Poisson homology, D-modules on Poisson varieties, and complex singularities

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joint work with Travis Schedler

1. Preliminaries.

- X - an affine algebraic variety over $\mathbb{C}$ (possibly singular)
- O_X - the algebra of regular functions on X; O_X^* - the space of algebraic densities
- $\mathrm{Vect} X = \mathrm{Der}(O_X)$ - the Lie algebra of vector fields on X
- $\mathfrak{g} \subset \mathrm{Vect} X$ - a Lie subalgebra.
- $(O_X^*)^{\mathfrak{g}}$ - the space of $\mathfrak{g}$-invariant densities.

Main Question. When is $(O_X^*)^{\mathfrak{g}}$ finite dimensional? What is its dimension?

Main example. X is a Poisson variety, $\mathfrak{g} = \mathrm{HVect} X$ is the Lie algebra of Hamiltonian vector fields.

Note that $(O_X^*)^{\mathfrak{g}} = (O_X/\mathfrak{g}O_X)^*$, so the main question is equivalent to the same question about the space of coinvariants $O_X/\mathfrak{g}O_X$. In the main example, this is the space

$$HP_0(X) := O_X/\{O_X, O_X\},$$

called the zeroth Poisson homology of X.

2. $\mathfrak{g}$-leaves and the main theorem.

In smooth or analytic geometry, a $\mathfrak{g}$-leaf of a point $x \in X$ is defined as the set of points which one can reach from x moving along the vector fields from $\mathfrak{g}$. We want to extend this definition to the setting of algebraic geometry.

To this end, define $\mathfrak{g}_x \subset T_xX$ to be the subspace spanned by specializations at x of vector fields from $\mathfrak{g}$, and let X_i be the set of $x \in X$

such that $\dim \mathfrak{g}_x = i$. Then X_i is locally closed in X, and each irreducible component $X_{i,j}$ of X_i has dimension $\geq i$, since $\mathfrak{g}_x \subset T_x X_{i,j}$ for all i, j, $x \in X_{i,j}$.

Proposition 0.1. *Suppose that one has* $\dim X_{i,j} = i$ *for all* i, j. *Then* $X_{i,j}$ *are smooth and* $\mathfrak{g}_x = T_x X_{i,j}$ *for all* i, j, $x \in X_{i,j}$.

Definition 0.2. In this situation, we say that $X_{i,j}$ are the $\mathfrak{g}$-leaves of X, and that X has finitely many $\mathfrak{g}$-leaves. If X is Poisson and $\mathfrak{g} = \mathrm{HVect} X$ then $\mathfrak{g}$-leaves are called symplectic leaves.

Theorem 0.3. *(E-Schedler, 2009) If X has finitely many $\mathfrak{g}$-leaves, then $O_X/\mathfrak{g}O_X$ is finite dimensional. In particular, if X is Poisson and has finitely many symplectic leaves then $HP_0(X)$ is finite dimensional.*

3. Examples. Here are some examples where this theorem applies.

Example 0.4. X is connected symplectic of dimension n, $\mathfrak{g} = \mathrm{HVect}X$. In this case, X is the only symplectic leaf, and $HP_0(X) = H^n(X, \mathbb{C})$ by Brylinski's theorem and Grothendieck's algebraic de Rham theorem.

Example 0.5. Let $Y = X/G$, where X is as in the previous example, and

G is a finite group of symplectomorphisms of X. Then the symplectic leaves are the connected components of the sets of points with a given stabilizer, so there are finitely many of them and the theorem says that $HP_0(Y)$ is finite dimensional. In the case when $X = \mathbb{C}^{2n}$ and $G \subset Sp(2n, \mathbb{C})$, this was a conjecture of Alev and Farkas, proved by Berest, Ginzburg, and myself in 2004. The dimension of $HP_0(Y)$ is unknown even in this special case.

Example 0.6. Let $Q(x, y, z)$ be a polynomial, and X be the surface defined by the equation

$$Q(x, y, z) = 0.$$

Suppose that Q is quasihomogeneous and $0 \in X$ is an isolated singularity. Then

$$HP_0(X) = \mathbb{C}[x, y, z]/(Q_x, Q_y, Q_z),$$

the local ring of the singularity, which is finite dimensional. Its dimension is the Milnor number μ of the singularity.

This example extends to surfaces in $\mathbb{C}^N$, $N > 3$, which are complete intersections, as well as to complete intersections of dimensions $d > 2$ (in which case $\mathfrak{g}$ is replaced by the Lie algebra of divergence-free vector fields arising from $d - 2$-forms).

Example 0.7. As a generalization of the previous example, consider the case when Q is any polynomial (not necessarily quasihomogeneous), such that X has isolated singularities.

Proposition 0.8. *One has*

$$HP_0(X) = H^2(X, \mathbb{C}) \oplus \bigoplus_s \mathbb{C}^{\mu_s},$$

where the sum is over singular points of X, and μ_s is the Milnor number of s.

Example 0.9. Let Q be quasihomogeneous, and consider the symmetric power $S^n X$ of the surface X defined by the equation $Q = 0$. For a partition $\lambda = (\lambda_1, ..., \lambda_m)$ of n, let $S_\lambda \subset S_m$ be the stabilizer of the vector λ. Let $S^\lambda V := (V^{\otimes m})^{S_\lambda}$.

Proposition 0.10. *One has*

$$HP_0(S^n X) = \bigoplus_\lambda S^\lambda HP_0(X).$$

For example, if $HP_0(X) = R$, then $HP_0(S^2X) = S^2R \oplus R$, $HP_0(S^3X) = S^3R \oplus R \otimes R \oplus R$, etc. For the generating functions, we have

$$\sum_{n \geq 0} \dim HP_0(S^nX)[i] z^i q^n =$$

$$\prod_i \prod_{n \geq 1} (1 - z^i q^n)^{-d_i},$$

where $d_i = \dim R[i]$.

Conjecture 0.11. If $\widetilde{X} \to X$ is a (homogeneous) symplectic resolution of dimension n (i.e., a birational map such that $\widetilde{X}$ is symplectic), then

$$\dim HP_0(X) = \dim H^n(\widetilde{X}, \mathbb{C}).$$

Only $\geq$ is known. By the last example, the Conjecture holds for symmetric powers of ADE singularities.

It also holds for Slodowy slices and hypertoric varieties, but is open for quiver varieties.

4. Idea of proof of the theorem.

The proof of the theorem is based on the theory of D-modules. Recall that $X \subset V = \mathbb{C}^n$. By a D-module on X we mean a module over the algebra D_V of differential operators on V which is set-theoretically supported on X as an O_V-module. We define the right D-module

$$M = M_{X,\mathfrak{g}} := (I_X D_V + \widetilde{\mathfrak{g}} D_V) \backslash D_V,$$

where $I_X \subset O_V$ is the ideal of X, and $\widetilde{\mathfrak{g}}$ is the Lie algebra of vector fields on V that are parallel to X and restrict on X to elements of $\mathfrak{g}$.

The proof is based on the following facts:

• The space $O_X/\mathfrak{g}O_X$ is the top de Rham cohomology of M, i.e.

$$O_X/\mathfrak{g}O_X = M \otimes_{D_V} O_V.$$

• M is a holonomic D-module (its singular support is the union of the conormal bundles of the $\mathfrak{g}$-leaves, i.e., is Lagrangian, since there are finitely many $\mathfrak{g}$-leaves).

• The cohomology of a holonomic D-module is holonomic (a standard theorem in D-module theory).