On the Van Est homomorphism for Lie groupoids

Eckhard Meinrenken (based on joint work with David Li-Bland)

Fields Institute, December 13, 2013
Let $G \rightrightarrows M$ be a Lie groupoid, with Lie algebroid $A = \text{Lie}(G)$.

Weinstein-Xu (1991) constructed a cochain map

$$\text{VE}: C^\bullet(G) \to C^\bullet(A)$$

from smooth groupoid cochains to the Chevalley-Eilenberg complex of the Lie algebroid $A$ of $G$. 

Crainic (2003) proved a Van Est Theorem for this map. 

Weinstein, Mehta (2006), and Abad-Crainic (2008, 2011) generalized to 

$$\text{VE}: W^\bullet_\bullet(G) \to W^\bullet_\bullet(A)$$

for suitably defined Weil algebras.
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### Overview

**Applications**

- Foliation theory
- Integration of (quasi-)Poisson manifolds and Dirac structures
- Multiplicative forms on groupoids (Mackenzie-Xu, Bursztyn-Cabrera-Ortiz)
- Index theory (Posthuma-Pflaum-Tang)
- Lie pseudogroups and Spencer operators (Crainic-Salazar-Struchiner),
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Using the **Fundamental Lemma** of homological perturbation theory, we’ll give a simple construction of VE (and its properties).
Let $G \rightrightarrows M$ be a Lie groupoid over $M \subseteq G$.

$$m_0 \overset{g}{\leftarrow} m_1.$$ 

Multiplication $(g_1, g_2) \mapsto g_1 g_2$ defined for *composable arrows*:

$$\left( m_0 \overset{g_1}{\leftarrow} m_1 \overset{g_2}{\leftarrow} m_2 \right) \mapsto \left( m_0 \overset{g_1 g_2}{\leftarrow} m_2 \right).$$
Let $G \rightrightarrows M$ be a Lie groupoid over $M \subseteq G$. 

$\begin{align*}
m_0 & \xleftarrow{g} m_1.
\end{align*}$

Multiplication $(g_1, g_2) \mapsto g_1 g_2$ defined for \textit{composable arrows}:

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\end{align*}$

**Examples**

- Lie group $G \rightrightarrows pt$
- Pair groupoid $\text{Pair}(M) = M \times M \rightrightarrows M$
- Fundamental groupoid $\Pi(M) \rightrightarrows M$
- Foliation groupoid(s), e.g., $\Pi_{\mathcal{F}}(M) \rightrightarrows M$
- Gauge groupoids of principal bundles
- Action groupoids $K \ltimes M \rightrightarrows M$
- Groupoids associated with hypersurfaces
Let $B_p G$ be the manifold of $p$-arrows $(g_1, \ldots, g_p)$:

$$m_0 \xleftarrow{g_1} m_1 \xleftarrow{g_2} m_2 \cdots \xleftarrow{g_p} m_p$$

It is a simplicial manifold, with face maps

$$\partial_i : B_p G \to B_{p-1} G, \ i = 0, \ldots, p$$

removing $m_i$ and degeneracies $\epsilon_i : B_p G \to B_{p+1} G$ repeating $m_i$.

For example

$$\partial_1 \left( m_0 \xleftarrow{g_1} m_1 \xleftarrow{g_2} m_2 \cdots \xleftarrow{g_p} m_p \right) = \left( m_0 \xleftarrow{g_1 g_2} m_2 \cdots \xleftarrow{g_p} m_p \right).$$
Groupoid cochain complex: $C^\bullet(G) := C^\infty(B\mathbf{\bullet}G)$ with differential

$$\delta = \sum_{i=0}^{p+1} (-1)^i \partial^*_i : C^\infty(B_pG) \to C^\infty(B_{p+1}G)$$

and algebra structure $C^p(G) \otimes C^{p'}(G) \to C^{p+p'}(G)$,

$$f \cup f' = \text{pr}^* f (\text{pr}')^* f',$$

where pr, pr' are the ‘front face’ and ‘back face’ projections.
Groupoid cochain complex: $C^\bullet(G) := C^\infty(B_\bullet G)$ with differential

