Group Actions and Cohomology in the Calculus of Variations

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Focused Research Workshop on
Exterior Differential Systems and Lie Theory

Fields Institute, Toronto, Canada, December 2013

Example: Integrable systems

Potential Kadomtsev-Petviashvili (PKP) equation

$$u_{tx} + \frac{3}{2} u_x u_{xx} + \frac{1}{4} u_{xxxx} + \frac{3}{4} s^2 u_{yy} = 0, \qquad s^2 = \pm 1.$$

Admits an infinite dimensional algebra of distinguished symmetries $\mathfrak{g}_{PKP}$ involving 5 arbitrary functions of time t.

(David, Kamran, Levi, Winternitz, *Symmetry reduction for the Kadomtsev-Petviashvili equation using a loop algebra*, J. Math. Phys. **27** (1986), 1225–1237.)

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PKP EQUATION

The symmetry algebra $\mathfrak{g}_{PKP}$ is spanned by the vector fields

$$X_f = f\frac{\partial}{\partial t} + \frac{2}{3}yf'\frac{\partial}{\partial y} + (\frac{1}{3}xf' - \frac{2}{9}s^2y^2f'')\frac{\partial}{\partial x} + (-\frac{1}{3}uf' + \frac{1}{9}x^2f'' - \frac{4}{27}s^2xy^2f''' + \frac{4}{243}y^4f'''')\frac{\partial}{\partial u},$$

$$Y_g = g\frac{\partial}{\partial y} - \frac{2}{3}s^2yg'\frac{\partial}{\partial x} + (-\frac{4}{9}s^2xyg'' + \frac{8}{81}y^3g''')\frac{\partial}{\partial u},$$

$$Z_h = h\frac{\partial}{\partial x} + (\frac{2}{3}xh' - \frac{4}{9}s^2y^2h'')\frac{\partial}{\partial u},$$

$$W_k = yk\frac{\partial}{\partial u}, \quad \text{and} \quad U_l = l\frac{\partial}{\partial u},$$

where $f = f(t)$, $g = g(t)$, $h = h(t)$, $k = k(t)$ and $l = l(t)$ are arbitrary smooth functions of t.

PKP EQUATION

Locally variational with the Lagrangian

$$L = -\frac{1}{2}u_t u_x - \frac{1}{4}u_x^3 + \frac{1}{8}u_{xx}^2 - \frac{3}{8}s^2 u_y^2.$$

But the PKP equation admits no Lagrangian that is invariant under $\mathfrak{g}_{PKP}$!

To what extent do these properties characterize the PKP-equation?

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EXAMPLE: VECTOR FIELD THEORIES

One-form $A = A_b(x^i)\, dx^b$ on $\mathbb{R}^m$ satisfying

$$T^a = T^a(x^i, A_b, A_{b,i_1}, A_{b,i_1 i_2}, \ldots, A_{b,i_1 i_2 \cdots i_k}) = 0, \quad a = 1, 2, \ldots, m.$$

SYMMETRIES

S_1: spatial translations

$$x^i \to x^i + a^i, \quad (a^i) \in \mathbb{R}^m.$$

S_2: Gauge transformations

$$A_a(x^i) \to A_a(x^i) + \frac{\partial \phi}{\partial x^a}(x^i), \quad \phi \in C^\infty(\mathbb{R}^m).$$

CONSERVATION LAWS

C_1: There are functions $t^i_j = t^i_j(x^i, A_a, A_{a,i_1}, A_{a,i_1 i_2}, \ldots, A_{a,i_1 i_2 \cdots i_l})$ such that, for each $j = 1, 2, \ldots, m$,

$$A_{a,j} T^a = D_i(t^i_j).$$

C_2: The divergence of T^a vanishes identically,

$$D_a T^a = 0.$$

THEOREM (ANDERSON, P.)

Suppose that the differential operator T^a admits symmetries S_1, S_2 and conservation laws C_1, C_2. Then T^a arises from a variational principle, $T^a = E^a(L)$ for some Lagrangian L, if

(i) $m = 2$, and T^a *is of third order;*

(ii) $m \geq 3$, and T^a *is of second order;*

(iii) the functions T^a are polynomials of degree at most m in the field variables A_a and their derivatives.

NATURAL QUESTION: Can the Lagrangian L be chosen to be invariant under [S1], [S2]?

The goal is to reduce these type of questions into algebraic problems.

Smooth *fiber bundle*

$$\begin{array}{ccc} F & \longrightarrow & E \\ & & \big\downarrow \pi \\ & & M \end{array}$$

Adapted coordinates

$$\{(x^1, x^2, \ldots, x^m, u^1, u^2, \ldots, u^p)\} = \{(x^i, u^\alpha)\}$$

such that

$$\pi(x^i, u^\alpha) = (x^i).$$

A *local section* is a smooth mapping

$$\sigma : \mathcal{U}^{\mathrm{op}} \subset M \to E$$

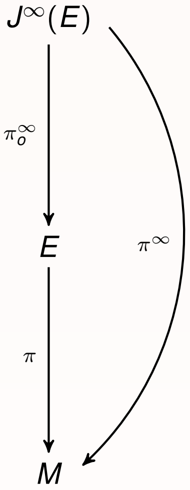
such that

$$\pi \circ \sigma = \mathrm{id}.$$

In adapted coordinates

$$\begin{aligned} &\sigma(x^1, x^2, \dots, x^m) \\ &= (x^1, x^2, \dots, x^m, f^1(x^1, x^2, \dots, x^m), \dots, f^p(x^1, x^2, \dots, x^m)). \end{aligned}$$

