Algebro-geometric approach to the Schlesinger equations
with V. Shramchenko

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Legacy of V. I. Arnold, Fields Institute, Toronto, November 25, 2014
The title could be "On a solution of a differential equation..." as suggested by V. I. Arnold.
Six Painlevé equations

- Paul Painlevé (1863-1933) classified all second order ODEs of the form $\frac{d^2y}{dx^2} = F\left(\frac{dy}{dx}, y, x\right)$ with $F$ rational in the first two arguments whose solutions have no movable singularities.

- Six new equations which cannot be solved in terms of known special functions.

- The sixth Painlevé equation, PVI, is the most general of them: $\text{PVI}(\alpha, \beta, \gamma, \delta)$.

$$\frac{d^2y}{dx^2} = \frac{1}{2} \left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x}\right) \left(\frac{dy}{dx}\right)^2 - \left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x}\right) \frac{dy}{dx}$$

$$+ \frac{y(y-1)(y-x)}{x^2(x-1)^2} \left(\alpha + \beta \frac{x}{y^2} + \gamma \frac{x-1}{(y-1)^2} + \delta \frac{x(x-1)}{(y-x)^2}\right).$$
Poncelet problem

- $C$ and $D$ are two smooth conics in $\mathbb{CP}^2$
- Question: Is there a closed trajectory inscribed in $C$ and circumscribed about $D$?
- Poncelet Theorem: Let $x \in C$ be a starting point. The Poncelet trajectory originating at $x$ closes up after $n$ steps iff so does a Poncelet trajectory originating at any other point of $C$. 
Solution of Poncelet problem
Griffiths, P., Harris, J., On Cayley’s explicit solution to Poncelet’s porism (1978)

* Let $C$ and $D$ be symmetric $3 \times 3$ matrices defining the conics $C$ and $D$ in $\mathbb{CP}^2$.

* $E = \{(x, y) \in \mathbb{CP}^1 \times \mathbb{CP}^1 : x \in C, y \in D^*, x \in y\}$ is an elliptic curve of the equation $v^2 = \det(D + uC)$.

* A closed Poncelet trajectory of length $k$ exists for two conics $C$ and $D$ iff the point $(u, v) = (0, \sqrt{\det D})$ is of order $k$ on $E$.

* $k \mathcal{A}_\infty(Q_0) \equiv 0 \iff \exists f \in L(-kP_\infty)$ with zero of order $k$ at $Q_0$. 
Hitchin’s work

Hitchin, N. Poncelet polygons and the Painlevé equations (1992)

For two conics and a Poncelet trajectory of length \( k \) there is an associated algebraic solution of \( PVI\left(\frac{1}{8}, -\frac{1}{8}, \frac{1}{8}, \frac{3}{8}\right) \).

- Existence of the Poncelet trajectory of length \( k \) implies \( kz_0 \equiv 0 \). \( (z_0 := 2w_1 \frac{m_1}{k} + 2w_2 \frac{m_2}{k} \). \)
- \( z_0 = A_\infty(Q_0) \), where \( A_\infty \) is the Abel map based at \( P_\infty \).
- A function \( g(u, v) \) on the curve \( v^2 = u(u - 1)(u - x) \) having a zero of order \( k \) at \( Q_0 \) and a pole of order \( k \) at \( P_\infty \).
Hitchin’s work

Hitchin, N. Poncelet polygons and the Painlevé equations (1992)

- The function

\[ s(u, v) = \frac{g(u, v)}{g(u, -v)} \]

has a zero of order \( k \) at \( Q_0 \) and a pole of order \( k \) at \( Q_0^* \) and no other zeros or poles.

- \( ds \) has exactly two zeros away from \( Q_0 \) and \( Q_0^* \).

- These two zeros are paired by the elliptic involution.

- Their \( u \)-coordinate as a function of \( x \) solves

\[ PVI\left(\frac{1}{8}, -\frac{1}{8}, \frac{1}{8}, \frac{3}{8}\right). \]
Picard solution to PVI \((0, 0, 0, \frac{1}{2})\)

- Transformed \(\wp\) satisfies:
  
  \[ (\wp'(z))^2 = \wp(z) (\wp(z) - 1) (\wp(z) - x). \]

- Define
  
  \[ z_0 := 2w_1 c_1 + 2w_2 c_2. \]

- \(z_0 = A_\infty(Q_0).\)

- Picard’s solution to PVI \((0, 0, 0, \frac{1}{2})\):
  
  \[ y_0(x) = \wp(z_0(x)). \]
Hitchin’s solution of PVI\(\left(\frac{1}{8}, -\frac{1}{8}, \frac{1}{8}, \frac{3}{8}\right)\)

