Vortex filament dynamics

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Legacy of Vladimir Arnold
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Outline

Vortex filaments

Natural questions in Hamiltonian dynamics

Hamiltonian PDEs

Invariant tori as critical points

Vortex filaments in $\mathbb{R}^3$

- **one vortex filament:**
  linear stationary, with uniform vortex strength $\gamma = 1$

$$b(s) = (0, 0, s)$$

It generates a flow field in $\mathbb{R}^3$ described by

$$\mathbf{u} = (\partial_{x_2}\psi, -\partial_{x_1}\psi, 0)$$

where the velocity field is given by a stream function

$$\psi = \frac{1}{2}\log(x_1^2 + x_2^2) = \frac{1}{2}\log(|z|^2)$$

and $z = x_1 + ix_2$ are complex horizontal coordinates.

Vortex filament pairs

Two exactly parallel linear vortex filaments evolve as described by point vortices in $\mathbb{R}^2$

- Opposite vorticity $\gamma_1 = 1 = -\gamma_2$, initial configuration

$$b_1(s) = (\tfrac{1}{2}a + i0, s)\ , \qquad b_2(s) = (-\tfrac{1}{2}a + i0, s)$$

  then ballistic linear evolution

$$b_1(s,t) = (\tfrac{1}{2}a + i\frac{t}{a}, s)\ , \qquad b_2(s,t) = (-\tfrac{1}{2}a + i\frac{t}{a}, s)$$

- Same vorticity $\gamma_1 = 1 = \gamma_2$ with the above initial configuration have circular orbits with angular frequency $\omega = a^{-2}$

$$b_1(s,t) = (\tfrac{1}{2}ae^{it/a^2}, s)\ , \qquad b_2(s,t) = (\tfrac{1}{2}ae^{i(t/a^2+\pi)}, s)$$

- **Question:** Consider two near-vertical vortex filaments, slightly perturbed from exactly vertical. Do there persist similar orbital motions, whose time evolution is periodic or quasi-periodic.

  Configuration to be 2π periodic in the vertical x_3 variables.
- In 'center of vorticity' coordinates, the horizontal separation of the two vortex filaments is

$$w(s,t) = x_1(s,t) + ix_2(s,t)$$

  In a frame rotating with angular frequency ω

$$i\partial_t w + \partial_s^2 w - \omega w + \frac{w}{|w|^2} = 0 \qquad (1)$$

  the Klein, Majda & Damodaran model of near parallel vortex filaments.
- NB: For configurations which are greatly deformed from vertical, this is not an accurate approximation

Hamiltonian PDE

- This is a PDE in the form of a Hamiltonian system
  Set $w = a + v(s,t)$ with $a \in \mathbb{R}$ and $v(s,t)$ a perturbation term,

$$i\partial_t v + \partial_s^2 v - \omega(a+v) + \frac{a+v}{|a+v|^2} = 0 \qquad (2)$$

  by the choice $\omega = a^{-2}$ then $v = 0$ is stationary
- The Hamiltonian is

$$H = \int_0^{2\pi} \tfrac{1}{2}|\partial_s v|^2 + \frac{1}{2a^2}|a+v|^2 - \tfrac{1}{2}\log|a+v|^2 \, ds \qquad (3)$$

  Writing $v(s,t) = X(s,t) + iY(s,t)$ the dynamics are given by Hamilton's canonical equations

$$\begin{aligned} \partial_t X &= grad_Y H \\ \partial_t Y &= -grad_X H \end{aligned}$$

- Small $\|v\|_{H^1}$ solutions exist globally in time (C. Kenig, G. Ponce & L. Vega (2003), V. Banica & E. Miot (2012))

Linearized equations

- The tangent plane approximation is given by the linearization

  The linearized equations at equilibrium $(X, Y) = 0$ are derived from the quadratic Hamiltonian

$$H^{(2)} = \int_0^{2\pi} \tfrac{1}{2}\left[(\partial_s X)^2 + (\partial_s Y)^2 + \frac{2}{a^2}X^2\right] ds \qquad (4)$$