Infinite jet bundle of sections



Infinite jet bundle

Adapted coordinates $\Longrightarrow$ locally

$$J^\infty(E) \approx \{(x^i, u^\alpha, u^\alpha_{x^{j_1}}, u^\alpha_{x^{j_1}x^{j_2}}, \ldots, u^\alpha_{x^{j_1}x^{j_2}\ldots x^{j_k}}, \ldots)\}.$$

Often write

$$u^\alpha_{x^{j_1}x^{j_2}\ldots x^{j_k}} = u^\alpha_{j_1 j_2 \cdots j_k} = u^\alpha_J,$$

where $J = (j_1, j_2, \ldots, j_k)$, $1 \leq j_l \leq m$, is a *multi-index*.

Cotangent bundle of $J^\infty(E)$

Horizontal forms: $dx^1, dx^2, \dots, dx^m$.

Contact forms: $\theta^\alpha_J = du^\alpha_J - u^\alpha_{Jk} dx^k$.

The space of differential forms $\Lambda^*(J^\infty(E))$ on $J^\infty(E)$ splits into a direct sum of spaces of horizontal degree r and vertical (or contact) degree s:

$$\Lambda^*(J^\infty(E)) = \sum_{r,s\geq 0} \Lambda^{r,s}(J^\infty(E)).$$

Here $\omega \in \Lambda^{r,s}(J^\infty(E))$ is a finite sum of terms of the form

$$f(x^i, u^\alpha, u^\alpha_j, \dots, u^\alpha_J)\, dx^{k_1} \wedge \cdots \wedge dx^{k_r} \wedge \theta^{\alpha_1}_{L_1} \wedge \cdots \wedge \theta^{\alpha_s}_{L_s}.$$

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Horizontal and Vertical Differentials

The horizontal connection generated by the total derivative operators

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij_1}^\alpha \frac{\partial}{\partial u_{j_1}^\alpha} + u_{ij_1j_2}^\alpha \frac{\partial}{\partial u_{j_1j_2}^\alpha} + \cdots$$

is flat $\Longrightarrow$

The exterior derivative splits as

$$d = d_H + d_V,$$

where

$$d_H : \Omega^{r,s} \to \Omega^{r+1,s}, \qquad d_V : \Omega^{r,s} \to \Omega^{r,s+1}.$$

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Horizontal and Vertical Differentials

$$d_H f(x^i, u^\alpha, \ldots, u^\alpha_J) = \sum_{j=1}^{m} D_j f(x^i, u^\alpha, \ldots, u^\alpha_J) dx^j,$$

$$d_V f(x^i, u^\alpha, \ldots, u^\alpha_J) = \sum_{\beta=1}^{p} \sum_{|K|\geq 0} \frac{\partial f}{\partial u^\beta_K}(x^i, u^\alpha, \ldots, u^\alpha_J)\theta^\beta_K.$$

$$d^2 = 0 \qquad \Longrightarrow$$

$$d_H^2 = 0, \qquad d_V^2 = 0, \qquad d_H d_V + d_V d_H = 0.$$

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\begin{array}{ccccccccccc}
 & & \vdots & & \vdots & & & & \vdots & & \vdots \\
 & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} \\
0 & \longrightarrow & \Lambda^{0,1} & \xrightarrow{d_H} & \Lambda^{1,1} & \xrightarrow{d_H} & \cdots & & \Lambda^{m-1,1} & \xrightarrow{d_H} & \Lambda^{m,1} \\
 & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} \\
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 & & \uparrow{\scriptstyle \pi^*} & & \uparrow{\scriptstyle \pi^*} & & & & \uparrow{\scriptstyle \pi^*} & & \uparrow{\scriptstyle \pi^*} \\
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\end{array}
$$

Functional Forms

Define

$$\partial^I_\alpha u^\beta_J = \begin{cases} \delta^\beta_\alpha \delta^{(i_1}_{j_1} \cdots \delta^{i_k)}_{j_k}, & \text{if } |I| = |J|, \\ 0, & \text{otherwise.} \end{cases}$$

Interior Euler operator $F^I_\alpha \colon \Lambda^{r,s} \to \Lambda^{r,s-1}$, $s \geq 1$,

$$F^I_\alpha(\omega) = \sum_{|J| \geq 0} \binom{|I| + |J|}{|I|} (-D)_J (\partial^{IJ}_\alpha \lrcorner\, \omega).$$

Integration-by-parts operator $I : \Lambda^{m,s} \to \Lambda^{m,s}, \quad s \geq 1,$

$$I(\omega) = \frac{1}{s} \theta^\alpha \wedge F_\alpha(\omega).$$

Spaces of *functional s-forms* $\mathcal{F}^s = I(\Lambda^{m,s}), \quad s \geq 1.$

Differentials $\quad \delta_V = I \circ d_V \colon \mathcal{F}^s \to \mathcal{F}^{s+1}$. Then $\quad \delta_V^2 = 0.$

Functional Forms

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$$\partial_\alpha^I u_J^\beta = \begin{cases} \delta_\alpha^\beta \delta_{j_1}^{(i_1} \cdots \delta_{j_k}^{i_k)}, & \text{if } |I| = |J|, \\ 0, & \text{otherwise.} \end{cases}$$

