Coherent adequate forcing and preserving CH

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Joint work with John Krueger

Forcing and its applications retrospective workshop

Introduction

The method of side conditions, invented by Todorcevic, describes a style of forcing in which elementary substructures are included in the conditions of a forcing poset P to ensure properness of P and hence, the preservation of ω_1.

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If $q \in P$ and $N \prec H(\theta)$ with $|N| = \aleph_0$, then

1. q is said to be (N, P)-generic iff for every dense subset D of P belonging to N, $D \cap N$ is predense below q.
2. q is said to be strongly (N, P)-generic iff for every dense subset D of $P \cap N$, D is predense below q.

R1 By elementarity, if D is a dense subset of P and $D, P \in N$, then $D \cap N$ is a dense subset of $P \cap N$. So, if $P \in N$, then $2 \Rightarrow 1$.
R2 If q is strongly (N, P)-generic, then q forces that $N \cap G$ is a V-generic filter on the ctble. set $N \cap P$. So, q adds a Cohen real.

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A typical condition of a forcing P equipped with side cond. is a pair (x, A) where x is an approximation to the desired generic object and A is a finite set of ctble. elementary substructures such that if $N \in A$, then (x, A) is (N, P)-generic.

Friedman and Mitchell independently took the first step in generalizing this method from adding generic objects of size ω_1 to adding larger objects by defining forcing posets with finite conditions for adding a club subset of ω_2. Neeman was the first to simplify the side conditions of F. and M. by presenting a general framework for forcing on ω_2 with side conditions.

The forcing posets of F, M, and N for adding a club of ω_2 with finite cond. all force that $2^\omega = \omega_2$. In fact, they can be factored in many ways so that the quotient forcing also has strongly generic cond. in the intermediate extensions.

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Friedman asked whether it is possible to add a club subset of ω_2 with finite conditions while preserving CH.

We solve this problem by defining a forcing poset which adds a club to a fat stationary set and falls in the class of coherent adequate type forcings.

Our main result is that any coherent adequate forcing preserves CH.

Moreover, any coherent adequate forcing on $H(\lambda)$ (meaning that our side conditions are ctble. elementary substructures of $H(\lambda)$) , where $2^\omega < \lambda$ is a cardinal of uncountable cofinality, collapses 2^ω to have size ω_1, preserves $(2^\omega)^+$, and forces CH.

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Coherent Adequate Sets (Development due to Krueger)

From now on, assume that $\lambda \geq \omega_2$ is a fixed cardinal of uncountable cofinality. Also fix a predicate $Y \subseteq H(\lambda)$, which we assume codes a well-ordering of $H(\lambda)$.

Let $\mathcal{X}$ be the set of countable elementary substructures $N \prec (H(\lambda), \in, Y)$ and let $\Gamma := S^{\omega_2}_{\omega_1}$ be the set of ordinals in ω_2 having uncountable cofinality. So, if N is in $\mathcal{X}$, then N is in $H(\lambda)$ and Γ is definable in N.

Now we introduce a way to compare members of $\mathcal{X}$: For $M \in X$, Γ_M denote the set of $\beta \in S^{\omega_2}_{\omega_1}$ such that

$$\beta = min(\Gamma \setminus sup(M \cap \beta))$$

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In particular, $\omega_1 \in \Gamma_M$, $|\Gamma_M| = \aleph_0$ and $\Gamma_M \subseteq \Gamma_N$ if $M \subseteq N$.

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If $M, N \in \mathcal{X}$, then $\beta_{M,N} := max(\Gamma_M \cap \Gamma_N)$ exists.

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We define the relations $<$, $\leq$ and $\sim$ on $\mathcal{X}$. Let $M < N$ if $M \cap \beta_{M,N} \in N$ (implying that $\beta_{M,N} = min(\Gamma \setminus (M \cap \beta_{M,N}) \in N)$. Let $M \sim N$ if $M \cap \beta_{M,N} = N \cap \beta_{M,N}$. Let $M \leq N$ if either $M < N$ or $M \sim N$.