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Variations:

- Normalized subcomplex $\tilde{C}^\bullet(G)$: kernel of degeneracy maps $\epsilon_i$.
- More generally, with coefficients in $G$-modules $S \to M$.
- $C^\bullet(G)_M := C^\infty(B_\bullet G)_M$, the germs along $M \subseteq B_p G$.
- Extends to double complex $W^\bullet,\bullet(G) := \Omega^\bullet(B_\bullet G)$. 

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Example (Alexander-Spanier complex)

$C^\bullet(\text{Pair}(M))_M = C^\infty(M^{p+1})_M$

$$(\delta f)(m_0, \ldots, m_{p+1}) = \sum_{i=0}^{p+1} (-1)^i f(m_0, \ldots, \hat{m_i}, \ldots, m_{p+1})$$
Let $A \to M$ be a Lie algebroid, with anchor $a: A \to TM$ and bracket $[\cdot , \cdot ]_A$ on $\Gamma(A)$. Thus

$$[X, fY] = f[X, Y] + (a(X)f) Y.$$

Examples

- Lie algebra $g$
- Tangent bundle $TM$
- Tangent bundle to foliation $T_F M \subset TM$
- Atiyah algebroid of principal bundle
- Cotangent Lie algebroid of Poisson manifold
- Action Lie algebroids $k \ltimes M$
- Lie algebroids associated with hypersurfaces

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Lie algebroid cohomology

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- ...
The Chevalley-Eilenberg complex is $C^\bullet(A) = \Gamma(\bigwedge^\bullet A^*)$ with differential

$$(d_{CE}\phi)(X_0, \ldots, X_p)$$

$$= \sum_{i=0}^p (-1)^i a(X_i)\phi(X_0, \ldots, \hat{X}_i, \ldots, X_p)$$

$$+ \sum_{i<j} (-1)^{i+j} \phi([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_p)$$

and with product the wedge product.
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and with product the wedge product.

- More generally, with coefficients in $A$-modules $S \to M$.
- Extends to double complex $W^{\bullet, \bullet}(A)$
- For $A = T_\mathcal{F}M$, get foliated de Rham complex $\Omega_\mathcal{F}(M)$. 

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A Lie groupoid $G \rightrightarrows M$ has an associated Lie algebroid:

- $\text{Lie}(G) = \nu(M, G)$
- anchor $a: \text{Lie}(G) \to TM$ induced from $Tt - Ts: TG \to TM$,
- $[\cdot, \cdot]$ from $\Gamma(\text{Lie}(G)) = \text{Lie}(\Gamma(G))$ where $\Gamma(G)$ is the group of bisections.

The Van Est map relates the corresponding cochain complexes.
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The Van Est map relates the corresponding cochain complexes.

We’ll explain this map using a double complex.
Define a principal $G$-bundle

$$E_p G \xrightarrow{\kappa_p} B_p G$$

where $E_p G \subseteq G^{p+1}$ consists of elements $(a_0, \ldots, a_p)$ with common source:
.. and where $\pi_p$ and $\kappa_p$ take such an element to the common source $m$, respectively to

$$m_0 \leftarrow m_1 \leftarrow \ldots \leftarrow m_p$$

with $g_i = a_i a_{i-1}^{-1}$. 
.. and where \( \pi_p \) and \( \kappa_p \) take such an element

\[
\begin{array}{c}
m \\
\downarrow \quad a_1 \\
\downarrow \\
m_1 \\
m_0 \\
\end{array}
\quad \downarrow \quad \leftarrow \quad \downarrow \\
\vdots \\
\downarrow \\
\quad \leftarrow \quad \downarrow \\
\quad \leftarrow \quad \leftarrow \quad \leftarrow \\
m_p \\
\end{array}
\]