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Free Variational Bicomplex

$$
\begin{array}{ccccccccccccc}
 & & \vdots & & \vdots & & & & \vdots & & \vdots & & \vdots \\
 & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle \delta_V} \\
0 & \longrightarrow & \Lambda^{0,2} & \xrightarrow{d_H} & \Lambda^{1,2} & \xrightarrow{d_H} & \cdots & & \Lambda^{m-1,2} & \xrightarrow{d_H} & \Lambda^{m,2} & \xrightarrow{I} & \mathcal{F}^2 \\
 & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle \delta_V} \\
0 & \longrightarrow & \Lambda^{0,1} & \xrightarrow{d_H} & \Lambda^{1,1} & \xrightarrow{d_H} & \cdots & & \Lambda^{m-1,1} & \xrightarrow{d_H} & \Lambda^{m,1} & \xrightarrow{I} & \mathcal{F}^1 \\
 & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & \nearrow{\scriptstyle E} & \\
\mathbb{R} & \longrightarrow & \Lambda^{0,0} & \xrightarrow{d_H} & \Lambda^{1,0} & \xrightarrow{d_H} & \cdots & & \Lambda^{m-1,0} & \xrightarrow{d_H} & \Lambda^{m,0} & & \\
 & & \uparrow{\scriptstyle \pi^*} & & \uparrow{\scriptstyle \pi^*} & & & & \uparrow{\scriptstyle \pi^*} & & \uparrow{\scriptstyle \pi^*} & & \\
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\end{array}
$$

Euler-Lagrange Complex

- Columns are locally exact
- Interior rows are globally exact!

Horizontal homotopy operator

$$h_H^{r,s}(\omega) = \frac{1}{s} \sum_{|I|\geq 0} c_I D_I \big[\theta^\alpha \wedge F_\alpha^{Ij}(D_j \lrcorner\, \omega)\big], \quad s \geq 1,$$

where $c_I = \frac{|I|+1}{n-r+|I|+1}$.

EULER-LAGRANGE COMPLEX

The edge complex

$$\mathbb{R} \longrightarrow \Lambda^{0,0} \xrightarrow{d_H} \Lambda^{1,0} \xrightarrow{d_H} \cdots$$

$$\xrightarrow{d_H} \Lambda^{m-1,0} \xrightarrow[\text{Div}]{d_H} \Lambda^{m,0} \xrightarrow[\text{E}]{\delta_V} \mathcal{F}^1 \xrightarrow[\mathcal{H}]{\delta_V} \mathcal{F}^2 \xrightarrow{\delta_V} \cdots$$

is called the *Euler-Lagrange* complex $\mathcal{E}^*(J^\infty(E))$.

Canonical representations

$$\omega = V^i(x^i, u^{[k]})(\partial_{x^i} \lrcorner \nu) \in \Lambda^{m-1,0},$$
$$\lambda = L(x^i, u^{[k]})\nu \in \Lambda^m,$$
$$\Delta = \Delta_\alpha(x^i, u^{[k]})\theta^\alpha \wedge \nu \in \mathcal{F}^1,$$
$$\mathcal{H} = \frac{1}{2}\mathcal{H}^I_{\alpha\beta}(x^i, u^{[k]})\theta^\alpha \wedge \theta^\beta_I.$$

Then

$$\lambda = d_H\omega \quad \Longleftrightarrow \quad L = D_i V^i,$$
$$\Delta = \delta_V\lambda \quad \Longleftrightarrow \quad \Delta_\alpha = \mathsf{E}_\alpha(L),$$
$$\mathcal{H} = \delta_V\Delta \quad \Longleftrightarrow \quad \mathcal{H}^I_{\alpha\beta} = -\partial^I_\beta\Delta_\alpha + (-1)^{|I|}\mathsf{E}^I_\alpha(\Delta_\beta),$$

where $\mathsf{E}^I_\alpha(F) = \sum_{|J|\geq 0}\binom{|I|+|J|}{|I|}(-D)_J(\partial^{IJ}_\alpha F)$.

COHOMOLOGY

Associated cohomology spaces:

$$H^r(\mathcal{E}^*(J^\infty(E))) = \frac{\ker \delta_V \colon \mathcal{E}^r \to \mathcal{E}^{r+1}}{\operatorname{im} \delta_V \colon \mathcal{E}^{r-1} \to \mathcal{E}^r} \,.$$

This complex is locally exact and its cohomology $H^*(\mathcal{E}^*(J^\infty(E)))$ is isomorphic with the de Rham cohomology of $E \approx$ singular cohomology of E.

Group Actions

A *Lie pseudo-group* $\mathcal{G}$ consists a collection of local diffeomorphisms on E satisfying

1. $\mathrm{id} \in \mathcal{G}$;
2. If $\psi_1, \psi_2 \in \mathcal{G}$, then $\psi_1 \circ (\psi_2)^{-1} \in \mathcal{G}$ where defined;
3. There is k_o such that the pseudo-group jets

   $$\mathcal{G}^k = \{ j_z^k \psi \,|\, \psi \in \mathcal{G}, z \in \mathrm{dom}\ \psi \}, \quad k \geq k_o,$$

   form a smooth bundle.
4. A local diffeomorphism $\psi \in \mathcal{G} \iff j_z^k \psi \in \mathcal{G}^k$, $k \geq k_o$, for all $z \in \mathrm{dom}\ \psi$.

