Coherent adequate forcing and preserving CH

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Joint work with John Krueger

Forcing and its applications retrospective workshop
Introduction
The method of side conditions, invented by Todorcevic, describes a style of forcing in which elementary substructures are included in the conditions of a forcing poset $P$ to ensure properness of $P$ and hence, the preservation of $\omega_1$.

Definition
If $q \in P$ and $N \prec H(\theta)$ with $|N| = \aleph_0$, then

1. $q$ is said to be $(N, P)$-generic iff for every dense subset $D$ of $P$ belonging to $N$, $D \cap N$ is predense below $q$.

2. $q$ is said to be strongly $(N, P)$-generic iff for every dense subset $D$ of $P \cap N$, $D$ is predense below $q$.

R1 By elementarity, if $D$ is a dense subset of $P$ and $D, P \in N$, then $D \cap N$ is a dense subset of $P \cap N$. So, if $P \in N$, then $2 \Rightarrow 1$.

R2 If $q$ is strongly $(N, P)$-generic, then $q$ forces that $N \cap G$ is a $V$-generic filter on the ctble. set $N \cap P$. So, $q$ adds a Cohen real.
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A typical condition of a forcing $P$ equipped with side cond. is a pair $(x, A)$ where $x$ is an approximation to the desired generic object and $A$ is a finite set of ctble. elementary substructures such that if $N \in A$, then $(x, A)$ is $(N, P)$-generic.

Friedman and Mitchell independently took the first step in generalizing this method from adding generic objects of size $\omega_1$ to adding larger objects by defining forcing posets with finite conditions for adding a club subset of $\omega_2$. Neeman was the first to simplify the side conditions of F. and M. by presenting a general framework for forcing on $\omega_2$ with side conditions.

The forcing posets of F, M, and N for adding a club of $\omega_2$ with finite cond. all force that $2^\omega = \omega_2$. In fact, they can be factored in many ways so that the quotient forcing also has strongly generic cond. in the intermediate extensions.
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Friedman asked whether it is possible to add a club subset of \( \omega_2 \) with finite conditions while preserving CH.

We solve this problem by defining a forcing poset which adds a club to a fat stationary set and falls in the class of coherent adequate type forcings.

Our main result is that any coherent adequate forcing preserves CH.

Moreover, any coherent adequate forcing on \( H(\lambda) \) (meaning that our side conditions are ctble. elementary substructures of \( H(\lambda) \)), where \( 2^\omega < \lambda \) is a cardinal of uncountable cofinality, collapses \( 2^\omega \) to have size \( \omega_1 \), preserves \( (2^\omega)^+ \), and forces CH.
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Coherent Adequate Sets (Development due to Krueger)

From now on, assume that $\lambda \geq \omega_2$ is a fixed cardinal of uncountable cofinality. Also fix a predicate $Y \subseteq H(\lambda)$, which we assume codes a well-ordering of $H(\lambda)$.

Let $\mathcal{X}$ be the set of countable elementary substructures $N \prec (H(\lambda), \in, Y)$ and let $\Gamma := S_{\omega_2}^{\omega_1}$ be the set of ordinals in $\omega_2$ having uncountable cofinality. So, if $N$ is in $\mathcal{X}$, then $N$ is in $H(\lambda)$ and $\Gamma$ is definable in $N$.

Now we introduce a way to compare members of $\mathcal{X}$: For $M \in \mathcal{X}$, $\Gamma_M$ denote the set of $\beta \in S_{\omega_1}^{\omega_2}$ such that

$$\beta = \min(\Gamma \setminus \sup(M \cap \beta))$$
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In particular, $\omega_1 \in \Gamma_M$, $|\Gamma_M| = \aleph_0$ and $\Gamma_M \subseteq \Gamma_N$ if $M \subseteq N$.

**Lemma**

If $M, N \in \mathcal{X}$, then $\beta_{M,N} := \max(\Gamma_M \cap \Gamma_N)$ exists.

**Lemma**

If $M, N \in \mathcal{X}$ and $M'$ denotes $(M \cap \omega_2) \cup \lim((M \cap \omega_2))$, then $M' \cap N' \subseteq \beta_{M,N}$.
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We define the relations $<$, $\leq$ and $\sim$ on $\mathcal{X}$. Let $M < N$ if $M \cap \beta_{M,N} \in N$ (implying that $\beta_{M,N} = \min(\Gamma \setminus (M \cap \beta_{M,N}) \in N)$. Let $M \sim N$ if $M \cap \beta_{M,N} = N \cap \beta_{M,N}$. Let $M \leq N$ if either $M < N$ or $M \sim N$.