to the common source \( m \), respectively to

\[
\begin{array}{c}
m_0 \leftarrow g_1 \\
m_1 \leftarrow g_2 \\
\vdots \\
m_0 \leftarrow g_p \\
m_p \\
\end{array}
\]

with \( g_i = a_ia_i^{-1} \). The groupoid action of an element \( m' \xleftarrow{g} m \) takes this element to

\[
\begin{array}{c}
m' \\
\downarrow \quad a_1g^{-1} \\
\downarrow \\
m_1 \\
m_0 \\
\end{array}
\quad \downarrow \quad \leftarrow \quad \downarrow \\
\vdots \\
\downarrow \\
\quad \leftarrow \quad \leftarrow \quad \leftarrow \\
m_p \\
\end{array}
\]
View $M$ as a simplicial manifold (with all $M_p = M$). Then

\[
\begin{array}{c}
E_p G \xrightarrow{\kappa_p} B_p G \\
\pi_p \downarrow \\
M
\end{array}
\]

is a simplicial principal $G$-bundle. The map

\[\iota_p : M \to E_p G, \ m \mapsto (m, \ldots, m)\]

is a simplicial inclusion; $\pi_p \circ \iota_p = \text{id}$. 
Van Est double complex

View $M$ as a simplicial manifold (with all $M_p = M$). Then

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**Theorem**

There is a (canonical) simplicial deformation retraction from $E \cdot G$ onto $M$.

See: G. Segal, Classifying spaces and spectral sequences (1968).
Van Est double complex

Since $E \cdot G$ is a simplicial manifold, have cochain complex

$$C^\infty(E \cdot G), \quad \delta = \sum_{i=0}^{p+1} (-1)^i \partial_i^*, \quad \partial_i^* = \partial_i + \cdots + \partial_{i+1},$$

with cochain maps

$$C^\infty(E \cdot G) \xleftarrow{\kappa^*} C^\infty(B \cdot G) = C^\bullet(G) \xrightarrow{\pi^*} C^\infty(M \cdot)$$

The map $h: C^\infty(E_{p+1}G) \to C^\infty(E_pG)$,

$$(hf)(a_0, \ldots, a_p) = \sum_{i=0}^{p} (-1)^{i+1} f(a_0, \ldots, a_i, m \ldots, m)$$

with $m = \pi_p(a_0, \ldots, a_p)$, is a $\delta$-homotopy: $h\delta + \delta h = 1 - \pi^* i^*$. 

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Let \( A = \text{Lie}(G) \). Since \( \kappa_p: E_pG \to B_pG \) is a principal \( G \)-bundle, have
\[
T_F E_p G \cong \pi^*_p A,
\]
and \( T_F E \bullet G \to A \) is a morphism of simplicial Lie algebroids.

Get double complex
\[
(\Omega^q_F(E_p G), \delta, d)
\]
with \( d = (-1)^p d_{CE} \).
Have morphism of double complexes

\[
\begin{align*}
\Omega^\bullet_F(E \cdot G) & \xleftarrow{\kappa^*} C^\infty(B \cdot G) = C^\bullet(G) \\
\Gamma(\wedge \bullet A^\bullet) & = C^\bullet(A_\bullet)
\end{align*}
\]

where \( d = 0 \) on \( C^\infty(B \cdot G) \).
Van Est double complex

Have morphism of double complexes

\[
\Omega_{\mathcal{F}}(E \cdot G) \xleftarrow{\kappa^*} C^\infty(B \cdot G) = C^\cdot(G)
\]

\[
\pi^* \quad \Gamma(\wedge \cdot A^*) = C^\cdot(A^*)
\]

where \(d = 0\) on \(C^\infty(B \cdot G)\).

Here \(\pi^*\) is a homotopy inverse to \(\iota^*\), with \(h\) as above.
Van Est double complex

Have morphism of double complexes

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Here \(\pi^\bullet\) is a homotopy inverse to \(\iota^\bullet\), with \(h\) as above.