Since $\beta_{M,N} \geq \omega_1$, $M < N$ implies that $M \cap \omega_1 < N \cap \omega_1$ and $M \sim N$ implies that $M \cap \omega_1 = N \cap \omega_1$.

A subset A of $\mathcal{X}$ is **adequate** iff every 2 elements of A are comparable under $\leq$.

Note that if A is finite and adequate, $N \in \mathcal{X}$ and $A \in \mathcal{X}$, then N has access to all the the initial segments of each $M \in A$. So, $A \cup \{N\}$ is adequate.

Next we define remainder points, which describe the overlap of models past their comparison point.

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Definition

If $\{M, N\}$ is adequate, then, the reminder points of N over M, denoted by $R_M(N)$, is defined as the set of β satisfying either:

a $N \leq M$ and $\beta = min(N \setminus \beta_{M,N})$, or

b there is $\gamma \in M \setminus \beta_{M,N}$, such that $\beta = min(N \setminus \gamma)$.

This remainder is always finite, since otherwise there would be a common limit point of M and N greater than $\beta_{M,N}$!!!!
Given an adequate A, define $R_A = \bigcup\{R_M(N) : M, N \in A\}$.

Given $S \subseteq \omega_2$ and an adequate A, A is said to be **(S)-adequate** if $R_A \subseteq S$.

A finite set A is said to be **coherent (S)-adequate** if A is (S)-adequate and A is symmetric (style Asperó-Mota).

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If $M, N \in \mathcal{X}$, then they are said to be **strongly isomorphic** iff there is an isomorphism $\sigma_{M,N} : (M, \in, Y) \longrightarrow (N, \in, Y)$ being the identity on $M \cap N$. Note that in such a case $M \cap \omega_1 = N \cap \omega_1$.

Definition

Let A be a finite subset of $\mathcal{X}$. A is said to be **coherent** **(S)-adequate** if A is an (S)-adequate set satisfying:

(1) Given M, N in A, if $M \cap \omega_1 = N \cap \omega_1$ (i.e., $M \sim N$), then there is a (unique) strong isomorphism between them.

(2) Given M, N in A, if $M \cap \omega_1 < N \cap \omega_1$ (i.e., $M < N$), then there is some P in A such that $N \cap \omega_1 = P \cap \omega_1$ and $M \in P$.

(3) A is closed under isomorphisms.

The rest of this talk is part of my joint work with K. From now on, fix $S \subseteq \omega_2$ such that $S \cap cof(\omega_1)$ is stationary and also fix $\mathcal{Y} \subseteq \mathcal{X}$ stationary in $[H(\lambda)]^\omega$ and closed under iso.
By the Tarski-Vaught test, the club $\mathcal{X}$ is closed under iso.

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(I) x is a finite subset of $H(\lambda)$,

(II) $A \subseteq \mathcal{Y}$ and A is a coherent (S)-adequate set,

(III) If $(y, B) \leq (x, A)$, N and N' are iso. sets in B, and $(x, A) \in N$, then $(y, B) \leq \sigma_{N,N'}((x, A)) \in P$ (symmetry),

(IV) If $\{M_0, \ldots, M_n\} \subseteq \mathcal{Y}$ is coherent (S)-adequate and $(x, A) \in M_0 \cap \ldots \cap M_n$, then there is a condition $(y, B) \leq (x, A)$ s.t. $\{M_0, \ldots, M_n\} \subseteq B$, and

(V) For all $M \in A$, (x, A) is strongly (M, P)-generic.

By clause (IV) and since $\mathcal{Y}$ is stat in $[H(\lambda)]^{\omega}$, any (S, $\mathcal{Y}$) coherent adequate poset preserves ω_1 and adds Cohen reals. We will see that we only add a small number of new reals.

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Let $\lambda > 2^\omega$ with $cof(\lambda) > \omega$. Let $\langle r_i : i < 2^\omega \rangle$ be the Y-first enumeration of the power set of ω. So, Y codes the relation Z, where $Z(i, n)$ holds if $i < 2^\omega$ and $n \in r_i$.