- Linearized equations

$$\partial_t X = grad_Y H^{(2)} = -\partial_s^2 Y$$
$$\partial_t Y = -grad_X H^{(2)} = \partial_s^2 X - \frac{2}{a^2}X$$

Linear flow

- Writing in a Fourier basis and using the Plancherel identity
  $X(s) = (1/\sqrt{2\pi}) \sum_{k\in\mathbb{Z}} \hat{X}_k e^{iks}$
  $Y(s) = (1/\sqrt{2\pi}) \sum_{k\in\mathbb{Z}} \hat{Y}_k e^{iks}$

$$H^{(2)} = \sum_{k\in\mathbb{Z}} \tfrac{1}{2}\Big((k^2 + \frac{2}{a^2})|\hat{X}_k|^2 + k^2|\hat{Y}_k|^2\Big)$$

  An infinte series of uncoupled harmonic oscillators, with frequencies $\omega_k(a) = \pm k\sqrt{k^2 + (2/a^2)}$.

- The solution operator, or the linear flow

$$\begin{aligned} \begin{pmatrix} X(s,t) \\ Y(s,t) \end{pmatrix} &= \Phi(t) \begin{pmatrix} X(s,0) \\ Y(s,0) \end{pmatrix} \\ &= \sum_{k\in\mathbb{Z}} e^{iks} \begin{pmatrix} \cos(\omega_k t) & k^2 \sin(\omega_k t)/\omega_k \\ -\omega_k \sin(\omega_k t)/k^2 & \cos(\omega_k t) \end{pmatrix} \begin{pmatrix} \hat{X}_k \\ \hat{Y}_k \end{pmatrix} \end{aligned}$$

  Angles evolve with linear motion $\theta_k \mapsto \theta_k + t\omega_k$

Elementary facts

- All solutions are Periodic, or Quasi-Periodic, or in general Almost Periodic functions of time
- More specifically, for initial data (X^0, Y^0) the active wavenumbers are $K := \{k \ : \ (\hat{X}_k^0, \hat{Y}_k^0) \neq 0\}$
  The dimension of the frequency basis is

$$m := \dim_{\mathbb{Q}}\big(\mathrm{span}_{\mathbb{Q}}\{\omega_k \ : \ k \in K\}\big)$$

- Orbit space consists of tori

$$\overline{\mathrm{orbit}}(X^0, Y^0) = \overline{\{\Phi(t)(X^0, Y^0) \ : \ t \in \mathbb{R}\}} = \mathbb{T}^m$$

Periodic (P): $m = 1$
Quasi-Periodic (QP): $1 < m < +\infty$
Almost Periodic (AP): $m = +\infty$

NB: For generic a then $\omega_k(a)$ satisfy $1 \leq m \leq +\infty$

Elementary facts

- Energy is conserved

$$H^{(2)}(X,Y) = \frac{1}{2}\sum_{k\in\mathbb{Z}} \overline{\begin{pmatrix}\hat{X}_k \\ \hat{Y}_k\end{pmatrix}} \begin{pmatrix} k^2 + \frac{2}{a^2} & 0 \\ 0 & k^2 \end{pmatrix} \begin{pmatrix}\hat{X}_k \\ \hat{Y}_k\end{pmatrix}$$

$$= \quad H^{(2)}(\Phi(t)(X,Y)) = \frac{1}{2}\sum_{k\in\mathbb{Z}} \overline{\begin{pmatrix}\hat{X}_k \\ \hat{Y}_k\end{pmatrix}} \Phi(t)^T \begin{pmatrix} k^2 + \frac{2}{a^2} & 0 \\ 0 & k^2 \end{pmatrix} \Phi(t) \begin{pmatrix}\hat{X}_k \\ \hat{Y}_k\end{pmatrix}$$