Example: Symmetry groups of differential equations, gauge groups, ...

The *graph* $\Gamma_\sigma \subset E$ of a local section σ of $E \to M$ is the set

$$\Gamma_\sigma = \{\sigma(x^i) \,|\, (x^i) \in \operatorname{dom}\sigma\}.$$

Let $\psi \in \mathcal{G}$. Define the transform $\psi\cdot\sigma$ of σ under ψ by

$$\Gamma_{\psi\cdot\sigma} = \psi(\Gamma_\sigma).$$

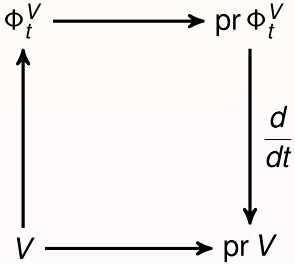
The *prolonged* action of $\mathcal{G}$ on $J^\infty(E)$ is then defined by

$$\begin{array}{ccc} j^\infty_{x_o}\sigma & \xrightarrow{\operatorname{pr}\psi} & j^\infty_{\psi(x_o)}(\psi\cdot\sigma) \\ \uparrow & & \uparrow \\ \sigma & \xrightarrow{\psi} & \psi\cdot\sigma \end{array}$$

A function F defined on a $\mathcal{G}$-invariant open $\mathcal{U} \subset J^{\infty}(E)$ is called a *differential invariant* of $\mathcal{G}$ if $F \circ \mathrm{pr}\,\psi = F$ for all $\psi \in \mathcal{G}$.

A k-form $\omega \in \Lambda^k(\mathcal{U})$ is $\mathcal{G}$ *invariant* if $(\mathrm{pr}\,\psi)^*\omega = \omega$ for all $\psi \in \mathcal{G}$.

The prolongation pr V of a local vector field V on E is defined by



A local vector field V on E is a $\mathcal{G}$ *vector field*, $V \in \mathfrak{g}$, if the flow $\Phi_t^V \in \mathcal{G}$ for all fixed t on some interval about 0.

The prolongation pr V of a local vector field V on E is defined by

$$\begin{array}{ccc} \Phi_t^V & \longrightarrow & \mathrm{pr}\,\Phi_t^V \\ \big\uparrow & & \big\downarrow \frac{d}{dt} \\ V & \longrightarrow & \mathrm{pr}\,V \end{array}$$

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Suppose that $\mathcal{G}$ consists of *projectable transformations*. Then the actions of $\mathcal{G}$ and $\mathfrak{g}$ both preserve the spaces $\Lambda^{r,s}(J^\infty(E))$ and commute with the horizontal and vertical differentials d_H, d_V, and the integration-by-parts operator I.

$\Longrightarrow$

The differentials d_H, d_V, δ_V map $\mathcal{G}$- and $\mathfrak{g}$-invariant forms to $\mathcal{G}$- and $\mathfrak{g}$-invariant forms, respectively.

$\mathfrak{g}$-INVARIANT VARIATIONAL BICOMPLEX:

$$
\begin{array}{ccccccccccc}
 & & \vdots & & \vdots & & & \vdots & & \vdots & & \vdots \\
 & & \uparrow {\scriptstyle d_V} & & \uparrow {\scriptstyle d_V} & & & \uparrow {\scriptstyle d_V} & & \uparrow {\scriptstyle d_V} & & \uparrow {\scriptstyle \delta_V} \\
0 & \longrightarrow & \Lambda_{\mathfrak{g}}^{0,2} & \xrightarrow{d_H} & \Lambda_{\mathfrak{g}}^{1,2} & \xrightarrow{d_H} & \cdots & \Lambda_{\mathfrak{g}}^{m-1,2} & \xrightarrow{d_H} & \Lambda_{\mathfrak{g}}^{m,2} & \xrightarrow{I} & \mathcal{F}_{\mathfrak{g}}^{2} \\
 & & \uparrow {\scriptstyle d_V} & & \uparrow {\scriptstyle d_V} & & & \uparrow {\scriptstyle d_V} & & \uparrow {\scriptstyle d_V} & & \uparrow {\scriptstyle \delta_V} \\
0 & \longrightarrow & \Lambda_{\mathfrak{g}}^{0,1} & \xrightarrow{d_H} & \Lambda_{\mathfrak{g}}^{1,1} & \xrightarrow{d_H} & \cdots & \Lambda_{\mathfrak{g}}^{m-1,1} & \xrightarrow{d_H} & \Lambda_{\mathfrak{g}}^{m,1} & \xrightarrow{I} & \mathcal{F}_{\mathfrak{g}}^{1} \\
 & & \uparrow {\scriptstyle d_V} & & \uparrow {\scriptstyle d_V} & & & \uparrow {\scriptstyle d_V} & & \uparrow {\scriptstyle d_V} & \nearrow {\scriptstyle E} & \\
\mathbb{R} & \longrightarrow & \Lambda_{\mathfrak{g}}^{0,0} & \xrightarrow{d_H} & \Lambda_{\mathfrak{g}}^{1,0} & \xrightarrow{d_H} & \cdots & \Lambda_{\mathfrak{g}}^{m-1,0} & \xrightarrow{d_H} & \Lambda_{\mathfrak{g}}^{m,0} & & \\
 & & \uparrow {\scriptstyle \pi^*} & & \uparrow {\scriptstyle \pi^*} & & & \uparrow {\scriptstyle \pi^*} & & \uparrow {\scriptstyle \pi^*} & & \\
\mathbb{R} & \longrightarrow & \Lambda_{M,\mathfrak{g}}^{0} & \xrightarrow{d} & \Lambda_{M,\mathfrak{g}}^{1} & \xrightarrow{d} & \cdots & \Lambda_{M,\mathfrak{g}}^{m-1} & \xrightarrow{d} & \Lambda_{M,\mathfrak{g}}^{m} & &
\end{array}
$$

