Subsymmetric sequences in large Banach spaces

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Introduction

A sequence (x_k) in a Banach space X is subsymmetric if there is $C \geq 1$ such that for all $(\lambda_i)_{i=1}^l$ and all increasing sequences $(k_i)_{i=1}^l$ and $(n_i)_{i=1}^l$ we have that

$$\|\sum_{i=1}^{l} \lambda_i x_{k_i}\| \leq C \|\sum_{i=1}^{l} \lambda_i x_{n_i}\|.$$

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Ramsey principles imply that large uncountable structures have infinite **indiscernible** sequences.

Questions

- What is the minimal cardinal κ such that any Banach space of density κ has a subsymmetric sequence?
- What is the minimal cardinal κ such that any reflexive Banach space of density κ has a subsymmetric sequence?

Define

$\mathfrak{ns} = \min\{\kappa : \text{every Banach space of density } \kappa \text{ has a subsymmetric seq.}\}$

and

$\mathfrak{ns}_{refl} = \min\{\kappa : \text{every refl. Banach space of density } \kappa \text{ has a subsymmetric seq.}\}.$

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- B., Lopez-Abad, Todorcevic, 2014: For every κ smaller than the first inaccessible cardinal, there is a reflexive Banach space of density κ with no subsymmetric sequences.

Large, compact, hereditary families

Given an index set I, a family $\mathcal{F}$ of finite subsets of I containing the singletons is said to be:

- **hereditary** if $t \subseteq s \in \mathcal{F}$ implies $t \in \mathcal{F}$;
- **compact** if is compact as a subspace of 2^I;
- **large** if for every infinite set M of I and every $k \geq 1$, $\mathcal{F} \cap [M]^k \neq \emptyset$.

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Given a large compact and hereditary family $\mathcal{F}$ on I, define in $c_{00}(I)$ the following norm:

$$\|x\|_{\mathcal{F}} = \max\{\|x\|_\infty, \sup\{\sum_{n\in s} |x_n| : s \in \mathcal{F}\}\}.$$

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Let $X_{\mathcal{F}}$ be the completion of $(c_{00}(\kappa), \|\cdot\|_{\mathcal{F}})$.

Large, compact, hereditary families

Theorem (Lopez-Abad, Todorcevic, 2013)

Given an infinite cardinal κ, TFAE:

- *κ is not ω-Erdös;*
- *there is a non-trivial weakly-null sequence $(x_\alpha)_{\alpha<\kappa}$ with no subsymmetric basic subsequence;*
- *there are large compact and hereditary families on κ.*

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However, the space $X_{\mathcal{F}}$ has subsymmetric subsequences.

Ingredients

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- Interpolation method
- Tsirelson space
- CL-sequences

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- e_α the element of $c_{00}(\kappa)$ which values 1 at α and 0 elsewhere.

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- $\| \cdot \|_X$ a norm on $c_{00}(\omega)$ such that (e_n) is a 1-unconditional basic sequence in the completion X of $(c_{00}(\omega), \| \cdot \|_X)$;
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Let $\mathfrak{X}$ be the completion of $(c_{00}(\kappa), \|\cdot\|_{\mathfrak{X}})$. (e_α) is a C-unconditional basis.

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- $\mathfrak{X}$ contains no copies of c_0 or ℓ_1 and conclude that $\mathfrak{X}$ is reflexive;
- any subsymmetric sequence has
  - either a "relevant part" which is in one of the X_n's;
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- any subsymmetric sequence has
  - either a "relevant part" which is in one of the X_n's;
  - or a "relevant part" which is in T;
- the second alternative cannot hold;
- the first alternative would give us a subsymmetric weakly null disjointly supported sequence in X_n, which in turn will give us a sequence in X_{n+1} equivalent to the unit basis of ℓ_1.

