Amenable minimal Cantor systems of free groups arising from diagonal actions

Yuhei SUZUKI

RIMS at Kyoto University

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Cantor set

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3. metrizability
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$\Rightarrow$ The property 'homeomorphic to the Cantor set' is preserved by many operations.
(E.g., finite direct sum, countable direct product, projective limit,...)
$\Rightarrow$ We can regard the Cantor set as a topological analogue of the Lebesgue space.

Classification theory of C^*-algebras

A : C^*-algebra.
$\rightsquigarrow$ $K_*(A) := (K_0(A), [1_A]_0, K_1(A))$: an invariant of A.
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Example :

- The Cuntz algebras $\mathcal{O}_n$ $(2 \leq n \leq \infty)$.
- The Cuntz–Krieger algebras $\mathcal{O}_A$.
- The boundary algebras $C(\partial\Gamma) \rtimes \Gamma$ of ICC hyperbolic groups.

Amenable dynamical systems

Amenability of discrete groups is generalized to that of topological dynamical systems.

Example

1. Any dynamical system of an amenable group.
2. The boundary action of a hyperbolic group.
3. $\mathrm{SL}(n, \mathbb{Z}) \curvearrowright \mathrm{SO}(n) = \mathrm{SL}(n, \mathbb{R})/P$.

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$\alpha\colon \Gamma \curvearrowright X$: amenable $\Rightarrow$ $C(X) \rtimes_{\mathrm{red}} \Gamma$ has nice properties.

- $C(X) \rtimes_{\mathrm{full}} \Gamma = C(X) \rtimes_{\mathrm{red}} \Gamma$ canonically.
- $C(X) \rtimes \Gamma$ is nuclear.
- $C(X) \rtimes \Gamma$ satisfies the universal coefficient theorem. (Tu 1999)

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Minimality = Topological analogue of ergodicity

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2. Give a new presentation for a *Kirchberg algebra* in the UCT class.

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For both purposes, it is important to construct concrete and tractable examples. Until now, only a few examples were known.

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Example

1. The boundary action $\beta_n\colon \mathbb{F}_n \curvearrowright \partial\mathbb{F}_n$. (Analysed by J. Spielberg.)
2. (G. A. Elliott and A. Sierakowski 2011)
   $\exists$ amenable minimal Cantor $\mathbb{F}_n$-system s.t. $K_0 = 0$.

Main Theorem

Theorem (S. 13)

Let $\mathbb{Z}^\infty \leq G \leq \mathbb{Q}^\infty$ *with* $[G : \mathbb{Z}^\infty] = \infty$, $2 \leq n < \infty$, $k \in \mathbb{Z}$.
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- $(K_0(C(X) \rtimes_\gamma \mathbb{F}_n), [1]_0) \cong (G \oplus \Lambda_{G,n}, 0 \oplus [k(n-1)^{-1}])$.
  Here

  $$\Lambda_{G,n} := \{x \in \mathbb{Q}/\mathbb{Z} : \exists \text{ finite } H \leq G, \text{ s.t. } \mathrm{ord}(x) | (n-1)\sharp H\}.$$

- $K_1(C(X) \rtimes_\gamma \mathbb{F}_n) \cong \mathbb{Z}^\infty$.
- *The crossed product is a Kirchberg algebra in the UCT class.*

Sketch of the construction

(We only deal the case $k = 1$.)
Take a decreasing sequence $(\Gamma_m)_{m=1}^{\infty}$ of finite index subgroups of $\mathbb{F}_n$.
We study $\varprojlim(\mathbb{F}_n \curvearrowright \partial\mathbb{F}_n \times \mathbb{F}_n/\Gamma_m)_{m=1}^{\infty}$.

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Computation of $(\mathbf{K_0}, [\mathbf{1}]_0)$
Each $K_0(C(\partial\mathbb{F}_n \times \mathbb{F}_n/\Gamma_m) \rtimes \mathbb{F}_n)$ is explicitly computable. [Spielberg (1991), Cuntz (1981)] Then determine K_0-maps of the inclusions

$$C(\partial\mathbb{F}_n \times \mathbb{F}_n/\Gamma_m) \rtimes \mathbb{F}_n \hookrightarrow C(\partial\mathbb{F}_n \times \mathbb{F}_n/\Gamma_{m+1}) \rtimes \mathbb{F}_n.$$

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By continuity of K_0-groups, the construction of suitable systems and the computation of K_0-group are now reduced to algebraic problems.
Computation of $\mathbf{K_1}$
Use the Pimsner–Voiculescu Exact Sequence for free groups.