Lemma

If M and N are in $\mathcal{X}$ and iso., then $\sigma_{M,N}(\alpha) = \alpha$ for all $\alpha \in M \cap 2^\omega$. Hence, $M \cap 2^\omega = N \cap 2^\omega$.

Proof. It is enough to check that $r_\alpha = r_{\sigma_{M,N}(\alpha)}$. But $n \in r_\alpha$ iff $M \models Z(\alpha, n)$ iff $N \models Z(\sigma_{M,N}(\alpha), n)$ iff $n \in r_{\sigma_{M,N}(\alpha)}$.

Note that if A is a coherent (S)-adequate set $M \cap \omega_1 < N \cap \omega_1$, then there is $N' \in A$ s.t. $N \cap \omega_1 = N' \cap \omega_1$ and $M \in N'$. Since A is closed, $\sigma_{N',N}(M) \in N \cap A$. So, $M \cap 2^\omega = \sigma_{N',N}(M) \cap 2^\omega \subseteq N$.

Corollary: Any (S, $\mathcal{Y}$)-coherent adeq. poset collapses 2^ω to ω_1.

Let $\lambda > 2^\omega$ with $cof(\lambda) > \omega$. Let $\langle r_i : i < 2^\omega \rangle$ be the Y-first enumeration of the power set of ω. So, Y codes the relation Z, where $Z(i, n)$ holds if $i < 2^\omega$ and $n \in r_i$.

Lemma

If M and N are in $\mathcal{X}$ and iso., then $\sigma_{M,N}(\alpha) = \alpha$ for all $\alpha \in M \cap 2^\omega$. Hence, $M \cap 2^\omega = N \cap 2^\omega$.

Proof. It is enough to check that $r_\alpha = r_{\sigma_{M,N}(\alpha)}$. But $n \in r_\alpha$ iff $M \models Z(\alpha, n)$ iff $N \models Z(\sigma_{M,N}(\alpha), n)$ iff $n \in r_{\sigma_{M,N}(\alpha)}$.

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If $R \subseteq H(\lambda)$ and $z \in H(\lambda)$, then there are $M, N \in \mathcal{Y}$ satisfying:

(1) $z \in M \cap N$,

(2) $\{M, N\}$ *is coherent (S)-adequate,*

(3) *the structures* $(M, \in, Y, R)$ *and* $(N, \in, Y, R)$ *are elementary in* $(H(\lambda), \in, Y, R)$ *and are isomorphic, and*

(4) *there are* $\alpha \in M \cap (2^\omega)^+$ *and* $\beta \in N \cap (2^\omega)^+$ *s.t.* $\alpha \neq \beta$ *and* $\sigma_{M,N}(\alpha) = \beta$.

Sketch of proof for the case $2^\omega \geq \omega_2$**:** For each $i \in (2^\omega)^+$ fix $N_i \in \mathcal{Y}$ s.t. z and i are in N_i and $N_i \prec (H(\lambda), \in, Y, R)$. By a Δ system, there is a cofinal $I \subseteq (2^\omega)^+$ s.t. for all i, j in I, N_i and N_j are strongly isomorphic.

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Fix $i \in I$ and let $M = N_i$. Now, fix $j \in I$ such that $sup(M \cap (2^\omega)^+) < j$ and let $N = N_j$. Let us check that M and N witness the lemma. Properties (1) and (3) are obvious.

Since $2^\omega \geq \omega_2$ and M and N are isomorphic and by the above lemma, $M \cap \omega_2 = N \cap \omega_2$. So, trivially $\{M, N\}$ is adequate.

Also, $R_M(N) = R_N(M) = \emptyset$ and hence, $\{M, N\}$ is (S) coherent adequate. This verifies (2).

For (4), let $\beta := j$ and use that $(2^\omega)^+$ is either equal to λ or definable in $H(\lambda)$. So,

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Lemma Let P be an (S, $\mathcal{Y}$)-coherent adeq. poset. If p forces that $\langle \dot{f}_i : i < (2^\omega)^+ \rangle$ is a sequence of functions from ω to ω, then there is $q \le p$ and $\alpha < \beta$ such that q forces that $\dot{f}_\alpha = \dot{f}_\beta$.