- Indeed each action variable is conserved

$$I_k = \frac{\sqrt{k^2 + (2/a^2)}}{2|k|}|X_k|^2 + \frac{|k|}{2\sqrt{k^2 + (2/a^2)}}|Y_k|^2$$

$$\frac{d}{dt}\left(\Phi_k(t)^T \begin{pmatrix} k^2 + \frac{2}{a^2} & 0 \\ 0 & k^2 \end{pmatrix} \Phi_k(t)\right) = 0$$

  Hence all Sobolev energy norms are preserved

$$H^{(2)} = \sum_{k\in\mathbb{Z}} \omega_k I_k \,, \qquad \|(X,Y)\|_r^2 := \sum_k |k|^{2r} I_k$$

Natural general questions

1. Whether any solutions of the nonlinear problem are Periodic, Quasi Periodic or Almost Periodic

   This refers to the KAM theory for PDEs

2. Whether the action variables $I_k(z)$ are approximately conserved (averaging theory), giving upper bounds on growth of action variables, or on higher Sobolev norms

   This is in the realm of averaging theory for PDEs, including Birkhoff normal forms and Nekhoroshev stability

3. Whether there exist some solutions which exhibit a growing lower bound on the growth of the action variables

   These would be cascade orbits, related to the question of Arnold diffusion

Results

Theorem (C Garcia, WC & CR Yang (2012))

There exist Cantor families of periodic (i.e. $m = 1$*) solutions of the vortex filament equations* (2) *near the uniformly rotating solution* $v = 0$

Theorem (C Garcia, WC & CR Yang (in progress))

Given wavenumbers $k_1, \dots k_m$ *there is a set* $a \in \mathcal{A}$ *of full measure and an* $\varepsilon_0 = \varepsilon_0(a, k_1, \dots k_m)$ *such that for a Cantor set of amplitudes* $(b_1, \dots b_m) \in B_{\varepsilon_0} \subseteq \mathbb{C}^m$ *there exist QP solutions of* (2) *with* m*-many* $\mathbb{Q}$ *independent frequencies* $\Omega_j(b)$*, of the form*

$$v(s,t) = \sum_{j=1}^{m} b_j e^{ik_j s} e^{i\Omega_j(b)t} + \mathcal{O}(\varepsilon^2)$$

Actually, these two theorems hold for any central configuration of vortices. The case of more complex configurations of near-vertical vortices is part of our future research program.

Hamiltonian PDEs

- Flow in *phase space*, where $z \in \mathcal{H}$ a Hilbert space with inner product $\langle X, Y \rangle_{\mathcal{H}}$,

$$\partial_t z = J grad_z H(z) \ , \quad z(x,0) = z^0(x) \ , \tag{5}$$

- Symplectic form

$$\omega(X,Y) = \langle X, J^{-1} Y \rangle_{\mathcal{H}} \ , \quad J^T = -J \ .$$

- The flow $\qquad z(x,t) = \varphi_t(z^0(x))$, defined for $z \in \mathcal{H}_0 \subseteq \mathcal{H}$

- Theorem
The flow of (5) *preserves the Hamiltonian function:*

$$H(\varphi_t(z)) = H(z) \ , \qquad z \in \mathcal{H}_0$$

Proof: $\frac{d}{dt} H(\varphi_t(z)) = \langle \mathrm{grad}_z H, \dot{z} \rangle = \langle \mathrm{grad}_z H, J \mathrm{grad}_z H(z) \rangle = 0 \ .$

Invariant tori

- Mapping a torus $S(\theta) : \mathbb{T}^m_\theta \mapsto \mathcal{H}$ to be flow invariant
$$S(\theta + t\Omega) = \varphi_t(S(\theta))$$
  Angles evolve linearly, with frequency vector $\Omega \in \mathbb{R}^m$
- This implies that both
$$\partial_t S = \Omega \cdot \partial_\theta S \ , \quad \text{and} \quad \partial_t S = J \, \mathrm{grad}_z H(S)$$
  hence
$$\Omega \cdot \partial_\theta S = J \, \mathrm{grad}_z H(S) \tag{6}$$
- Problem of KAM tori: Solve (6) for $(S(\theta), \Omega)$.
  This is generally a small divisor problem.