$\mathfrak{g}$-invariant Euler-Lagrange complex $\mathcal{E}^*_{\mathfrak{g}}(J^\infty(E))$:

$$\mathbb{R} \longrightarrow \Lambda^{0,0}_{\mathfrak{g}} \xrightarrow{d_H} \Lambda^{1,0}_{\mathfrak{g}} \xrightarrow{d_H} \cdots$$

$$\xrightarrow{d_H} \Lambda^{m-1,0}_{\mathfrak{g}} \xrightarrow[\text{Div}]{d_H} \Lambda^{m,0}_{\mathfrak{g}} \xrightarrow[\text{E}]{\delta_V} \mathcal{F}^1_{\mathfrak{g}} \xrightarrow[\mathcal{H}]{\delta_V} \mathcal{F}^2_{\mathfrak{g}} \xrightarrow{\delta_V} \cdots$$

Associated cohomology spaces:

$$H^r(\mathcal{E}^*_{\mathfrak{g}}(J^\infty(E))) = \frac{\ker \delta_V \colon \mathcal{E}^r_{\mathfrak{g}} \to \mathcal{E}^{r+1}_{\mathfrak{g}}}{\operatorname{im} \delta_V \colon \mathcal{E}^{r-1}_{\mathfrak{g}} \to \mathcal{E}^r_{\mathfrak{g}}}\,.$$

Exactness of the Interior Horizontal Rows

Theorem

Let $\mathfrak{g}$ be a pseudo-group of projectable transformations acting on $E \to M$, and let ω^i and θ^α be $\mathfrak{g}$ invariant horizontal frame and zeroth order contact frame defined on some $\mathcal{G}$-invariant open set $\mathcal{U} \subset J^\infty(E)$ contained in an adapted coordinate system. Then the interior rows of the $\mathfrak{g}$-invariant augmented variational bicomplex restricted to $\mathcal{U}$ are exact,

$$H^*(\Lambda_{\mathfrak{g}}^{*,s}(\mathcal{U}), d_H) = \{0\}, \qquad s \geq 1.$$

Corollary: Under the above hypothesis

$$H^*(\mathcal{E}_{\mathfrak{g}}^*(\mathcal{U}), \delta_V) \cong H^*(\Lambda_{\mathfrak{g}}^*(\mathcal{U}), d).$$

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EXPLICIT DESCRIPTION OF THE INVARIANT VARIATIONAL BICOMPLEX.

Given a local cross section $\mathcal{K}^{(k)} \subset J^k(E)$ to the action of $\mathcal{G}^k$ on $J^k(E)$, let

$$\mathcal{H}^k_{|\mathcal{K}^{(k)}} = \{(g^k, z^k) \mid z^k \in \mathcal{K}^{(k)},\ g^k, z^k \text{ based at the same point}\},$$

and let

$$\mu^k \colon \mathcal{H}^k_{|\mathcal{K}^{(k)}} \to J^k(E), \quad \mu^k(g^k, z^k) = g^k \cdot z^k.$$

Then, if the action is locally free, μ^k will be a $\mathcal{G}$-equivariant local diffeomorphism with the action of $\mathcal{G}$ on $\mathcal{H}^k_{|\mathcal{K}^{(k)}}$ given by $\varphi \cdot (g^k, z^k) = (\varphi \cdot g^k, z^k)$.

Upshot: Locally one can find a complete set of differential invariants $\{I_\alpha\}$ and a coframe on $\mathcal{U} \subset J^k(E)$ consisting of $\{dI_\alpha\}$ and $\mathfrak{g}$-invariant 1-forms $\{\vartheta_\beta\}$ such that the algebra $\mathcal{A}$ generated by $\{\vartheta_\beta\}$ is closed under $d \Longrightarrow$

$$H^*_{\mathfrak{g}}(\mathcal{U}, d) \cong H^*(\mathcal{A}, d).$$

(Apply the $\mathfrak{g}$-equivariant homotopy $I_\alpha \to t I_\alpha$, $dI_\alpha \to t\, dI_\alpha$, $\vartheta_\beta \to \vartheta_\beta$, $0 \leq t \leq 1$.)

Gelfand-Fuks Cohomology

Formal power series vector fields on $\mathbb{R}^m$:

$$W_m = \left\{ \sum_{l=1}^{m} a^l \frac{\partial}{\partial x^l} \mid a^l \in \mathbb{R}[[x^1, \ldots, x^m]] \right\}.$$

Lie bracket $[\ ,\ ]\colon W_m \times W_m \to W_m$.

Give W_m a topology relative to the ideal $\mathbf{m} = < x^1, x^2, \ldots, x^m >$.

$\Lambda_c^*(W_m)$: continuous alternating functionals on W_m.