CL-sequences

$\mathcal{F}$ is $\mathcal{G}$-**large** if every infinite sequence (s_n) in $\mathcal{G}$ has an infinite subsequence $(s_n)_{n\in M}$ such that $\bigcup_{i\in t} s_i \in \mathcal{F}$ for every $t \in \mathcal{S}$, where $\mathcal{S}$ is the Schreier family.
A sequence of families $(\mathcal{F}_n)$ is **consecutively large** (CL) if:

- $\mathcal{F}_0 = [\kappa]^{\leq}1$;
- each $\mathcal{F}_n$ is compact and hereditary;
- $\mathcal{F}_n \subseteq \mathcal{F}_{n+1}$;
- $\mathcal{F}_{n+1}$ is $\mathcal{F}_n$-large.

CL-sequences

Example:

- $\mathcal{F}_0 = [\omega]^{\leq 1}$
- $\mathcal{F}_{n+1} = \mathcal{F}_n \otimes ([\omega]^{\leq 1} \oplus \mathcal{S})$

$$\mathcal{F} \oplus \mathcal{G} = \{s \cup t : s \in \mathcal{F}, t \in \mathcal{G} \text{ and } \max s < \min t\}$$

$$\mathcal{F} \otimes \mathcal{G} = \{s_1 \cup \cdots \cup s_n : (s_i) \subseteq \mathcal{F}, \quad \max s_i < \min s_{i+1}$$

$$\text{and } \{\min s_1, \ldots, \min s_n\} \in \mathcal{G}\}$$

Stepping up from κ to 2^κ

If $\mathcal{F}$ is a family on κ and T is a tree of height κ, let

$$\mathcal{C}(\mathcal{F}) = \{c \subseteq T : c \text{ is a chain and } \{ht(t) : t \in c\} \in \mathcal{F}\}.$$

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$$\mathcal{C}(\mathcal{F}) = \{c \subseteq T : c \text{ is a chain and } \{ht(t) : t \in c\} \in \mathcal{F}\}.$$

- If $\mathcal{F}$ is hereditary, then $\mathcal{C}(\mathcal{F})$ is hereditary.
- If $\mathcal{F}$ is compact, then $\mathcal{C}(\mathcal{F})$ is compact.
- If $(\mathcal{F}_n)$ is CL-sequence on κ, then $(\mathcal{C}(\mathcal{F}_n))$ is CL-sequence **on chains of** T.

Key lemma

Lemma

If T supports a CL-sequence on chains of T and the set of immediate successors of every node of T supports a CL-sequence, then T supports a CL-sequence.

- Given a family $\mathcal{C}$ on chains of T and, for each $t \in T$, a family $\mathcal{A}_t$ on the immediate successors of t, let $\mathcal{F}(\mathcal{C}, (\mathcal{A}_t)_{t \in T})$ be the family on T of all $s \subseteq T$ such that every chain in the "generated subtree" belongs to $\mathcal{C}$ and for every $t \in T$, the set of "immediate successors" of t with respect to s belong to $\mathcal{A}_t$.

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- Given a family $\mathcal{C}$ on chains of T and, for each $t \in T$, a family $\mathcal{A}_t$ on the immediate successors of t, let $\mathcal{F}(\mathcal{C}, (\mathcal{A}_t)_{t \in T})$ be the family on T of all $s \subseteq T$ such that every chain in the "generated subtree" belongs to $\mathcal{C}$ and for every $t \in T$, the set of "immediate successors" of t with respect to s belong to $\mathcal{A}_t$.
- Given CL-sequences $(\mathcal{C}_n)$ on chains of T and, for each $t \in T$, CL-sequences $(\mathcal{A}_t^n)$ on the immediate successors of t, we define suitable $(\bar{\mathcal{C}}_n)$ and $(\bar{\mathcal{A}}_t^n)$ such that $\mathcal{F}_n = \mathcal{F}(\bar{\mathcal{C}}_n, (\bar{\mathcal{A}}_t^n)_{t \in T})$ is a CL-sequence on T.

Final remarks

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For every κ smaller than the first ω-Mahlo cardinal, there is a reflexive Banach space of density κ with no subsymmetric sequences.

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So, first ω-Mahlo cardinal $\leq \mathfrak{ns}_{refl} \leq \omega$-Erdös cardinal.

Lemma

If a regular inaccessible cardinal κ supports a small C-sequence and every $\lambda < \kappa$ supports a CL-sequence, then κ supports a CL-sequence.