Twistor spaces, Einstein metrics and isomonodromic deformations (1995)

\[
y(x) = \frac{\theta'''_1(0)}{3\pi^2\theta^4_4(0)\theta'_1(0)} + \frac{1}{3} \left( 1 + \frac{\theta^4_3(0)}{\theta^4_4(0)} \right) \\
\quad + \frac{\theta'''_1(\nu)\theta_1(\nu) - 2\theta''_1(\nu)\theta'_1(\nu) + 4\pi i c_2 [\theta'''_1(\nu)\theta(\nu) - \theta''_1(\nu)]}{2\pi^2\theta^4_4(0)\theta_1(\nu)[\theta'_1(\nu) + 2\pi i c_2 \theta_1(\nu)]}.
\]

Here \(\nu = c_2 \tau + c_1\) with \(\tau = \frac{w_2}{w_1}\); and

\[
x = \frac{\theta^4_3(0)}{\theta^4_4(0)}.
\]
Okamoto transformations \( \sim 1980 \)

- a group of symmetries of \( \text{PVI}(\alpha, \beta, \gamma, \delta) \).

Lemma (V. D., V. Shramchenko): Okamoto transformation from \( \text{PVI}(0, 0, 0, \frac{1}{2}) \) to \( \text{PVI}(\frac{1}{8}, -\frac{1}{8}, \frac{1}{8}, \frac{3}{8}) \):

\[ y_0 \text{ - Picard's solution} \]

\[ y \text{ - Hitchin's solution} \]

\[ y(x) = y_0 + \frac{y_0(y_0 - 1)(y_0 - x)}{x(x - 1)y'_0 - y_0(y_0 - 1)}. \]
Our construction

- \( z_0 = 2w_1c_1 + 2w_2c_2, \quad z_0 = A_\infty(Q_0), \quad y_0(x) = \wp(z_0(x)). \)
- Differential of the third kind on the elliptic curve \( C: \)
  \[
  \Omega(P) = \Omega_{Q_0, Q_0^*}(P) - 4\pi i c_2 \omega(P).
  \]
- \( \omega(P) \) -holomorphic normalized differential on \( C \) in terms of \( z \) has the form: \( \omega = \frac{dz}{2w_1} \).
- \( \Omega \) has two simple poles at \( Q_0 \) et \( Q_0^* \) which project to \( y_0 \), Picard’s solution of PVI \((0, 0, 0, 1/2)\).
- \( \Omega \) has two simple zeros at \( P_0 \) et \( P_0^* \) which project to \( y \), Hitchin’s solution of PVI\((\frac{1}{8}, -\frac{1}{8}, \frac{1}{8}, \frac{3}{8})\).
Ω_{Q_0, Q_0*} as the Okamoto transformation

- Write the differential Ω in terms of the coordinate u:

\[ \Omega(P) = \frac{\omega(P)}{\omega(Q_0)} \left[ \frac{1}{u(P) - y_0} - \frac{l}{2w_1} \right] - 4\pi ic_2\omega(P). \]

where \( l = \oint_{a} \frac{du}{(u-y_0)\sqrt{u(u-1)(u-x)}}. \)

\( y = u(P) \) is projection of zeros of \( \Omega \) iff

\[ \frac{1}{y - y_0} = \frac{l}{2w_1} + 4\pi ic_2\omega(Q_0). \]

- By differentiating the relation \( \int_{P_\infty}^{Q_0} \omega = c_1 + c_2\tau \) with respect to \( x \) we find the derivative \( \frac{dy_0}{dx} : \)

\[ \frac{dy_0}{dx} = -\frac{1}{4} \Omega(P_x) \frac{\omega(P_x)}{\omega(Q_0)} \]

\[ = \frac{1}{4} \frac{\omega^2(P_x)}{\omega^2(Q_0)} \left[ 4\pi ic_2\omega(Q_0) - \frac{1}{x - y_0} + \frac{l}{2w_1} \right]. \]
\( \Omega_{Q_0, Q_0^*} \) as the Okamoto transformation

- Thus we get for the relationship between \( y \) and \( y_0 \):
  \[
  \frac{1}{y - y_0} = 4 \frac{\omega^2(Q_0)}{\omega^2(P_x)} \frac{dy_0}{dx} + \frac{1}{x - y_0}.
  \]

- The holomorphic normalized differential in terms of the \( u \)-coordinate has the form
  \[
  \omega(P) = \frac{du}{2w_1 \sqrt{u(u - 1)(u - x)}}.
  \]

- Therefore
  \[
  \omega(P_x) = \frac{2}{2w_1 \sqrt{x(x - 1)}} \quad \text{and} \quad \omega(Q_0) = \frac{1}{2w_1 \sqrt{y_0(y_0 - 1)(y_0 - x)}}.
  \]