Want to turn this into homotopy equivalence with respect to \(d + \delta\).
Set-up:

- $(C^{\bullet,\bullet}, d, \delta)$ be a double complex
- $i: D \hookrightarrow C$ a sub-double complex
- $r: C \to D$ a (bigraded) projection
- $h: C^{\bullet,\bullet} \to C^{\bullet-1,\bullet}$ with $h\delta + \delta h = 1 - i \circ r$. 

Lemma (Fundamental Lemma of homological perturbation theory)

Put $i' = (1 + h)d - 1$, $r' = r(1 + dh) - 1$, $h' = h(1 + dh) - 1$.

Then $i' \circ r'$ is a cochain map for $d + \delta$, and $h'(d + \delta) + (d + \delta)h' = 1 - i' \circ r'$. 

References: Gugenheim-Lambe-Stasheff, Brown, Crainic, ...
Perturbation theory

Set-up:

- \( (C \bullet, \bullet, d, \delta) \) be a double complex
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Lemma (Fundamental Lemma of homological perturbation theory)

Put

\[
i' = (1 + hd)^{-1}i, \quad r' = r(1 + dh)^{-1}, \quad h' = h(1 + dh)^{-1}.
\]

Then \( i' \circ r' \) is a cochain map for \( d + \delta \), and

\[
h'(d + \delta) + (d + \delta)h' = 1 - i' \circ r'.
\]

References: Gugenheim-Lambe-Stasheff, Brown, Crainic, ....
In our case, this shows that

\[ \iota^* \circ (1 + d h)^{-1} : \Omega^\bullet_F(E \bullet G) \to \Gamma(\wedge^\bullet A^\bullet) \]

is a homotopy equivalence, with homotopy inverse \((1 + h d)^{-1} \circ \pi^*\).
In our case, this shows that

\[ \iota^* \circ (1 + dh)^{-1} : \Omega^*_F(E\cdot G) \to \Gamma^{\bullet}(A^*) \]

is a homotopy equivalence, with homotopy inverse \((1 + hd)^{-1} \circ \pi^*\). But we also have obvious homotopy equivalences

\[ \Gamma^{\bullet}(A^*) \leftrightarrow \Gamma^{\bullet}(A^*) \]

Hence:

**Theorem**

The map

\[ \iota_0^* \circ (1 + dh)^{-1} : \Omega^*_F(EG) \to \Gamma^{\bullet}(A^*) \]

is a homotopy equivalence (for \(d + \delta\)), with homotopy inverse \(\pi_0^*\).
In our case, this shows that

\[ \iota^* \circ (1 + dh)^{-1} : \Omega^*_\mathcal{F}(EG) \to \Gamma(\wedge^* A^*_\bullet) \]

is a homotopy equivalence, with homotopy inverse \( (1 + hd)^{-1} \circ \pi^* \).

But we also have obvious homotopy equivalences

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Hence:

**Theorem**

The map

\[ \iota_0^* \circ (1 + dh)^{-1} : \Omega^*_\mathcal{F}(EG) \to \Gamma(\wedge A^*_\bullet) \]

is a homotopy equivalence (for \( d + \delta \)), with homotopy inverse \( \pi_0^* \).

Using \( \kappa^* : C^\infty(BG) \to \Omega^*_\mathcal{F}(EG) \) we get the desired cochain map:
Definition

The composition

\[ \text{VE} := \iota_0^* \circ (1 + dh)^{-1} \circ \kappa^* : C^\infty(BG) \to \Gamma(\wedge A^*) \]

is called the Van Est map.