  Rewrite (6) in self-adjoint form
$$J^{-1}\Omega \cdot \partial_\theta S - \mathrm{grad}_z H(S) = 0 \ . \tag{7}$$

Invariant tori - linear theory - small divisors

- The tangent space approximation for the mapping S
  Linearize at S, set $\delta S = Z$ and use the self adjoint form

$$\Omega \cdot J^{-1} \partial_\theta Z - \partial_z^2 H(S) Z = F \tag{8}$$

  Frequencies of the linearized flow are $\omega_k = \pm k \sqrt{k^2 + (2/a^2)}$.

- The eigenvalues of (8), the linearized operator for a solution with m temporal quasi-periods $\Omega = (\Omega_1, \dots \Omega_m) \in \mathbb{R}^m$

$$\lambda_{jk}^{\pm} := k^2 + \frac{1}{a^2} \pm \sqrt{(\Omega \cdot j)^2 + \frac{1}{a^4}}$$

  Eigenvalues $\lambda_{jk}^{\pm}$ are the small divisors.

  Analysis: resolvant expansion methods developed by Fröhlich & Spencer, WC & Wayne, Bourgain, Berti & Bolle, …

Space of torus mappings

Consider the space of mappings $S \in \mathcal{X} := \{S(\theta) : \mathbb{T}^m \mapsto \mathcal{H}\}$

- Define average action functionals

$$\begin{aligned} \bar{I}_j(S) &= \frac{1}{2}\int_{\mathbb{T}^m} \langle S, J^{-1}\partial_{\theta_j} S\rangle \, d\theta \\ \delta_S \bar{I}_j &= J^{-1}\partial_{\theta_j} S \end{aligned}$$

  The moment map for *mappings*

- The average Hamiltonian

$$\begin{aligned} \overline{H}(S) &= \int_{\mathbb{T}^m} H(S(\theta))\, d\theta \\ \delta_S \overline{H} &= \mathrm{grad}_z H(S) \end{aligned}$$

A variational formulation

Consider the subvariety of $\mathcal{X}$ defined by fixed actions

$$\mathcal{M}_a = \{S \in \mathcal{X} : \overline{I}_1(S) = a_1, \dots \overline{I}_m(S) = a_m\} \subseteq \mathcal{X}$$

Variational principle: critical points of $\overline{H}(S)$ on $\mathcal{M}_a$ correspond to solutions of equation (7), with Lagrange multiplier Ω.

NB: All of $\overline{H}(S)$, $\overline{I}_j(S)$ and $\mathcal{M}_a$ are invariant under the action of the torus $\mathbb{T}^m$; that is $\tau_\alpha : S(\theta) \mapsto S(\theta + \alpha)\ ,\ \alpha \in \mathbb{T}^m$.

Two questions

- Two questions.
  1. Do critical points exist on $\mathcal{M}_a$?
     Note that the following operators are degenerate on the space of mappings $\mathcal{X}$:

     $$\Omega \cdot J^{-1}\partial_\theta S\,, \qquad \Omega \cdot J^{-1}\partial_\theta S - \delta_S^2 \overline{H}(0)$$

  2. How to understand questions of multiplicity of solutions?
- Answers – proposal in some cases:
  1. Use infinite dimensional KAM theory or the Nash – Moser method, with parameters
     The latter relies on solutions of the linearized equations, via resolvant expansions (Fröhlich – Spencer estimates)
  2. Morse – Bott theory of critical $\mathbb{T}^m$ orbits.