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By (IV), there is $q = (y, B) \le p$ such that $M, N \in B$. Check that q, α and β work. This follows from the (M, P)-strongly genericity of q, the symmetric clause (III) and the fact that if $z \in M \cap P$ and $n, m \in \omega$: $z \Vdash \dot{f}_\alpha(n) = m$ iff $\sigma(z) \Vdash \dot{f}_\beta(n) = m$.

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Corollary

Any (S, $\mathcal{Y}$)-coherent adeq. P collapses $(2^\omega)^V$ to ω_1, forces CH and forces that the successor of $(2^\omega)^V$ in V is equal to ω_2.

Proof. If $p \in P$ collapses the successor of $(2^\omega)^V$, then there is a sequence of names which p forces that is an enumeration of ω_1 many distinct functions from ω to ω in order type $(2^\omega)^+$ **!!!**

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A (psychoanalytic) retrospective analysis

Prior to this work, Asperó and Mota proved that for any cardinal $\lambda \geq \omega_2$ of uncountable cofinality , the λ-*symmetric forcing* consisting of finite symmetric systems of countable elementary substructures of $H(\lambda)$ ordered by reverse inclusion preserves CH. This is one of the the two forcings that they currently use in the first step of their finite support iterations.

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By a result of Miyamoto from 2013, the λ-symmetric poset as well as any coherent adequate forcing on $H(\lambda)$ adds an ω_1–tree with λ many cofinal branches, for any regular $\lambda \geq \omega_2$.

In an unpublished work from the 80's Todorcevic also noticed that the ω_2-symm. poset preserves *CH* and adds a Kurepa tree.

Certainly, the *CH* preservation argument of Asperó and Mota slightly intersects the *CH* preservation argument of Krueger and Mota, but the former do not show how to force with side cond. together with another finite set of objects to preserve *CH*.

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Recall that a stationary set $S \subseteq \omega_2$ is said to be *fat* iff for every club $C \subseteq \omega_2$, $S \cap C$ contains a closed subset with o. t. $\omega_1 + 1$.

Corollary

Assume CH. If $S \subseteq \omega_2$ is fat stationary (for every club $C \subseteq \omega_2$, $S \cap C$ contains a closed subset with order type $\omega_1 + 1$), then there is an $(S, \mathcal{Y})$-coherent adeq. $P \subseteq H(\omega_2)$ preserving ω_1, ω_2, CH and s.t. $V^P \models S$ contains a club.

Sketch of proof. W.lo.g. we may assume that $S \cap cof(\omega_1)$ is stationary and that for all $\alpha \in S \cap cof(\omega_1)$, $S \cap \alpha$ contains a closed cofinal subset of α.

Let $\lambda = \omega_2$ and let Y code S together with a well-order of $H(\omega_2)$. In particular, isomorphisms between members of $\mathcal{X}$ preserve membership in S.

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Let $\mathcal{Y}$ denote the stationary set of $M \in \mathcal{X}$ such that for all $\alpha \in (M \cap S) \cup \{\omega_2\}$, $sup(M \cap \alpha) \in S$.

If $N \cap \omega_2 \not\subseteq \alpha$, let $\alpha_N := min(N \setminus \alpha)$.

P is the poset consisting of conditions $p = (x_p, A_p)$ satisfying:

(i) x_p is a finite set of nonoverlapping pairs whose first coordinate is in S,

(ii) A_p is a finite coherent adequate subset of $\mathcal{Y}$,

(iii) if $\langle \alpha, \alpha' \rangle \in x_p$, $N \in A_p$ and $N \cap \omega_2 \not\subseteq \alpha$, then $N \cap [\alpha, \alpha'] \neq \emptyset$ implies $\alpha, \alpha' \in N$, and $N \cap [\alpha, \alpha'] = \emptyset$ implies $\langle \alpha_N, \alpha_N \rangle \in x_p$,

(iv) if γ in R_{A_p}, then $\langle \gamma, \gamma \rangle \in x_p$, and

(v) p is symmetric