Purely infinite C^*-algebras associated to étale groupoids

joint with J. Brown (Kansas) and L. Clark (Otago)

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Introduction

In this talk we shall only consider certain groupoids:
SHÉL = Second countable Hausdorff Étale and Locally compact

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2. If $\gamma \in G$ then $(\gamma^{-1})^{-1} = \gamma$ and $(\gamma^{-1}, \gamma) \in G^{(2)}$.

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2. If $\gamma \in G$ then $(\gamma^{-1})^{-1} = \gamma$ and $(\gamma^{-1}, \gamma) \in G^{(2)}$.
3. If $(\gamma, \eta) \in G^{(2)}$ then $\gamma^{-1}(\gamma\eta) = \eta$ and $(\gamma\eta)\eta^{-1} = \gamma$.

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Remark

An equivalent definition can be made when regarding G as a set of arrows together with $G^{(0)}$ (the set of endpoints) and the associated source and range maps

$$r, s : G \to G^0 = r(G) = s(G), \quad r(\gamma) = \gamma\gamma^{-1}, \quad s(\gamma) = \gamma^{-1}\gamma, \quad \gamma \in G.$$

The composition and the inverse can then be regarded as a composition of arrows and as an inverse arrow.

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Definition

Let G be a topological groupoid. If the topology on G is second countable locally compact Hausdorff then G is called a *second countable locally compact Hausdorff* groupoid.

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An *étale groupoid* is a topological groupoid where s is a local homeomorphism.

Remark

Let G be a second countable locally compact Hausdorff groupoid. Then G is étale if and only if it has a countable basis of open bisections[1] *with compact closure.*

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Example 1

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Given $n \geq 2$. Define $S = \{1, \ldots, n\}$ and let Z and W be the infinite and finite sequences (words) in S.

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$$(\alpha\gamma, l(\alpha) - l(\beta), \beta\gamma),$$

for finite α, β and infinite γ and $l(\cdot)$ the length a word.

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for finite α, β and infinite γ and $l(\cdot)$ the length a word. With

$$G_n^2 = \{((x, k, x'), (y, l, y')) \in G_n \times G_n \ : \ x' = y\},$$

$$(x, k, x')(x', l, x'') = (x, k + l, x''), \quad (x, k, x')^{-1} = (x', -k, x),$$

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G_n is a groupoid. For S discrete and Z with the product topology, the basis $U_{\alpha,\beta} = \{(\alpha\gamma, l(\alpha) - l(\beta), \beta\gamma) \;:\; \gamma \in Z\}$ for α, β finite, makes G_n into a SHÉL groupoid.

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$$G^2_{(X,G)} = \{((x,t),(y,s)) \ : \ x.t = y\},$$

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- Oriented transformation groupoid algebras $C_r^*(\Gamma_\phi^+)$
- Bunce-Deddens C^*-algebras
- Partial crossed products $C_0(X)xG$ (with countable G)

Properties of groupoid algebras.

[2] A cpc (completely positive contractive) map s.t. $E(ba) = bE(a)$, $E(ab) = E(a)b$, $E(b) = b$ $(b \in B, a \in A)$

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Lemma (cf. Renault (1980))

Let G be SHÉL groupoid. Then

1. *The extension map from $C_c(G^{(0)})$ into $C_c(G)$ (where a function is defined to be zero on $G - G^{(0)}$) extends to an embedding of $C_0(G^{(0)})$ into $C_r^*(G)$.*

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2. *The restriction map $E_0 : C_c(G) \to C_c(G^{(0)})$ extends to a conditional expectation[2] $E : C_r^*(G) \to C_0(G^{(0)})$.*

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3. *The map E from item (2) is faithful. That is, $E(b^*b) = 0$ implies $b = 0$ for $b \in C_r^*(G)$.*

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4. *The subalgebra $C_c(G^{(0)})$ contains an approximate unit for $C_r^*(G)$.*