Consequence of the Main Theorem

Induced dynamical system construction
$\rightsquigarrow$Similar results for virtually free groups
(Ex : $\mathrm{SL}(2,\mathbb{Z})$, $G_1 * G_2 * \cdots * G_n$; G_i finite or $\mathbb{Z}$.).
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Corollary (S. 13)

G : torsion free abelian group of infinite rank. A : Kirchberg algebra in the UCT class s.t.

$$K_*(A) \cong (G \oplus \mathbb{Q}/\mathbb{Z}, 0, \mathbb{Z}^\infty).$$

Then $\forall\ \Gamma$: virtually free group, A is decomposed as the crossed product of an amenable minimal Cantor Γ-system.

Free Examples

$\Gamma \curvearrowright X$: Free $\Leftrightarrow \forall g \in \Gamma \setminus \{e\}, \nexists$ fixed point.
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Theorem

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Remark

We also can prove the same result for non f.g. case. In this case, we use $\mathbb{F}_\infty \cong [\mathbb{F}_2, \mathbb{F}_2] \curvearrowright \partial\mathbb{F}_2$ instead of the boundary action.

Application to classification of Cantor systems

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Odometer transformations

$(n_k)_{k=1}^{\infty}$: sequence of positive integers ≥ 2.
Consider

$$\varinjlim(\alpha_k \colon \mathbb{Z} \curvearrowright \mathbb{Z}/n_1 \cdots n_k\mathbb{Z})_{k=1}^{\infty}.$$

This only depends on the formal infinite product $N = \prod_{k=1}^{\infty} n_k$.
Denote it by α_N and call it the odometer transformation of type N.

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Example

$N = p^{\infty}$, p : prime number.
Then $(X, \alpha_N) = (\mathbb{Z}_p, +1)$. ($\mathbb{Z}_p$: the ring of p-adic integers.)

Application to classification of Cantor systems

For $2 \leq n < \infty$ and $N_1, \ldots, N_k$: sequence of infinite supernatural numbers with $k \leq n$, consider the Cantor $\mathbb{F}_n$-system

$$\gamma_{N_1,\ldots,N_k}{}^{(n)} := \beta_n \times \prod_{i=1}^{k} \alpha_{N_i} \circ \pi_i .$$

Here $\pi_i \colon \mathbb{F}_n = \langle s_1, \ldots, s_n \rangle \to \mathbb{Z}$ is a homomorphism given by $s_i \mapsto 1$ and $s_j \mapsto 0$ for $j \neq i$.

Application to classification of Cantor systems

Definition

$\gamma_i \colon \Gamma_i \curvearrowright X_i$: minimal topologically free Cantor system (i=1, 2).
γ_1 and γ_2 are orbit equivalent $\Leftrightarrow \exists h \colon X_1 \to X_2$ homeomorphism, s.t.
$\forall x \in X_1$, $h(\Gamma_1.x) = \Gamma_2.h(x)$.

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$\gamma : \Gamma \curvearrowright X$: minimal topologically free Cantor system.
Topological full group $[[\gamma]] :=$ the group consists of all $h \in Homeo(X)$ which are "**locally**" given by $s \in \Gamma$.

Classification results for γ's

Theorem (S. 13)

For two Cantor systems $\varphi := \gamma^{(n)}_{N_1,\ldots,N_k}$ *and* $\psi := \gamma^{(m)}_{M_1,\ldots,M_l}$, *T.F.A.E.*

1. *They are strong orbit equivalent.*
2. *They are continuous orbit equivalent.*
3. $[[\varphi]] \cong [[\psi]]$.
4. $C(X) \rtimes_\varphi \mathbb{F}_n \cong C(X) \rtimes_\psi \mathbb{F}_m$.
5. $K_*(C(X) \rtimes_\varphi \mathbb{F}_n) \cong K_*(C(X) \rtimes_\psi \mathbb{F}_m)$.
6. $k = l$, $n = m$, *and* $\exists \sigma \in \mathfrak{S}_k$ *and* $\exists (n_1, \ldots, n_k)$, $\exists (m_1, \ldots, m_k)$ *s.t.* $\prod_{j=1}^k n_j = \prod_{j=1}^k m_j$ *and* $n_i N_i = m_i M_{\sigma(i)}$ $\forall i$.

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