- Okamoto transformation:
  \[
  y(x) = y_0 + \frac{y_0(y_0 - 1)(y_0 - x)}{x(x - 1)y'_0 - y_0(y_0 - 1)}.
  \]
Remark on $\frac{dy_0}{dx}$

$y_0(x) = \varphi(z_0(x))$ - the Picard solution to PVI $(0, 0, 0, \frac{1}{2})$

$$\frac{dy_0}{dx} = -\frac{1}{4} \Omega(P_x) \frac{\omega(P_x)}{\omega(Q_0)}$$

$$(z_0 = 2w_1 c_1 + 2w_2 c_2) \quad \Omega(P) = \Omega_{Q_0, Q_0^*}(P) - 4\pi i c_2 \omega(P)$$
Normalization of the differential $\Omega$

$z_0 = 2w_1 c_1 + 2w_2 c_2$.

$\Omega(P) = \Omega_{Q_0, Q_0^*}(P) - 4\pi i c_2 \omega(P)$.

The constants $c_1$ and $c_2$ determine the periods of $\Omega$:

$\int_a \Omega = -4\pi i c_2 \quad \int_b \Omega = 4\pi i c_1$.

$\Omega$ does not depend on the choice of $a$- and $b$-cycles.

Therefore our construction is global on the space of elliptic two-fold coverings of $\mathbb{C}P^1$ ramified above the point at infinity.
Schlesinger system (four points)

- Linear matrix system

\[
\frac{d\Phi}{du} = A(u)\Phi, \quad A(u) = \frac{A^{(1)}}{u} + \frac{A^{(2)}}{u - 1} + \frac{A^{(3)}}{u - x}
\]

\(u \in \mathbb{C}, \Phi \in M(2, \mathbb{C}), A \in sl(2, \mathbb{C})\)

- Isomonodromy condition (Schlesinger system)

\[
\frac{dA^{(1)}}{dx} = \frac{[A^{(3)}, A^{(1)}]}{x},
\]

\[
\frac{dA^{(2)}}{dx} = \frac{[A^{(3)}, A^{(2)}]}{x - 1},
\]

\[
\frac{dA^{(3)}}{dx} = -\frac{[A^{(3)}, A^{(1)}]}{x} - \frac{[A^{(3)}, A^{(2)}]}{x - 1}.
\]

\(A^{(1)} + A^{(2)} + A^{(3)} = \text{const.}\)
Solution to the Schlesinger system (four points)

- By conjugating, assume \( A^{(1)} + A^{(2)} + A^{(3)} = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} \).

- Then the term \( A_{12} \) is of the form:

\[
A_{12}(u) = \kappa \frac{(u - y)}{u(u - 1)(u - x)}
\]

- The zero \( y \) as a function of \( x \) satisfies the

\[
\text{PVI} \left( \frac{(2\lambda - 1)^2}{2}, \ -\text{tr}(A^{(1)})^2, \ \text{tr}(A^{(2)})^2, \ \frac{1 - 2\text{tr}(A^{(3)})^2}{2} \right)
\]

- For \( \text{PVI}(\frac{1}{8}, -\frac{1}{8}, \frac{1}{8}, \frac{3}{8}) \) \( \lambda = -\frac{1}{4} \). Our construction implies

\[
A_{12}(u) = \frac{\Omega(P)}{\omega(P)} \frac{(u - y_0)}{u(u - 1)(u - x)}, \quad P \in \mathcal{L}, \quad u = u(P).
\]
Solution to the Schlesinger system (four points)

Let $\phi(P) = \frac{du}{\sqrt{u(u-1)(u-x)}}$ - a non-normalized holomorphic differential.

$$A_{12}^{(1)} = -\frac{1}{4} y_0 \Omega(P_0) \phi(P_0), \quad \beta_1 := -\frac{y_0}{4} (\Omega(P_0))^2,$$

$$A_{12}^{(2)} = \frac{1}{4} (1 - y_0) \Omega(P_1) \phi(P_1), \quad \beta_2 := \frac{1 - y_0}{4} (\Omega(P_1))^2,$$

$$A_{12}^{(3)} = \frac{1}{4} (x - y_0) \Omega(P_x) \phi(P_x), \quad \beta_3 := \frac{x - y_0}{4} (\Omega(P_x))^2.$$

Then the following matrices solve the Schlesinger system

$$A^{(i)} := \begin{pmatrix}
-\frac{1}{4} - \frac{\beta_i}{2} & A_{12}^{(i)} \\
-\frac{1}{4} \frac{\beta_i + \beta_i^2}{A_{12}^{(i)}} & \frac{1}{4} + \frac{\beta_i}{2}
\end{pmatrix}, \quad i = 1, 2, 3.$$

Eigenvalues of matrices $A^{(i)}$ are $\pm 1/4$.

