\omega_1}$$

$$\{\alpha \in S : F(f \restriction \alpha) = g(\alpha)\} \text{ is stationary.}$$

Observations:

- $\diamondsuit_S^{\omega_1}(\omega_1, =) + ||A, B, \to|| \leq \omega_1 \ \Rightarrow \ \diamondsuit_S^{\omega_1}(A, B, \to)$
- $\diamondsuit_S \Leftrightarrow \ CH + \diamondsuit_S^{\omega_1}(\omega_1, =)$.

Theorem

$\forall S \in NS(\omega_1)^+ \ \diamondsuit_S^{\omega_1}(\omega_1, =)$ holds in all canonical models.

"All" Borel weak diamonds hold in the Sacks model

Theorem

$\forall S \in NS(\omega_1)^+ \ \diamondsuit_S^{\omega_1}(\omega_1, =)$ holds in any canonical model.

combined with

Theorem (Zapletal 2008)

For every Borel cardinal invariant $(A, B, \to)$ if $||A, B, \to|| < \mathfrak{c}$ can be forced then $V^{\mathbb{S}_{\omega_2}} \models ||A, B, \to|| \leq \omega_1$.

gives

Corollary

$V^{\mathbb{S}_{\omega_2}} \models \diamondsuit^{\omega_1}(A, B, \to)$ for every Borel cardinal invariant $(A, B, \to)$ such that $||A, B, \to|| \leq \omega_1$ can be forced over any model without collapsing ω_2.

Contents

Canonical models

Question

What can be said about all canonical models? Or, which problems can not be solved in any canonical model?

Canonical models

The following hold in all canonical models:

- All Whitehead groups of size ω_1 are free (Shelah - $\diamondsuit_S^{\omega_1}(2,=)$)
- Baumgartner's theorem fails (Baumgartner - $\diamondsuit^{\omega_1}(2,=)$)
- $\mathfrak{p} = \mathfrak{q} = \omega_1$, $\mathfrak{a} = \mathfrak{b}$, $\mathfrak{r} = \mathfrak{u}$, $\mathfrak{s} = \mathfrak{s}_\omega$ ... (MHD)
- There is a non-metrizable separable Fréchet group (H.-Ramos - $\diamondsuit(2,=)$)
- There is a Cohen indestructible MAD family (H.-Guzmán - $\mathfrak{b} = \mathfrak{c} + \diamondsuit(\mathfrak{b})$)
- There is a compact sequential space of sequential order > 2 (Dow - $\mathfrak{b} = \mathfrak{c}$ + Gaspar-Hernandez-H. - $\diamondsuit(\mathfrak{b})$)
- There is a compact weakly first countable space that is not first countable (Gorelic-Juhasz-Weis - $\mathfrak{b} = \mathfrak{c}$ + Gaspar-Hernandez-H. - $\diamondsuit(\mathfrak{b})$)
- There is a ccc forcing adding a real and not adding either random or a Cohen real (Brendle - $cof(\mathcal{M}) = \mathfrak{c}$ + Guzmán - $\diamondsuit(cof(\mathcal{M}))$).

A few more results

- (Gaspar-Hernandez-H. 2015) $\diamondsuit(\mathfrak{s}) \Rightarrow$ Counterexample to the Scarborough-Stone problem.
- (Fernández-H. 2015) $\diamondsuit(\mathfrak{r}_{Hindman}) \Rightarrow$ There is a union-ultrafilter of character ω_1.
- (Fernández-H. 2015) $\diamondsuit(\mathfrak{r}_{Fin\times scattered}) \Rightarrow$ There is a gruff ultrafilter of character ω_1.
- (Cancino-Guzmán-Miller 2014) $\diamondsuit(\mathfrak{r};\mathfrak{d}) \Rightarrow$ There is an ideal independent maximal family of size ω_1.

Questions

Questions

1. Is $\diamondsuit^{\omega_1}(\omega_1, <)$ consistent with $\neg\diamondsuit^{\omega_1}(\omega_1, =)$?
2. What happens on ω_2?
3. Clarify what happens in canonical ccc models.
4. Can there be a canonical model without P-points? Suslin trees?
5. Is there a non-trivial invariant whose diamond produces $\clubsuit$?

Thank you for your attention!!!

Questions

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