The linearized vortex filament equations

Illustrate this with the linearized vortex filament equations

- The quadratic Hamiltonian

$$H^{(2)} = \int_0^{2\pi} \tfrac{1}{2}\left[(\partial_s X)^2 + (\partial_s Y)^2 + \frac{2}{a^2}X^2\right] ds$$

  with frequencies $\omega_k = \pm k\sqrt{k^2 + (2/a^2)}$

- Linearized equations for an invariant torus

$$\Omega \cdot \partial_\theta X = grad_Y H^{(2)} = -\partial_s^2 Y$$

$$\Omega \cdot \partial_\theta Y = -grad_X H^{(2)} = \partial_s^2 X - \frac{2}{a^2}X$$

- Fourier representation of torus mappings $S(\theta) : \mathbb{T}^m \mapsto \mathcal{H}$

$$S(x,\theta) = \sum_{k\in\mathbb{Z}} S_k(\theta)e^{iks} = \sum_{k\in\mathbb{Z}, j\in\mathbb{Z}^m} S_{jk}e^{ij\cdot\theta}e^{iks}$$

  Eigenvalues $\lambda_{jk}^{\pm}(\Omega) = k^2 + \frac{1}{a^2} \pm \sqrt{(\Omega\cdot j)^2 + \frac{1}{a^4}}$

Null space

- Choose $(\omega_{k_1}, \dots \omega_{k_m})$ linear frequencies, and a frequency vector $\Omega^0 = (\Omega^0_1, \dots \Omega^0_m)$ solving the resonance relations

$$\lambda^-_{jk}(\Omega^0) = 0 \ .$$

- This identifies a null eigenspace in the space of mappings

$$\mathcal{X}_1 \subseteq \mathcal{X} \ .$$

Proposition

$\mathcal{X}_1 \subseteq \mathcal{X}$ *is even dimensional;* $\dim(\mathcal{X}_1) = 2M \geq 2m$. *It is possibly infinite dimensional*

- Nonresonant case: $M = m$.
- Resonant case: $M > m$.

Lyapunov - Schmidt decomposition

- Decompose $\mathcal{X} = \{S : \mathbb{T}^m \mapsto M\} = \mathcal{X}_1 \oplus \mathcal{X}_2 = Q\mathcal{X} \oplus P\mathcal{X}$.
- Equation (7) is equivalent to

$$Q\big(J^{-1}\Omega \cdot \partial_\theta S \quad - \quad \mathrm{grad}_z H(S)\big) = 0\ , \tag{9}$$

$$P\big(J^{-1}\Omega \cdot \partial_\theta S \quad - \quad \mathrm{grad}_z H(S)\big) = 0\ . \tag{10}$$

- Decompose the mappings $S = S_1 + S_2$ as well.
- Small divisor problem for $S_2 = S_2(S_1, \Omega)$, which one solves for $(S_1, \Omega) \in \mathcal{E}$ a Cantor set.

Variational problem reduced to a link

It remains to solve the Q-equation (9). This can be posed variationally (with analogy to Weinstein - Moser theory).

- Define

$$\begin{aligned} \bar{I}_j^1(S_1) &= \bar{I}_j(S_1 + S_2(S_1, \Omega)) \\ \overline{H}^1(S_1) &= \overline{H}(S_1 + S_2(S_1, \Omega)) \\ \mathcal{M}_a^1 &= \{S_1 \in \mathcal{X}_1 \ : \ \bar{I}_j^1(S_1) = a_j \ , \ j = 1 \ldots m\} \end{aligned}$$

- Critical points of $\overline{H}^1(S_1)$ on $\mathcal{M}_a^1$ are solutions of (9) with action vector a.

equivariant Morse – Bott theory

The group $\mathbb{T}^m$ acts on $\mathcal{M}_a^1$ leaving $\overline{H}^1(S_1)$ invariant.

One seeks critical $\mathbb{T}^m$ orbits.