Proposition

This map agrees with the Van Est map of Weinstein-Xu.
Equivalently, we may write $\text{VE} = \iota_0^* \circ (1 + dh)^{-1} \circ \kappa^*$ as

$$\text{VE} = (-1)^p \iota_0^* \circ (d \circ h)^p \circ \kappa_p^* : C^\infty(B_p G) \to \Gamma(\wedge^p A^*)$$

corresponding to a ‘zig-zag’: E.g., for $p = 2$

$$C^\infty(B_2 G) \xrightarrow{\kappa_2^*} \Omega_F^0(E_2 G) \xrightarrow{h} \Omega_F^0(E_1 G) \xrightarrow{d} \Omega_F^1(E_1 G) \xrightarrow{h} \Omega_F^1(E_0 G) \xrightarrow{d} \Omega_F^2(E_0 G) \xrightarrow{\iota_2^*} \Gamma(\wedge^2 A^*).$$
Let \( j_p : B_p G \to E_p G \) be the inclusion as submanifold for which \( a_0 \in M \).
Let $j_p: B_p G \rightarrow E_p G$ be the inclusion as submanifold for which $a_0 \in M$.

**IF** given retraction of $G$ onto $M$ along $t$-fibers
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**IF** given retraction of $G$ onto $M$ along $t$-fibers

$\rightsquigarrow$ retraction of $E_p G$ onto $j_p(B_p G)$ along $\kappa_p$-fibers,
Let $j_p : B_p G \to E_p G$ be the inclusion as submanifold for which $a_0 \in M$.

**IF** given retraction of $G$ onto $M$ along $t$-fibers

$\rightsquigarrow$ retraction of $E_p G$ onto $j_p(B_p G)$ along $\kappa_p$-fibers,

$\rightsquigarrow$ homotopy operator $k : \Omega^{-\bullet}_{\mathcal{F}}(E \bullet G) \to \Omega^{-1}_{\mathcal{F}}(E \bullet G)$ with $kd + dk = 1 - \kappa^* j^*$. 

Recall that $C^\infty(BG)$ denotes 'germs'.

**Corollary** For any (local) Lie groupoid, $\text{VE}: C^\infty(BG) \to \Gamma(\wedge A^\bullet)$ is a quasi-isomorphism.
Let $j_p : B_p G \to E_p G$ be the inclusion as submanifold for which $a_0 \in M$.

**IF** given retraction of $G$ onto $M$ along $t$-fibers

\[ \rightsquigarrow \text{retraction of } E_p G \text{ onto } j_p(B_p G) \text{ along } \kappa_p\text{-fibers}, \]

\[ \rightsquigarrow \text{homotopy operator } k : \Omega^{\bullet}(E\cdot G) \to \Omega^{\bullet-1}(E\cdot G) \text{ with } \]

\[ kd + dk = 1 - \kappa^* j^*. \]

\[ \rightsquigarrow \text{‘integration’ } j^* \circ (1 + \delta k)^{-1} \circ \pi^* : \Gamma(\wedge^{\bullet} A^*) \to C^\infty(B\cdot G). \]
Van Est map

Let \( j_p : B_p G \to E_p G \) be the inclusion as submanifold for which \( a_0 \in M \).

IF given retraction of \( G \) onto \( M \) along \( t \)-fibers

\[ \mapsto \text{retraction of } E_p G \text{ onto } j_p(B_p G) \text{ along } \kappa_p \text{-fibers}, \]

\[ \mapsto \text{homotopy operator } k : \Omega^\bullet_F(E_* G) \to \Omega^\bullet_{F^{-1}}(E_* G) \text{ with } \]

\[ kd + dk = 1 - \kappa^* j^*. \]

\[ \mapsto \text{‘integration’ } j^* \circ (1 + \delta k)^{-1} \circ \pi^* : \Gamma(\wedge^\bullet A^*) \to C^\infty(B_* G). \]

Recall that \( C^\infty(BG)_M \) denotes ‘germs’.