Generalization to hyperelliptic curves

Let $z_0 \in \text{Jac}(\mathcal{L})$, $z_0 = c_1 + c_2^t \mathcal{B}$, and $\sum_{j=1}^{g} A_\infty(Q_j) = z_0$. Define the differential

$$\Omega(P) = \sum_{j=1}^{g} \Omega_{Q_j} Q_j^*(P) - 4\pi i c_2^t \omega(P).$$

Let $q_j = u(Q_j)$. Then

$$\frac{\partial q_j}{\partial u_k} = -\frac{1}{4} \Omega(P_k)v_j(P_k),$$

where

$$v_j(P) = \frac{\phi(P) \prod_{\alpha=1, \alpha \neq j}^{g} (u - q_\alpha)}{\phi(Q_j) \prod_{\alpha=1, \alpha \neq j}^{g} (q_j - q_\alpha)}, \quad j = 1, \ldots, g$$
Normalization of the differential $\Omega$

$$\Omega(P) = \sum_{j=1}^{g} \Omega_{Q_j Q_j^t}(P) - 4\pi i \, c_2^t \omega(P)$$

where $z_0 = c_1 + c_2^t \mathbb{B}$ and $\sum_{j=1}^{g} A_\infty(Q_j) = z_0$;

c_1, c_2 \in \mathbb{R}^g.$

- The constant vectors $c_1 = (c_{11}, \ldots, c_{1g})^t$ and $c_2 = (c_{21}, \ldots, c_{2g})^t$ determine the periods of $\Omega$:

$$\int_{a_k} \Omega = -4\pi i c_{2k} \quad \int_{b_k} \Omega = 4\pi i c_{1k}.$$  

- $\Omega$ does not depend on the choice of $a$- and $b$-cycles.
Schlesinger system \((n\) points\)

\[
\frac{d\Phi}{du} = A(u)\Phi, \quad A(u) = \sum_{j=1}^{2g+1} \frac{A(j)}{u - u_j},
\]

where \(u \in \mathbb{C}, \Phi(u) \in \text{M}(2, \mathbb{C}), \ A^{(j)} \in \text{sl}(2, \mathbb{C})\).

- Schlesinger system for residue-matrices \(A^{(i)} \in \text{sl}(2, \mathbb{C})\):

  \[
  \frac{\partial A^{(j)}}{\partial u_k} = \frac{[A^{(k)}, A^{(j)}]}{u_k - u_j}, \quad A^{(1)} + \cdots + A^{(2g+1)} = -A^{(\infty)} = \text{const}
  \]

- by removing the conjugation freedom assume

\[
A^{(\infty)} = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}.
\]
Solution to the Schlesinger system ($n$ points)

- Let $\phi(P) = \frac{du}{\sqrt{\prod_{i=1}^{2g+1} (u-u_i)}}$ - a non-normalized holomorphic differential.

- Use the differential $\Omega$ to construct an analogue of $A_{12}$ in the hyperelliptic case

$$A_{12}(u) = \frac{\Omega(P) \prod_{\alpha=1}^{g} (u - q_{\alpha})}{\phi(P) \prod_{j=1}^{2g+1} (u - u_j)},$$

- Its residues at the simple poles:

$$A_{12}^{(n)} = \frac{\kappa}{4} \Omega(P_n) \phi(P_n) \prod_{\alpha=1}^{g} (u_n - q_{\alpha}). \quad (1)$$

- Introduce the following quantities:

$$\beta_n := \frac{1}{4} \Omega(P_n) \sum_{j=1}^{g} v_j(P_n) - \frac{1}{2} \Omega(\infty) A_{12}^{(n)}.$$
The following matrices $A^{(i)}$ with $i = 1, \ldots, 2g + 1$ solve the Schlesinger system

$$A^{(i)} := \begin{pmatrix} \frac{-1}{4} - \frac{\beta_i}{2} & A^{(i)}_{12} \\ -\frac{1}{4} \frac{\beta_i + \beta_i^2}{A^{(i)}_{12}} & \frac{1}{4} + \frac{\beta_i}{2} \end{pmatrix};$$

$$A^{(1)} + \cdots + A^{(2g+1)} = -A^{(\infty)} = \begin{pmatrix} -1/4 & 0 \\ 0 & 1/4 \end{pmatrix}.$$


Zeros of $\Omega$ are zeros of $A_{12}(u)$ and are solutions of the multidimensional Garnier system.
Consider the case of a point $z_0$ with rational coordinates $c_1, c_2 \in \mathbb{Q}^g$ with respect to the Jacobian of the hyperelliptic curve of genus $g$. It corresponds to a periodic trajectory of a billiard ordered game associated to $g$ quadrics from a confocal family in $d = g + 1$ dimensional space. For billiard ordered games see V. Dragović, M. Radnović, JMPA 2006.