$\Lambda_c^*(W_m)$ is generated by $\delta^i_{j_1 j_2 \cdots j_k}$, where

$$\delta^i_{j_1 j_2 \cdots j_k}\left(a^l \frac{\partial}{\partial x^l}\right) = \frac{\partial^k a^i}{\partial x^{j_1} \partial x^{j_2} \cdots \partial x^{j_k}}(0).$$

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The differential $d_{GF}\colon \Lambda^r_c(W_m) \to \Lambda^{r+1}_c(W_m)$ is induced by Lie bracket of vector fields so that

$$d_{GF}\omega(X, Y) = -\omega([X, Y]), \qquad \omega \in \Lambda^1_c(W_m).$$

$d^2_{GF} = 0!$

Let $\mathfrak{g}_o \subset \mathfrak{g} \subset W_m$ be subalgebras. Define

$$\begin{aligned} \Lambda^*_c(\mathfrak{g}) &= \Lambda^*_c(W_m)_{|\mathfrak{g}}, \\ \Lambda^*_c(\mathfrak{g}, \mathfrak{g}_o) &= \{\omega \in \Lambda^*_c(\mathfrak{g}) \,|\, X \lrcorner\, \omega = 0, \\ &\qquad X \lrcorner\, d_{GF}\omega = 0, \quad \text{for all } X \in \mathfrak{g}_o\}. \end{aligned}$$

The *Gelfand-Fuks cohomology* $H^*_{GF}(\mathfrak{g}, \mathfrak{g}_o)$ of $\mathfrak{g}$ relative to $\mathfrak{g}_o$ is the cohomology of the complex $(\Lambda^*_c(\mathfrak{g}, \mathfrak{g}_o), d_{GF})$.

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Evaluation Mapping

Pick $\sigma^\infty \in J^\infty(E)$.

For a given infinitesimal transformation group $\mathfrak{g}$ acting on E, let

$$\mathfrak{g}_o = \{X \in \mathfrak{g} \,|\, \mathrm{pr}\, X(\sigma^\infty) = 0\}.$$

Define $\rho \colon \Lambda^*_{\mathfrak{g}}(J^\infty(E)) \to \Lambda^*_c(\mathfrak{g}, \mathfrak{g}_o)$ by

$$\rho(\omega)(X_1, \ldots, X_r) = (-1)^r \omega(\mathrm{pr}\, X_1, \ldots, \mathrm{pr}\, X_r)(\sigma^\infty).$$

Then ρ is a cochain mapping, that is, it commutes with the application of d and d_{GF}, and thus induces a mapping

$$\overline{\rho} \colon H^*(\Lambda^*_{\mathfrak{g}}(J^\infty(E)), d) \to H^*_{GF}(\mathfrak{g}, \mathfrak{g}_o).$$

Goal is to show that $\overline{\rho}$ is an isomorphism (moving frames!).

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EQUIVARIANT DEFORMATIONS

Construct a submanifold $\mathcal{P}^\infty \subset \mathcal{U} \subset J^\infty(E)$ such that

1. $\operatorname{pr}\mathfrak{g}$ acts transitively on $\mathcal{P}^\infty$, and
2. $\mathcal{P}^\infty$ is $\operatorname{pr}\mathfrak{g}$-equivariant strong deformation retract of $\mathcal{U}$, that is, there is a smooth map $H\colon \mathcal{U} \times [0,1] \to \mathcal{U}$ such that

$$
\begin{aligned}
H(\sigma^\infty, 0) &= \sigma^\infty, && \text{for all } \sigma^\infty \in \mathcal{U},\\
H(\sigma^\infty, 1) &\in \mathcal{P}^\infty, && \text{for all } \sigma^\infty \in \mathcal{U},\\
H(\sigma^\infty, t) &= \sigma^\infty, && \text{for all } (\sigma^\infty, t) \in \mathcal{P}^\infty \times [0,1],\\
(H_t)_*(\operatorname{pr} V_{|\sigma^\infty}) &= \operatorname{pr} V_{|H(\sigma^\infty,t)}, && \text{for all } V \in \mathfrak{g},\\
& && \quad (\sigma^\infty, t) \in \mathcal{U} \times [0,1].
\end{aligned}
$$

Equivariant deformations

Under these circumstances the inclusion map

$$\iota\colon \mathcal{P}^\infty \to \mathcal{U}$$

and the evaluation map

$$\rho\colon \Lambda^*_{\mathfrak{g}}(\mathcal{P}^\infty) \to \Lambda^*_c(\mathfrak{g}, \mathfrak{g}_o)$$

induce isomorphisms in cohomology.

PKP EQUATION AGAIN

The symmetry algebra $\mathfrak{g}_{PKP}$ of the PKP equation

$$u_{tx} + \frac{3}{2} u_x u_{xx} + \frac{1}{4} u_{xxxx} + \frac{3}{4} s^2 u_{yy} = 0.$$

is spanned by the vector fields

$$X_f = f \frac{\partial}{\partial t} + \frac{2}{3} y f' \frac{\partial}{\partial y} + \left(\frac{1}{3} x f' - \frac{2}{9} s^2 y^2 f''\right) \frac{\partial}{\partial x}$$
$$+ \left(-\frac{1}{3} u f' + \frac{1}{9} x^2 f'' - \frac{4}{27} s^2 x y^2 f''' + \frac{4}{243} y^4 f''''\right) \frac{\partial}{\partial u},$$

$$Y_g = g \frac{\partial}{\partial y} - \frac{2}{3} s^2 y g' \frac{\partial}{\partial x} + \left(-\frac{4}{9} s^2 x y g'' + \frac{8}{81} y^3 g'''\right) \frac{\partial}{\partial u},$$

$$Z_h = h \frac{\partial}{\partial x} + \left(\frac{2}{3} x h' - \frac{4}{9} s^2 y^2 h''\right) \frac{\partial}{\partial u},$$

$$W_k = yk \frac{\partial}{\partial u}, \quad \text{and} \quad U_l = l \frac{\partial}{\partial u},$$

where $f = f(t)$, $g = g(t)$, $h = h(t)$, $k = k(t)$ and $l = l(t)$ are arbitrary smooth functions of t.