Since $\beta_{M,N} \geq \omega_1$, $M < N$ implies that $M \cap \omega_1 < N \cap \omega_1$ and $M \sim N$ implies that $M \cap \omega_1 = N \cap \omega_1$.

A subset $A$ of $\mathcal{X}$ is **adequate** iff every 2 elements of $A$ are comparable under $\leq$.

Note that if $A$ is finite and adequate, $N \in \mathcal{X}$ and $A \in \mathcal{X}$, then $N$ has access to all the the initial segments of each $M \in A$. So, $A \cup \{N\}$ is adequate.

Next we define remainder points, which describe the overlap of models past their comparison point.
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If \( \{M, N\} \) is adequate, then, the reminder points of \( N \) over \( M \), denoted by \( R_M(N) \), is defined as the set of \( \beta \) satisfying either:

a. \( N \leq M \) and \( \beta = \min(N \setminus \beta_{M,N}) \), or

b. there is \( \gamma \in M \setminus \beta_{M,N} \), such that \( \beta = \min(N \setminus \gamma) \).

This remainder is always finite, since otherwise there would be a common limit point of \( M \) and \( N \) greater than \( \beta_{M,N} \) !!!!

Given an adequate \( A \), define \( R_A = \bigcup \{R_M(N) : M, N \in A \} \).

Given \( S \subseteq \omega_2 \) and an adequate \( A \), \( A \) is said to be \((S)\)-adequate if \( R_A \subseteq S \).

A finite set \( A \) is said to be coherent \((S)\)-adequate if \( A \) is \((S)\)-adequate and \( A \) is symmetric (style Asperó-Mota).
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If $M, N \in \mathcal{X}$, then they are said to be **strongly isomorphic** iff there is an isomorphism $\sigma_{M,N} : (M, \in, Y) \to (N, \in, Y)$ being the identity on $M \cap N$. Note that in such a case $M \cap \omega_1 = N \cap \omega_1$.

**Definition**

Let $A$ be a finite subset of $\mathcal{X}$. $A$ is said to be **coherent (S)-adequate** if $A$ is an (S)-adequate set satisfying:

1. Given $M, N$ in $A$, if $M \cap \omega_1 = N \cap \omega_1$ (i.e., $M \sim N$), then there is a (unique) strong isomorphism between them.
2. Given $M, N$ in $A$, if $M \cap \omega_1 < N \cap \omega_1$ (i.e., $M < N$), then there is some $P$ in $A$ such that $N \cap \omega_1 = P \cap \omega_1$ and $M \in P$.
3. $A$ is closed under isomorphisms.

The rest of this talk is part of my joint work with K. From now on, fix $S \subseteq \omega_2$ such that $S \cap \text{cof}(\omega_1)$ is stationary and also fix $\mathcal{Y} \subseteq \mathcal{X}$ stationary in $[H(\lambda)]^\omega$ and closed under iso.

By the Tarski-Vaught test, the club $\mathcal{X}$ is closed under iso.
If $M, N \in \mathcal{X}$, then they are said to be strongly isomorphic iff there is an isomorphism $\sigma_{M,N} : (M, \in, Y) \rightarrow (N, \in, Y)$ being the identity on $M \cap N$. Note that in such a case $M \cap \omega_1 = N \cap \omega_1$.

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A poset $P$ is said to be an $(S, \mathcal{Y})$-coherent adequate type forcing if its conditions are pairs $(x, A)$ satisfying:

(I) $x$ is a finite subset of $H(\lambda)$,

(II) $A \subseteq \mathcal{Y}$ and $A$ is a coherent $(S)$-adequate set,

(III) If $(y, B) \leq (x, A)$, $N$ and $N'$ are iso. sets in $B$, and $(x, A) \in N$, then $(y, B) \leq \sigma_{N,N'}((x, A)) \in P$ (symmetry),

(IV) If $\{M_0, \ldots, M_n\} \subseteq \mathcal{Y}$ is coherent $(S)$-adequate and $(x, A) \in M_0 \cap \ldots \cap M_n$, then there is a condition $(y, B) \leq (x, A)$ s.t. $\{M_0, \ldots, M_n\} \subseteq B$, and

(V) For all $M \in A$, $(x, A)$ is strongly $(M, P)$-generic.