Question: How many critical orbits of $\overline{H}^1$ on $\mathcal{M}_a^1$?
Depends upon its topology.

Conjecture (a reasonable guess)

For given a there exist integers $p_1, \ldots p_m$ such that $\sum_j p_j = M$ and

$$\mathcal{M}_a^1 \simeq \otimes_{j=1}^m \mathbb{S}^{2p_j - 1}$$

Check this fact, in endpoint cases.

- Periodic orbits $m = 1$, resonant case $M > 1$.

$$\mathcal{M}_a^1 \simeq \mathbb{S}^{2M-1} \,, \qquad \mathcal{M}_a^1/\mathbb{T}^1 \simeq \mathbb{C}P_w(M-1)$$

  This restates the estimate of Weinstein and Moser

$$\#\{\text{critical } \mathbb{T}^1 \text{ orbits}\} \geq M$$

- Nonresonant quasi-periodic orbits $m = M$.

$$\mathcal{M}_a^1 \simeq \otimes_{j=1}^{M} \mathbb{S}^1 \,, \qquad \mathcal{M}_a^1/\mathbb{T}^m \simeq \text{a point}$$

  In case this corresponds to a Lagrangian KAM torus
  Percival's variational principle.

equivariant Morse – Bott theory

- The case $m = 2 \leq M$ occurs in the problem of doubly periodic traveling wave patterns on the surface of water.

$$\mathcal{M}_a^1 \simeq \mathbb{S}^{2p-1} \otimes \mathbb{S}^{2(M-p)-1}$$

- The case $m = M - 1$

$$\mathcal{M}_a^1 \simeq \mathbb{S}^1 \otimes \cdots \otimes \mathbb{S}^3$$

topology of links

Theorem (Chaperon, Bosio & Meersmann (2006))

The topology of links $\mathcal{M}_a^1$ can be complex. There are cases in which

$$\mathcal{M}_a^1 \simeq \#_{\ell=1}^{q}(\mathbb{S}^{2p_{\ell 1}-1} \otimes \cdots \otimes \mathbb{S}^{2p_{\ell k}-1}) , \quad \sum_j p_{\ell j} = M$$

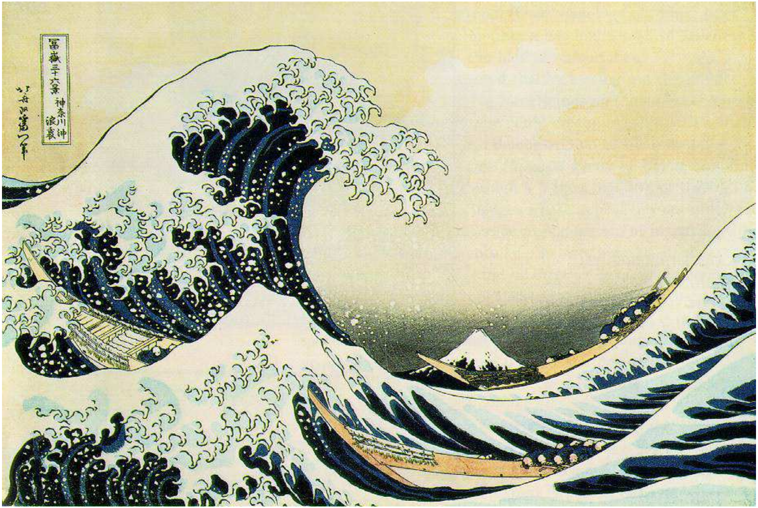
Furthermore, there are more complex quantities than this.

Proof: combinatorics and cohomolological calculations.

Conjecture (revised opinion)

The number of distinct critical $\mathbb{T}^m$ orbits of $\overline{H}^1$ on $\mathcal{M}_a^1$ is bounded below:

$$\#\{\text{critical orbits of } \overline{H}^1 \text{ on } \mathcal{M}_a^1\} \geq (M - m + 1) .$$



Thank you