PKP equation

Now $E = \{(t,x,y,u)\} \to \{(t,x,y)\}$. The *PKP source form*

$$\Delta_{PKP} = \left(u_{tx} + \frac{3}{2}u_x u_{xx} + u_{xxxx} + \frac{3}{4}s^2 u_{yy}\right)\theta \wedge dt \wedge dx \wedge dy$$

generates non-trivial cohomology in $H^4(\mathcal{E}_{\mathfrak{g}_{PKP}}(J^\infty(E)))$!

The characterization problem of the PKP-equation by its symmetry algebra amounts to the computation of $H^4(\mathcal{E}^*_{\mathfrak{g}_{PKP}}(\mathcal{U}))$.

For a suitable $\mathcal{U} \subset J^\infty(\mathcal{U})$, $H^*(\mathcal{E}^*_{\mathfrak{g}_{PKP}}(\mathcal{U}))$ can be computed by an explicit description of differential invariants and an invariant coframe arising from the moving frames construction.

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The Gelfand-Fuks complex for $\mathfrak{g}_{PKP}$ admits a basis α^n, β^n, γ^n, υ^n, ϑ^n, $n = 0, 1, 2, \ldots$, of invariant forms so that

$$d\alpha^n = \sum_{k=0}^{n} \binom{n}{k} \alpha^k \wedge \alpha^{n-k+1},$$

$$d\beta^n = \sum_{k=0}^{n} \binom{n}{k} \{\alpha^k \wedge \beta^{n-k+1} - \frac{2}{3}\alpha^{k+1} \wedge \beta^{n-k}\},$$

$$d\gamma^n = \sum_{k=0}^{n} \binom{n}{k} \{\alpha^k \wedge \gamma^{n-k+1} - \frac{1}{3}\alpha^{k+1} \wedge \gamma^{n-k} - \frac{2}{3}s^2\beta^k \wedge \beta^{n-k+1}\},$$

$$d\upsilon^n = \sum_{k=0}^{n+1} \binom{n+1}{k} \{\alpha^k \wedge \upsilon^{n-k+1} + \frac{4}{9}s^2(\beta^{k+1} \wedge \gamma^{n-k+1} - 2\beta^k \wedge \gamma^{n-k+2})\},$$

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The complex splits into a direct sum of simultaneous eigenspaces of 2 Lie derivative operators.

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PKP EQUATION

Let A be a non-vanishing differential function on some open set $\mathcal{U} \subset J^\infty(E)$ satisfying

$$\operatorname{pr} X_f(A) + \frac{1}{3} A f'(t) = 0, \quad \frac{\partial A}{\partial y} = 0, \quad \text{for every smooth } f(t),$$

and let B be a differential function on $\mathcal{U}$ satisfying

$$\operatorname{pr} X_f(B) + \frac{2}{3} y A^{-1} f''(t) = 0, \qquad \frac{\partial B}{\partial y} = 0, \qquad \text{for every smooth } f(t).$$

For example, one can choose

$$A = (u_{x^n})^{\frac{1}{n+1}} \quad \text{and} \quad B = -\frac{3}{2} s^2 u_{x^{n-1}y} (u_{x^n})^{-\frac{n+2}{n+1}}, \qquad n \geq 3,$$

on $\mathcal{U} = \{u_{x^n} > 0\}$.

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Theorem

Suppose that differential functions A and B, defined on an open $\mathcal{U} \subset J^\infty(E)$, are chosen as above. Then the dimensions of the cohomology spaces $H^r(\mathcal{E}^*_{\mathfrak{g}_{PKP}}(\mathcal{U}), \delta_V)$ are

r	1	2	3	4	5	6	7	≥ 8
dim	0	1	1	3	3	2	3	0

Representatives of the Cohomology Classes

Let $\{\alpha^0, \beta^0, \gamma^0\}$ be the $\mathfrak{g}_{PKP}$ invariant horizontal frame defined by

$$\alpha^0 = A^3 dt, \qquad \beta^0 = A^2 dy + A^3 B dt,$$
$$\gamma^0 = A dx - \frac{2}{3} s^2 A^2 B dy + A^3 C dt,$$

where

$$C = -\frac{3}{2} u_x A^{-2} - \frac{1}{3} s^2 B^2,$$

and let K be the $\mathfrak{g}_{PKP}$ differential invariant

$$K = (u_{tx} + \frac{3}{4} s^2 u_{yy} + \frac{3}{2} u_x u_{xx}) A^{-5}.$$

Representatives of the Cohomology Classes

Let Δ^1, $\Delta^2 \in \mathcal{E}^4_{\mathfrak{g}_{PKP}}(\mathcal{U})$ be the source forms

$$\Delta^1 = (u_{tx} + \frac{3}{2} u_x u_{xx} + \frac{3}{4} s^2 u_{yy})\, dt \wedge dx \wedge dy \wedge du,$$
$$\Delta^2 = u_{xxxx}\, dt \wedge dx \wedge dy \wedge du,$$

and let $\Delta^3 \in \mathcal{E}^4_{\mathfrak{g}_{PKP}}(\mathcal{U})$ be the source form which is the Euler-Lagrange expression

$$\Delta^3 = \mathrm{E}(BK\alpha^0 \wedge \beta^0 \wedge \gamma^0).$$

Then $H^4(\mathcal{E}^*(\mathcal{U}), \delta_V) = < \Delta^1, \Delta^2, \Delta^3 >$.