By clause (IV) and since $\mathcal{Y}$ is stat in $[H(\lambda)]^\omega$, any $(S, \mathcal{Y})$-coherent adequate poset preserves $\omega_1$ and adds Cohen reals. We will see that we only add a small number of new reals.
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(I) $x$ is a finite subset of $H(\lambda)$,

(II) $A \subseteq \mathcal{Y}$ and $A$ is a coherent $(S)$-adequate set,

(III) If $(y, B) \leq (x, A)$, $N$ and $N'$ are iso. sets in $B$, and $(x, A) \in N$, then $(y, B) \leq \sigma_{N,N'}((x, A)) \in P$ (symmetry),

(IV) If $\{M_0, \ldots, M_n\} \subseteq \mathcal{Y}$ is coherent $(S)$-adequate and $(x, A) \in M_0 \cap \ldots \cap M_n$, then there is a condition $(y, B) \leq (x, A)$ s.t. $\{M_0, \ldots, M_n\} \subseteq B$, and

(V) For all $M \in A$, $(x, A)$ is strongly $(M, P)$-generic.

By clause (IV) and since $\mathcal{Y}$ is stat in $[H(\lambda)]^{\omega}$, any $(S, \mathcal{Y})$ coherent adequate poset preserves $\omega_1$ and adds Cohen reals. We will see that we only add a small number of new reals.
Let $\lambda > 2^\omega$ with $\text{cof}(\lambda) > \omega$. Let $\langle r_i : i < 2^\omega \rangle$ be the $Y$-first enumeration of the power set of $\omega$. So, $Y$ codes the relation $Z$, where $Z(i, n)$ holds if $i < 2^\omega$ and $n \in r_i$.

**Lemma**

If $M$ and $N$ are in $\mathcal{X}$ and iso., then $\sigma_{M,N}(\alpha) = \alpha$ for all $\alpha \in M \cap 2^\omega$. Hence, $M \cap 2^\omega = N \cap 2^\omega$.

**Proof.** It is enough to check that $r_\alpha = r_{\sigma_{M,N}(\alpha)}$. But $n \in r_\alpha$ iff $M \models Z(\alpha, n)$ iff $N \models Z(\sigma_{M,N}(\alpha), n)$ iff $n \in r_{\sigma_{M,N}(\alpha)}$.

Note that if $A$ is a coherent ($S$)-adequate set $M \cap \omega_1 < N \cap \omega_1$, then there is $N' \in A$ s.t. $N \cap \omega_1 = N' \cap \omega_1$ and $M \in N'$. Since $A$ is closed, $\sigma_{N',N}(M) \in N \cap A$. So, $M \cap 2^\omega = \sigma_{N',N}(M) \cap 2^\omega \subseteq N$.

**Corollary:** Any ($S$, $Y$)-coherent adeq. poset collapses $2^\omega$ to $\omega_1$. 
Let $\lambda > 2^\omega$ with $\text{cof}(\lambda) > \omega$. Let $\langle r_i : i < 2^\omega \rangle$ be the $Y$-first enumeration of the power set of $\omega$. So, $Y$ codes the relation $Z$, where $Z(i, n)$ holds if $i < 2^\omega$ and $n \in r_i$.

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Lemma

If $M$ and $N$ are in $\mathcal{X}$ and iso., then $\sigma_{M,N}(\alpha) = \alpha$ for all $\alpha \in M \cap 2^\omega$. Hence, $M \cap 2^\omega = N \cap 2^\omega$.

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**Lemma**

*If $M$ and $N$ are in $\mathcal{X}$ and iso., then $\sigma_{M,N}(\alpha) = \alpha$ for all $\alpha \in M \cap 2^\omega$. Hence, $M \cap 2^\omega = N \cap 2^\omega$.***

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**Corollary:** Any ($S$, $Y$)-coherent adeq. poset collapses $2^\omega$ to $\omega_1$. 
Lemma

If $R \subseteq H(\lambda)$ and $z \in H(\lambda)$, then there are $M, N \in \mathcal{Y}$ satisfying:

1. $z \in M \cap N$,
2. $\{M, N\}$ is coherent $(S)$-adequate,
3. the structures $(M, \in, Y, R)$ and $(N, \in, Y, R)$ are elementary in $(H(\lambda), \in, Y, R)$ and are isomorphic, and
4. there are $\alpha \in M \cap (2^\omega)^+$ and $\beta \in N \cap (2^\omega)^+$ s.t. $\alpha \neq \beta$ and $\sigma_{M,N}(\alpha) = \beta$.