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\begin{array}{ccccccccc}
0 & \longrightarrow & C_r^*(G|_U) & \xrightarrow{\iota_r} & C_r^*(G) & \xrightarrow{\rho_r} & C_r^*(G|_D) & \longrightarrow & 0\,, \\
 & & \downarrow{\scriptstyle E_U} & & \downarrow{\scriptstyle E} & & \downarrow{\scriptstyle E_D} & & \\
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where $U = G^{(0)} - D$, ι_r and ρ_r are determined on continuous functions by extension and restriction respectively. Moreover, $\mathrm{image}(\iota_r) \subseteq \ker \rho_r$.

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Lemma

Let G be a SHÉL groupoid. and $E : C_r^*(G) \to C_0(G^{(0)})$ be the faithful conditional expectation extending restriction. Suppose that G is topologically principal. For every $\epsilon > 0$ and $c \in C_r^*(G)^+$, there exists $f \in C_0(G^{(0)})^+$ s.t.:

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Let G be a SHÉL groupoid. Suppose that G is topologically principal. For every nonzero $a \in C_r^*(G)^+$, there exists nonzero $h \in C_0(G^{(0)})^+$ s.t. $h \precsim a$.

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1. *$C_r^*(G)$ is purely infinite*

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Theorem

Let G be a SHÉL groupoid. Suppose that G is minimal and topologically principal. TFAE

1. *$C_r^*(G)$ is purely infinite*
2. *Every nonzero positive element of $C_0(G^{(0)})$ is infinite in $C_r^*(G)$.*

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Let G be a SHÉL groupoid. TFAE

1. $C^*(G)$ *is a Kirchberg algebra*
2. G *is minimal, topologically principal, measure-wise amenable and every non-zero positive element of* $C_0(G^{(0)})$ *is infinite in* $C^*(G)$.

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Corollary

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Remark

(1)-(2) $\Rightarrow$ UCT.

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Kirchberg algebras in UCT are (stably isomorphic to) SHÉL ample groupoids.

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Theorem

Let G be a SHÉL ample groupoid. Suppose that G is topologically principal, minimal and that $\mathbb{B}$ is a basis of $G^{(0)}$ consisting of compact open sets. TFAE

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1. $C_r^*(G)$ *is purely infinite*
2. *Every nonzero projection p in $C_0(G^{(0)})$ with $\operatorname{supp}(p) \in \mathbb{B}$ is infinite in $C_r^*(G)$.*

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Corollary

Let G be a SHÉL ample groupoid. Suppose that G is topologically principal and minimal. TFAE

1. $C_r^*(G)$ *is purely infinite*
2. *There exists a point* $x \in G^{(0)}$ *and a neighbourhood basis* $\mathcal{D}$ *at* x *consisting of compact open sets s.t. every nonzero projection* q *in* $C_0(G^{(0)})$ *with* $\mathrm{supp}(q) \in \mathcal{D}$ *is infinite in* $C_r^*(G)$.

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Corollary

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1. $C^*(\Lambda)$ *is purely infinite.*
2. *For every vertex* $v \in \Lambda^0$ *the projection is infinite.*
3. *There exists* $x \in \Lambda^\infty$ *s.t.* p_v *is infinite for every vertex* v *on* x.

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1. *For every closed invariant set $D \subseteq G^{(0)}$*

$$C^*(G|_D) = C_r^*(G|_D).$$

2. *For every closed invariant set $D \subseteq G^{(0)}$ the sequence*

$$0 \longrightarrow C_r^*(G|_{G^{(0)}-D}) \xrightarrow{\iota_r} C_r^*(G) \xrightarrow{\rho_r} C_r^*(G|_D) \longrightarrow 0$$

is exact where ι_r and ρ_r are determined on continuous functions by extension and restriction respectively.

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1. *The C^*-algebra $C_r^*(G)$ is strongly purely infinite, and*

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References

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