Corollary

For any (local) Lie groupoid,

\[ \text{VE} : C^\infty(BG)_M \to \Gamma(\wedge A^*) \]

is a quasi-isomorphism.
Consider following situation:

- $(C, d, \delta)$ be a bigraded bidifferential algebra
- $i: D \hookrightarrow C$ a sub-bidifferential algebra
- $r: C \to D$ a projection preserving products
- $h: C^{\bullet, \bullet} \to C^{\bullet-1, \bullet}$ with

$$h\delta + \delta h = 1 - i \circ r.$$ 

Lemma (Gugenheim-Lambe-Stasheff)

Suppose $h$ is a twisted derivation

$$h(\omega \cup \omega') = h(\omega) \cup (i \circ r)(\omega') + (-1)^{|\omega|} \omega \cup h(\omega'),$$

and that it satisfies the side conditions $h \circ h = 0$ and $h \circ i = 0$. Then

$$i' = (1 + hd)^{-1}i, \quad r' = r(1 + dh)^{-1}$$

are morphisms of graded differential algebras (w.r.t. $d + \delta$).
In our case, these conditions hold once we restrict to the normalized subcomplex

$$\tilde{C}^{\infty}(B\dot{G}) \subset C^{\infty}(B\dot{G})$$

(i.e. kernel of the degeneracy maps $\epsilon_i^*$). Hence we obtain

The map

$$\text{VE}: \tilde{C}^{\infty}(BG) \rightarrow \Gamma(\wedge A^*)$$

preserves products.
The discussion also applies to the more general Van Est map

$$\text{VE}: W^{p,q}(G) = \Omega^q(B_p G) \to W^{p,q}(A).$$

In particular:

- given a retraction of $G$ along $t$-fibers there is a canonical ‘integration map’ in opposite direction
- over the normalized complex, VE preserves products
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In particular:

- given a retraction of $G$ along $t$-fibers there is a canonical ‘integration map’ in opposite direction
- over the normalized complex, VE preserves products

Another definition: Note that $\Gamma(\wedge^p A^*)$ are skew-symmetric multilinear functions on $A \times_M A \cdots \times_M A$ ($p$ factors).

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Definition

$W^{p,q}(A)$ are the skew-symmetric multi-linear $q$-forms on $A \times_M A \cdots \times_M A$ ($p$ factors).
Another definition: ‘Kähler differentials’. Start with any vector bundle $A \to M$. 

\[ \Omega^1_R = \text{hom}_R(X^1_R, R) \]

$\Omega^q_R$ skew-symmetric $R$-multilinear $q$-forms $W^q(A) = \Gamma(\Omega^q_R)$. $W(A)$ has a ‘de Rham’ differential of degree $(0, 1)$.

Any degree $k$ derivation $X$ of $\Gamma(R)$ extends to a degree $(k, 0)$ derivation $L_X$ of $W(A)$. If $A$ is a Lie algebroid, apply this to $X = d_{CE}$. 

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- $\mathcal{X}_R^1 = \text{der}(R)$. I.e., $\Gamma(\mathcal{X}_R^1) = \text{der}(\Gamma(R))$. 

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- $\mathcal{X}_R^1 = \text{der}(R)$. I.e., $\Gamma(\mathcal{X}_R^1) = \text{der}(\Gamma(R))$.
- $\Omega^1_R = \text{hom}_R(\mathcal{X}_R^1, R)$. 

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Another definition: ‘Kähler differentials’. Start with any vector bundle \( A \rightarrow M \).

- \( R := \wedge A^* \)
- \( \mathcal{X}^1_R = \text{der}(R) \). I.e., \( \Gamma(\mathcal{X}^1_R) = \text{der}(\Gamma(R)) \).
- \( \Omega^1_R = \text{hom}_R(\mathcal{X}^1_R, R) \).
- \( \Omega^q_R \) skew-symmetric \( R \)-multilinear \( q \)-forms
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- $W^{\bullet,q}(A) = \Gamma(\Omega_R^q)$.
- $W(A)$ has a ‘de Rham’ differential of degree $(0, 1)$.
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Any degree $k$ derivation $X$ of $\Gamma(R)$ extends to a degree $(k, 0)$ derivation $\mathcal{L}_X$ of $W(A)$. If $A$ is a Lie algebroid, apply this to $X = d_{CE}$. 

Eckhard Meinrenken (based on joint work with David Li-Bland)
Thanks.