Sketch of proof for the case $2^\omega \geq \omega_2$: For each $i \in (2^\omega)^+$ fix $N_i \in \mathcal{Y}$ s.t. $z$ and $i$ are in $N_i$ and $N_i \prec (H(\lambda), \in, Y, R)$. By a $\Delta$ system, there is a cofinal $I \subseteq (2^\omega)^+$ s.t. for all $i, j$ in $I$, $N_i$ and $N_j$ are strongly isomorphic.
**Lemma**

If \( R \subseteq H(\lambda) \) and \( z \in H(\lambda) \), then there are \( M, N \in \mathcal{Y} \) satisfying:

1. \( z \in M \cap N \),
2. \( \{ M, N \} \) is coherent \((S)\)-adequate,
3. the structures \((M, \in, Y, R)\) and \((N, \in, Y, R)\) are elementary in \((H(\lambda), \in, Y, R)\) and are isomorphic, and
4. there are \( \alpha \in M \cap (2^\omega)^+ \) and \( \beta \in N \cap (2^\omega)^+ \) s.t. \( \alpha \neq \beta \) and \( \sigma_{M,N}(\alpha) = \beta \).

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Fix \( i \in I \) and let \( M = N_i \). Now, fix \( j \in I \) such that 
\[ \sup(M \cap (2^\omega)^+) < j \] and let \( N = N_j \). Let us check that \( M \) and \( N \) witness the lemma. Properties (1) and (3) are obvious.

Since \( 2^\omega \geq \omega_2 \) and \( M \) and \( N \) are isomorphic and by the above lemma, \( M \cap \omega_2 = N \cap \omega_2 \). So, trivially \( \{M, N\} \) is adequate.

Also, \( R_M(N) = R_N(M) = \emptyset \) and hence, \( \{M, N\} \) is (S) coherent adequate. This verifies (2).

For (4), let \( \beta := j \) and use that \( (2^\omega)^+ \) is either equal to \( \lambda \) or definable in \( H(\lambda) \). So,
\[ \alpha := \sigma_{M,N}(\beta) < \sup(M \cap (2^\omega)^+) < j = \beta \]
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Lemma Let $P$ be an $(S, Y)$-coherent adeq. poset. If $p$ forces that $\langle f_i : i < (2^\omega)^+ \rangle$ is a sequence of functions from $\omega$ to $\omega$, then there is $q \leq p$ and $\alpha < \beta$ such that $q$ forces that $\dot{f}_\alpha = \dot{f}_\beta$.

Sketch of proof. Define $R \subset H(\lambda)$ by letting $R(z, i, n, m)$ if $z \in P$ and $z \Vdash \dot{f}_i(n) = m$. Fix $M$ and $N$ in $Y$ satisfying:

1. $p \in M \cap N$,
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By (IV), there is $q = (y, B) \leq p$ such that $M, N \in B$. Check that $q, \alpha$ and $\beta$ work. This follows from the $(M, P)$-strongly genericity of $q$, the symmetric clause (III) and the fact that if $z \in M \cap P$ and $n, m \in \omega$: $z \Vdash \dot{f}_\alpha(n) = m$ iff $\sigma(z) \Vdash \dot{f}_\beta(n) = m$. 
Lemma Let $P$ be an $(S, \mathcal{Y})$-coherent adeq. poset. If $p$ forces that $\langle f_i : i < (2^\omega)^+ \rangle$ is a sequence of functions from $\omega$ to $\omega$, then there is $q \leq p$ and $\alpha < \beta$ such that $q$ forces that $f_\alpha = f_\beta$.

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Corollary

Any \((S, \mathcal{V})\)-coherent adeq. \(P\) collapses \((2^\omega)^V\) to \(\omega_1\), forces CH and forces that the successor of \((2^\omega)^V\) in \(V\) is equal to \(\omega_2\).

Proof. If \(p \in P\) collapses the successor of \((2^\omega)^V\), then there is a sequence of names which \(p\) forces that is an enumeration of \(\omega_1\) many distinct functions from \(\omega\) to \(\omega\) in order type \((2^\omega)^+\) !!!
Corollary

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A (psychoanalytic) retrospective analysis

Prior to this work, Asperó and Mota proved that for any cardinal \( \lambda \geq \omega_2 \) of uncountable cofinality, the \( \lambda \)-symmetric forcing consisting of finite symmetric systems of countable elementary substructures of \( H(\lambda) \) ordered by reverse inclusion preserves \( CH \). This is one of the two forcings that they currently use in the first step of their finite support iterations.