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COROLLARY:

Let $\Delta \in \mathcal{E}^4_{\mathfrak{g}_{PKP}}(\mathcal{U})$ be a $\mathfrak{g}_{PKP}$ invariant source form that is the Euler-Lagrange expression of some Lagrangian 3-form $\lambda \in \mathcal{E}^3(\mathcal{U})$. Then there are constants c_1, c_2, c_3 and a $\mathfrak{g}_{PKP}$-invariant Lagrangian 3-form $\lambda_0 \in \mathcal{E}^3_{\mathfrak{g}_{PKP}}(\mathcal{U})$ such that

$$\Delta = c_1\Delta^1 + c_2\Delta^2 + c_3\Delta^3 + \mathrm{E}(\lambda_0).$$

Vector Field Theories

Here $E = T^*\mathbb{R}^m = \{(x^i, A_j)\} \to \{(x^i)\}$.

Now the infinitesimal transformation group $\mathfrak{g}$ is spanned by

$$T_i = \frac{\partial}{\partial x^i}, \quad V_\phi = \phi_{,i}\frac{\partial}{\partial A_i},$$

where ϕ is an arbitrary smooth function on $\mathbb{R}^m$.

Need to compute $H^{m+1}(\mathcal{E}^*_{\mathfrak{g}}(J^\infty(T^*\mathbb{R}^m)))$!

The standard horizontal homotopy operator for the free variational bicomplex commutes with the action of $\mathfrak{g}$ $\Longrightarrow$

$$H^{*,s}(\Lambda^{*,*}_{\mathfrak{g}}(J^\infty(E)), d_H, I) = \{0\}, \qquad s \geq 1.$$

So it suffices to compute $H^*(\Lambda^*_{\mathfrak{g}}(J^\infty(E)), d)$.

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Vector Field Theories

Parametrize $J^\infty(T^*\mathbb{R}^m)$ by

$(x^i,\, A_a,\, A_{(a,b_1)},\, F_{ab_1},\, A_{(a,b_1b_2)},\, F_{a(b_1,b_2)},\, A_{(a,b_1b_2b_3)},\, F_{a(b_1,b_2b_3)},\, \ldots),$

where $F_{ab} = A_{a,b} - A_{b,a}$.

Now the variables $F_{a(b_1,b_2\cdots b_r)}$ are invariant under the action of $\mathfrak{g}$ $\Longrightarrow$

$$\mathcal{P}^\infty = \{\,\sigma^\infty \in J^\infty(T^*\mathbb{R}^m)\,|\, F_{ij}(\sigma^\infty) = 0, F_{i(j,h)}(\sigma^\infty) = 0, \ldots\,\}$$

is a $\mathfrak{g}$-equivariant strong deformation retract of $J^\infty(T^*M)$ on which $\mathfrak{g}$ acts transitively.

Vector Field Theories

In conclusion,

$$H^*(\mathcal{E}^*_{\mathfrak{g}}(J^\infty(T^*M))) \cong H^*_{GF}(\widetilde{\mathfrak{g}}),$$

where the Lie algebra of formal vector fields $\widetilde{\mathfrak{g}}$ is spanned by the vector fields T_i and

$$V^{j_1 j_2 \cdots j_k} = x^{(j_1} x^{j_2} \cdots x^{j_{k-1}} \partial_A^{j_k)}, \qquad \partial_A^j = \frac{\partial}{\partial A_j}.$$

A basis for $H^*(\mathcal{E}^*_{\mathfrak{g}}(J^\infty(T^*M)))$ is given by

$$dx^{i_1} \wedge \cdots \wedge dx^{i_k} \wedge F^l \in \Lambda^{r,0}_{\mathfrak{g}}(J^\infty(T^*M)), \quad k + 2l = r,$$

$$dx^{i_1} \wedge \cdots \wedge dx^{i_k} \wedge F^l \wedge (d_V A)^s \in \mathcal{F}^s_{\mathfrak{g}}(J^\infty(T^*M)), \quad k + 2l + s = m.$$

$(A = A_i dx^i,\ F = F_{ij} dx^i \wedge dx^j.)$

Generators for $H^{m+1}(\mathcal{E}^*_{\mathfrak{g}}(J^\infty(T^*M)))$

$$\Delta^{i_1 i_2 \cdots i_k} = dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k} \wedge F^l \wedge d_V A, \quad k + 2l = m - 1,$$

$$\dim H^{m+1}(\mathcal{E}^*_{\mathfrak{g}}(J^\infty(T^*M))) = 2^m - 1.$$

Note that when $m = 2r + 1$, $\Delta = F^r \wedge d_V A$ is the Chern-Simons mass term with components

$$\Delta^i = \epsilon^{i j_1 k_1 j_2 k_2 \cdots j_r k_r} F_{j_1 k_1} F_{j_2 k_2} \cdots F_{j_r k_r}.$$