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A symmetric system is similar to a coherent adequate set, except that it does not have the adequate structure.
By a result of Miyamoto from 2013, the $\lambda$-symmetric poset as well as any coherent adequate forcing on $H(\lambda)$ adds an $\omega_1$–tree with $\lambda$ many cofinal branches, for any regular $\lambda \geq \omega_2$.

In an unpublished work from the 80’s Todorcevic also noticed that the $\omega_2$-symm. poset preserves CH and adds a Kurepa tree.

Certainly, the CH preservation argument of Asperó and Mota slightly intersects the CH preservation argument of Krueger and Mota, but the former do not show how to force with side cond. together with another finite set of objects to preserve CH.

This may be an empirical evidence that Krueger’s adequacy is crucial for this kind of constructions.
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Recall that a stationary set \( S \subseteq \omega_2 \) is said to be *fat* iff for every club \( C \subseteq \omega_2 \), \( S \cap C \) contains a closed subset with o. t. \( \omega_1 + 1 \).

**Corollary**

Assume CH. If \( S \subseteq \omega_2 \) is fat stationary (for every club \( C \subseteq \omega_2 \), \( S \cap C \) contains a closed subset with order type \( \omega_1 + 1 \)), then there is an \((S, Y)\)-coherent adeq. \( P \subseteq H(\omega_2) \) preserving \( \omega_1, \omega_2, CH \) and s.t. \( V^P \models S \) contains a club.

**Sketch of proof.** W.l.o.g. we may assume that \( S \cap \text{cof}(\omega_1) \) is stationary and that for all \( \alpha \in S \cap \text{cof}(\omega_1) \), \( S \cap \alpha \) contains a closed cofinal subset of \( \alpha \).

Let \( \lambda = \omega_2 \) and let \( Y \) code \( S \) together with a well-order of \( H(\omega_2) \). In particular, isomorphisms between members of \( \mathcal{X} \) preserve membership in \( S \).
Recall that a stationary set $S \subseteq \omega_2$ is said to be fat iff for every club $C \subseteq \omega_2$, $S \cap C$ contains a closed subset with order type $\omega_1 + 1$.

**Corollary**

Assume CH. If $S \subseteq \omega_2$ is fat stationary (for every club $C \subseteq \omega_2$, $S \cap C$ contains a closed subset with order type $\omega_1 + 1$), then there is an $(S, \mathcal{Y})$-coherent adeq. $P \subseteq H(\omega_2)$ preserving $\omega_1, \omega_2$, CH and s.t. $V^P \models S$ contains a club.

**Sketch of proof.** W.l.o.g. we may assume that $S \cap \operatorname{cof}(\omega_1)$ is stationary and that for all $\alpha \in S \cap \operatorname{cof}(\omega_1)$, $S \cap \alpha$ contains a closed cofinal subset of $\alpha$.

Let $\lambda = \omega_2$ and let $\mathcal{Y}$ code $S$ together with a well-order of $H(\omega_2)$. In particular, isomorphisms between members of $\mathcal{X}$ preserve membership in $S$. 
Let $\mathcal{Y}$ denote the stationary set of $M \in \mathcal{X}$ such that for all $\alpha \in (M \cap S) \cup \{\omega_2\}$, $\sup(M \cap \alpha) \in S$.

If $N \cap \omega_2 \notin \alpha$, let $\alpha_N := \min(N \setminus \alpha)$.

$P$ is the poset consisting of conditions $p = (x_p, A_p)$ satisfying:

(i) $x_p$ is a finite set of nonoverlapping pairs whose first coordinate is in $S$,

(ii) $A_p$ is a finite coherent adequate subset of $\mathcal{Y}$,

(iii) if $\langle \alpha, \alpha' \rangle \in x_p$, $N \in A_p$ and $N \cap \omega_2 \notin \alpha$, then $N \cap [\alpha, \alpha'] \neq \emptyset$ implies $\alpha, \alpha' \in N$, and $N \cap [\alpha, \alpha'] = \emptyset$ implies $\langle \alpha_N, \alpha_N \rangle \in x_p$,

(iv) if $\gamma$ in $R_{A_p}$, then $\langle \gamma, \gamma \rangle \in x_p$, and

(v) $p$ is symmetric